ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 38, Number 2, 2008

ON SUMS OF SECOND ORDER LINEAR RECURRENCES BY HESSENBERG MATRICES

E. KILIÇ AND D. TAŞCI

ABSTRACT. In this paper, we derive some relationships between the sums of second order linear recurrences and permanents or determinants of certain Hessenberg matrices.

1. Introduction. The Fibonacci sequence, $\{F_n\}$, is defined by the recurrence relation, for $n \geq 1$,

$$F_{n+1} = F_n + F_{n-1}, \tag{1.1}$$

where $F_0 = 0$, $F_1 = 1$. The Lucas sequence, $\{L_n\}$ is defined by the recurrence relation, for $n \geq 1$,

$$L_{n+1} = L_n + L_{n-1}, \tag{1.2}$$

where $L_0 = 2$, $L_1 = 1$.

The well-known Fibonacci, Lucas and Pell numbers can be generalized as follows: Let A and B be nonzero, relatively prime integers such that $D = A^2 - 4B \neq 0$. Define sequences $\{u_n\}$ and $\{v_n\}$ by, for all $n \geq 2$, see [**14**],

$$u_n = Au_{n-1} - Bu_{n-2} \tag{1.3}$$

$$v_n = Av_{n-1} - Bv_{n-2} \tag{1.4}$$

where $u_0 = 0$, $u_1 = 1$ and $v_0 = 2$, $v_1 = A$. If $A = 1$ and $B = -1$, then $u_n = F_n$ (the nth Fibonacci number) and $v_n = L_n$ (the nth Lucas number). If $A = 2$ and $B = -1$, then $u_n = P_n$ (the nth Pell number).

An alternative is to let the roots of the equation $t^2 - At + B = 0$ be α and β, for $n \geq 0$,

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad v_n = \alpha^n + \beta^n. \tag{1.5}$$

2000 AMS *Mathematics subject classification.* Primary 11B37, 15A15, 15A36.
Keywords and phrases. Second order linear recurrences, Hessenberg matrix, permanent, determinant, matrix methods.
Received by the editors on January 2, 2006.

DOI:10.1216/RMJ-2008-38-2-531

There are many relationships between permanents or determinants of tridiagonal matrices and the Fibonacci and Lucas numbers. For example, Minc [**12**] defined an $n \times n$ super diagonal $(0,1)$-matrix $F(n,k)$ for $n > k \geq 2$ and showed that the permanent of $F(n,k)$ equals the generalized order-k Fibonacci numbers. Also he gave some relations involving the permanents of some $(0,1)$-circulant matrices and the usual Fibonacci numbers.

In [**8**], the authors derived a nice result involving the permanent of a $(-1,0,1)$-matrix and the Fibonacci number F_{n+1}. The authors then explored similar directions involving the positive subscripted Fibonacci and Lucas numbers as well as their uncommon negatively subscripted counterparts. Finally, the authors obtained a relation between permanents of a certain matrix and the generalized Lucas number, see [**7, 17**] for more details on the generalized Fibonacci and Lucas numbers.

In [**9, 10**], the authors gave the relations involving the generalized Fibonacci and Lucas numbers and the permanent of the $(0,1)$-matrices. The results of Minc, [**12**], and the result of Lee, [**10**], on the generalized Fibonacci numbers are the same because they use the same matrix. However, Lee proved the same result by a different method, the contraction method for the permanent (for more detail of the contraction method, see [**1**]).

In [**11**], Lehmer proved a very general result on permanents of tridiagonal matrices whose main diagonal and super-diagonal elements are ones and whose subdiagonal entries are somewhat arbitrary.

Also in [**15, 16**], the authors defined a family of tridiagonal matrices $M(n)$ and showed that the determinants of $M(n)$ are the Fibonacci numbers F_{2n+2}. In [**3, 4**], the family of tridiagonal matrices $H(n)$ are defined and the authors showed that the determinants of $H(n)$ are the Fibonacci numbers F_n. In a similar family of matrices, the $(1,1)$ element of $H(n)$ is replaced with a 3. The determinants, [**2**], now generate the Lucas numbers L_n.

In [**5**], the authors obtained the families of $(0,1)$-matrices such that permanents of the matrices, equal to the sums of Fibonacci and Lucas numbers.

In [**6**], the authors defined two tridiagonal matrices and then gave relationships between the permanents and determinants of these matrices and second order linear recurrences.

The *permanent* of an n-square matrix $A = (a_{ij})$ is defined by

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} \tag{1.6}$$

where the summation extends over all permutations σ of the symmetric group S_n.

The following facts can be found in [**6**]: Let

$$T_n = [t_{ij}] = \begin{pmatrix} \alpha+\beta & \beta & & & 0 \\ \alpha & \alpha+\beta & \beta & & \\ & \alpha & \alpha+\beta & \ddots & \\ & & \ddots & \ddots & \beta \\ 0 & & & \alpha & \alpha+\beta \end{pmatrix}_{n\times n} \tag{1.7}$$

where α, β are real or complex numbers such that $\alpha\beta \neq 0$ and $(\alpha+\beta)^2 \neq 4\alpha\beta$. Then, for all $n \geq 1$,

$$\det T_n = u_{n+1} \tag{1.8}$$

where u_n is the nth term of the sequence $\{u_n\}$ given by (1.3). Let also

$$K_n = \begin{pmatrix} \alpha+\beta & 2\beta & & & 0 \\ \alpha & \alpha+\beta & \beta & & \\ & \alpha & \alpha+\beta & \ddots & \\ & & \ddots & \ddots & \beta \\ 0 & & & \alpha & \alpha+\beta \end{pmatrix}_{n\times n}. \tag{1.9}$$

Then, for $n \geq 1$,

$$\det K_n = v_n \tag{1.10}$$

where v_n is the nth term of the sequence $\{v_n\}$ given by (1.4) and α, β are as before.

An $m \times m$ matrix $A = (a_{ij})$ is a Hessenberg matrix or upper Hessenberg matrix if $a_{ij} = 0$ for $i > j+1$, that is,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,m-1} & a_{1,m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,m-1} & a_{2,m} \\ 0 & a_{32} & a_{33} & \cdots & a_{3,m-1} & a_{3,m} \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & a_{m-1,m-1} & a_{m-1,m} \\ 0 & 0 & 0 & \cdots & a_{m,m-1} & a_{m,m} \end{bmatrix}$$

In this paper, we give some relations involving the sums of second order linear recurrences and some special Hessenberg permanents and determinants.

2. Sums of the second order linear recurrence $\{u_n\}$ by Hessenberg matrices. In this section, we define certain $n \times n$ Hessenberg matrices and then determine some of the relationships between the sums of second order linear recurrence $\{u_n\}$ and certain Hessenberg permanents and determinants.

We define an $n \times n$ upper Hessenberg matrix $H_n = (h_{ij})$ with $h_{1i} = 1$ for $1 \le i \le n$, $h_{i+1,i} = 1$ for $1 \le i \le n-1$, $h_{ii} = \alpha+\beta$ and $h_{i,i+1} = -\alpha\beta$ for $2 \le i \le n$, and 0 otherwise. Clearly,

(2.1)

$$H_n = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & \alpha+\beta & -\alpha\beta & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \alpha+\beta & -\alpha\beta & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & \alpha+\beta & -\alpha\beta & \ddots & \dots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 & \alpha+\beta & -\alpha\beta & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & \alpha+\beta & -\alpha\beta \\ 0 & 0 & 0 & \dots & \dots & 0 & 1 & \alpha+\beta \end{bmatrix}.$$

Considering the tridiagonal matrix T_n and its permanent $\operatorname{per} T_n$ which are given by (1.7) and (1.8), and using the results of Minc [**13**] and Zhang [**18**] on tridiagonal permanents and determinants, we give the following theorem without proof.

Theorem 1. *Let α, β be the roots of the equation $t^2 - At + B = 0$, and the $n \times n$ tridiagonal matrix A_n has the form*

$$A_n = \begin{bmatrix} \alpha+\beta & -\alpha\beta & & & 0 \\ 1 & \alpha+\beta & -\alpha\beta & & \\ & 1 & \alpha+\beta & \ddots & \\ & & \ddots & \ddots & -\alpha\beta \\ 0 & & & 1 & \alpha+\beta \end{bmatrix}.$$

Then, for $n \ge 1$,

$$\operatorname{per} A_n = u_{n+1}$$

where u_n is the nth term of the sequence $\{u_n\}$.

Considering the definitions of the matrices H_n and A_n, we may write

$$H_n = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & & & & \\ 0 & & A_n & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}.$$

Now we have the following theorem.

Theorem 2. *Let α, β be the roots of the equation $t^2 - At + B = 0$, and the upper Hessenberg matrix H_n has the form (2.1). Then, for $n \geq 2$,*

$$\operatorname{per} H_n = \sum_{i=0}^{n} u_i,$$

where u_n is the nth term of the sequence $\{u_n\}$.

Proof. We will use the induction method to prove that $\operatorname{per} H_n = \sum_{i=0}^{n} u_i$. If $n = 2$,

$$\operatorname{per} H_2 = \operatorname{per} \begin{bmatrix} 1 & 1 \\ 1 & \alpha + \beta \end{bmatrix} = \alpha + \beta + 1.$$

If we multiply and divide the term $(\alpha + \beta)$ in the value of $\operatorname{per} H_2$ by $(\alpha - \beta)$, then we have

$$\operatorname{per} H_2 = \frac{(\alpha + \beta)(\alpha - \beta)}{(\alpha - \beta)} + 1 = \frac{\alpha^2 - \beta^2}{\alpha - \beta} + 1.$$

From the Binet formula of the sequence $\{u_n\}$, and since $u_1 = 1$ and $u_0 = 0$, we obtain $\operatorname{per} H_2 = \sum_{i=0}^{2} u_i = u_2 + u_1 + u_0$.

If $n = 3$, then we have

$$\begin{aligned}\operatorname{per} H_3 &= \operatorname{per}\begin{bmatrix}1 & 1 & 1\\ 1 & \alpha+\beta & -\alpha\beta\\ 0 & 1 & \alpha+\beta\end{bmatrix}\\ &= \left(\alpha^2+\alpha+\beta+\beta^2\right)+(\alpha+\beta)+1.\end{aligned}$$

If we multiply and divide the term $(\alpha^2+\alpha+\beta+\beta^2)+(\alpha+\beta)$ in the value of $\operatorname{per} H_3$ by $(\alpha-\beta)$, then we have

$$\operatorname{per} H_3 = \frac{\alpha^3-\beta^3}{\alpha-\beta}+\frac{\alpha^2-\beta^2}{\alpha-\beta}+1.$$

Using the Binet formula of the sequence $\{u_n\}$, and since $u_1 = 1$ and $u_0 = 0$,

$$\operatorname{per} H_3 = \sum_{i=0}^{3} u_i = u_3+u_2+u_1+u_0.$$

We suppose that the equation holds for n. Then we show that the equation holds for $n+1$. Thus, if we compute $\operatorname{per} H_{n+1}$ by the Laplace expansion of permanent with respect to the first column, then we obtain

$$\begin{aligned}\operatorname{per} H_{n+1} &= \operatorname{per}\begin{bmatrix}1 & 1 & 1 & 1 & \cdots & 1 & 1\\ 1 & \alpha+\beta & -\alpha\beta & 0 & \cdots & 0 & 0\\ 0 & 1 & \alpha+\beta & -\alpha\beta & \ddots & \cdots & 0\\ 0 & 0 & 1 & \alpha+\beta & \ddots & \cdots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & -\alpha\beta & 0\\ 0 & \cdots & \cdots & 0 & 1 & \alpha+\beta & -\alpha\beta\\ 0 & 0 & \cdots & \cdots & 0 & 1 & \alpha+\beta\end{bmatrix}\\ &= \operatorname{per}\begin{bmatrix}\alpha+\beta & -\alpha\beta & 0 & & & 0\\ 1 & \alpha+\beta & -\alpha\beta & 0 & & \\ 0 & 1 & \alpha+\beta & -\alpha\beta & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & 0\\ & & 0 & 1 & \alpha+\beta & -\alpha\beta\\ 0 & & & 0 & 1 & \alpha+\beta\end{bmatrix}\end{aligned}$$

$$
+\operatorname{per}\begin{bmatrix}
1 & 1 & 1 & \dots & 1 & 1\\
1 & \alpha+\beta & -\alpha\beta & 0 & \dots & 0\\
0 & 1 & \alpha+\beta & -\alpha\beta & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & 0\\
0 & \dots & 0 & 1 & \alpha+\beta & -\alpha\beta\\
0 & 0 & \dots & 0 & 1 & \alpha+\beta
\end{bmatrix}.
$$

From the definitions of the matrices H_n and A_n, we can write that

$$
\operatorname{per} H_{n+1} = \operatorname{per} A_n + \operatorname{per} H_n. \tag{2.2}
$$

Combining the result of Theorem 1 and our assumption, we have

$$
\operatorname{per} H_{n+1} = u_{n+1} + \sum_{i=0}^{n} u_i = \sum_{i=0}^{n+1} u_i.
$$

So the proof is complete. □

For example, when $\alpha = 1+\sqrt{5}/2$ and $\beta = 1-\sqrt{5}/2$, the sequence $\{u_n\}$ is reduced to the Fibonacci sequence $\{F_n\}$ and, by Theorem 2, we have

$$
\operatorname{per}\begin{bmatrix}
1 & 1 & 1 & 1 & \dots 1 & 1 & 1 & 1\\
1 & 1 & 1 & 0 & \dots & \dots & \dots & 0\\
0 & 1 & 1 & 1 & 0 & \dots & \dots & 0\\
0 & 0 & 1 & 1 & 1 & \ddots & & \vdots\\
\vdots & & \ddots & \ddots & \ddots & \ddots & 0 & \vdots\\
0 & \dots & \dots & 0 & 1 & 1 & 1 & 0\\
0 & 0 & \dots & \dots & 0 & 1 & 1 & 1\\
0 & 0 & 0 & \dots & \dots & 0 & 1 & 1
\end{bmatrix}_{n\times n} = \sum_{i=0}^{n} F_i,
$$

where F_n is the nth Fibonacci number. This result is given in [**5**].

A matrix A is called *convertible* if there is an $n\times n$ $(1,-1)$-matrix H such that $\operatorname{per} A = \det(A \circ H)$, where $A\circ H$ denotes the Hadamard product of A and H. Such a matrix H is called a *converter* of A.

Let S be a $(1,-1)$-matrix of order n, defined by

$$S = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ -1 & 1 & 1 & \dots & 1 & 1 \\ 1 & -1 & 1 & \dots & 1 & 1 \\ 1 & 1 & -1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \dots & -1 & 1 \end{bmatrix}.$$

Denote the matrix $H_n \circ S$ by G_n, that is,

(2.3)

$$G_n = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ -1 & \alpha+\beta & -\alpha\beta & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & \alpha+\beta & -\alpha\beta & 0 & \dots & \dots & 0 \\ 0 & 0 & -1 & \alpha+\beta & -\alpha\beta & \ddots & \dots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -1 & \alpha+\beta & -\alpha\beta & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & \alpha+\beta & -\alpha\beta \\ 0 & 0 & 0 & \dots & \dots & 0 & -1 & \alpha+\beta \end{bmatrix}.$$

Then we have the following theorem without proof.

Theorem 3. *Let α, β be the roots of the equation $t^2 - At + B = 0$, and let the upper Hessenberg matrix G_n have the form (2.3). Then, for $n \geq 2$,*

$$\det G_n = \sum_{i=0}^{n} u_i$$

where u_n is the nth term of the sequence $\{u_n\}$.

Also, when $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$, the sequence $\{u_n\}$ is reduced to the Pell sequence $\{P_n\}$ and, by Theorem 3, we have

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ -1 & 2 & 1 & 0 & & \dots & \dots & 0 \\ 0 & -1 & 2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & -1 & 2 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & \dots & 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & \dots & \dots & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & \dots & \dots & 0 & -1 & 2 \end{bmatrix}_{n\times n} = \sum_{i=0}^{n} P_i$$

where P_n is the nth Pell number.

3. Sums of the second order linear recurrence $\{v_n\}$ by Hessenberg matrices. In this section, we define certain $n \times n$ Hessenberg matrices and then determine some of the relationships between the sums of second order linear recurrence $\{v_n\}$ and certain Hessenberg permanents and determinants.

We define an $n \times n$ upper Hessenberg matrix $D_n = (d_{ij})$ with $d_{1i} = 1$ for $1 \le i \le n-1$, $d_{1n} = 2$, $d_{i+1,i} = 1$ for $1 \le i \le n-1$, $d_{ii} = \alpha+\beta$ for $2 \le i \le n$, $d_{i,i+1} = -\alpha\beta$ for $2 \le i \le n-2$, $d_{n-1,n} = -2\alpha\beta$ and 0 otherwise. Clearly,

(3.1)
$$D_n = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \dots & 1 & 2 \\ 1 & \alpha+\beta & -\alpha\beta & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \alpha+\beta & -\alpha\beta & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & \alpha+\beta & -\alpha\beta & \ddots & \dots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 & \alpha+\beta & -\alpha\beta & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & \alpha+\beta & -2\alpha\beta \\ 0 & 0 & 0 & \dots & \dots & 0 & 1 & \alpha+\beta \end{bmatrix}.$$

Considering the tridiagonal matrix K_n and its permanent $\operatorname{per} K_n$ which are given by (1.9) and (1.10), and using the results of Minc [**13**] and Zhang [**18**] on tridiagonal permanents and determinants, we give the following theorem without proof.

Theorem 4. *Let α, β be the roots of the equation $t^2 - At + B = 0$, and let the $n \times n$ tridiagonal matrix B_n have the form*

$$B_n = \begin{bmatrix} \alpha+\beta & -\alpha\beta & & & 0 \\ 1 & \alpha+\beta & -\alpha\beta & & \\ & 1 & \alpha+\beta & \ddots & \\ & & \ddots & \ddots & -2\alpha\beta \\ 0 & & & 1 & \alpha+\beta \end{bmatrix}.$$

Then, for $n \geq 1$,

$$\operatorname{per} B_n = v_n$$

where v_n is the nth term of the sequence $\{v_n\}$.

Considering the definitions of the matrices D_n and B_n, we may write

$$D_n = \begin{bmatrix} 1 & 1 & \dots & 1 & 2 \\ 1 & & & & \\ 0 & & B_n & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}$$

Now we have the following theorem.

Theorem 5. *Let α, β be the roots of the equation $t^2 - At + B = 0$, and let the upper Hessenberg matrix D_n have the form (3.1). Then, for $n \geq 3$,*

$$\operatorname{per} D_n = \sum_{i=0}^{n-1} v_i$$

where v_n is the nth term of the sequence $\{v_n\}$.

Proof. We will use the induction method to prove that $\operatorname{per} D_n = \sum_{i=0}^{n-1} v_i$. If $n = 3$,

$$\operatorname{per} D_3 = \operatorname{per} \begin{bmatrix} 1 & 1 & 2 \\ 1 & \alpha+\beta & -2\alpha\beta \\ 0 & 1 & \alpha+\beta \end{bmatrix} = (\alpha^2+\beta^2) + (\alpha+\beta) + 2.$$

From the Binet formula of the sequence $\{v_n\}$, and since $v_0 = 2$, we obtain $\operatorname{per} D_3 = \sum_{i=0}^{2} v_i = v_2 + v_1 + v_0$.

If $n = 4$, then we have

$$\begin{aligned}\operatorname{per} D_4 &= \operatorname{per}\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & \alpha+\beta & -\alpha\beta & 0 \\ 0 & 1 & \alpha+\beta & -2\alpha\beta \\ 0 & 0 & 1 & \alpha+\beta \end{bmatrix} \\ &= \left(\alpha^3+\beta^3\right) + \left(\alpha^2+\beta^2\right) + (\alpha+\beta) + 2.\end{aligned}$$

Using the Binet formula of the sequence $\{v_n\}$ and, since $v_0 = 2$,

$$\operatorname{per} D_4 = \sum_{i=0}^{3} v_i = v_3 + v_2 + v_1 + v_0.$$

Suppose that the equation holds for n. Then we show that the equation holds for $n+1$. If we compute $\operatorname{per} D_{n+1}$ by the Laplace expansion of permanent with respect to the first column, then we have

$$\begin{aligned}\operatorname{per} D_{n+1} &= \operatorname{per}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 2 \\ 1 & \alpha+\beta & -\alpha\beta & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \alpha+\beta & -\alpha\beta & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \alpha+\beta & -\alpha\beta & \ddots & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 & \alpha+\beta & -\alpha\beta & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \alpha+\beta & -2\alpha\beta \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & \alpha+\beta \end{bmatrix} \\ &= \operatorname{per}\begin{bmatrix} \alpha+\beta & -\alpha\beta & 0 & 0 & \cdots & 0 & 0 \\ 1 & \alpha+\beta & -\alpha\beta & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \alpha+\beta & -\alpha\beta & \ddots & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & \alpha+\beta & -\alpha\beta & 0 \\ 0 & 0 & \cdots & 0 & 1 & \alpha+\beta & -2\alpha\beta \\ 0 & 0 & \cdots & \cdots & 0 & 1 & \alpha+\beta \end{bmatrix}\end{aligned}$$

$$+\operatorname{per}\begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 2 \\ 1 & \alpha+\beta & -\alpha\beta & 0 & \dots & 0 & 0 \\ 0 & 1 & \alpha+\beta & -\alpha\beta & \dots & \dots & 0 \\ 0 & 0 & 1 & \alpha+\beta & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & -\alpha\beta & 0 \\ 0 & 0 & 0 & \dots & 1 & \alpha+\beta & -2\alpha\beta \\ 0 & 0 & 0 & \dots & 0 & 1 & \alpha+\beta \end{bmatrix}$$

From the definitions of the matrices D_n and B_n, we may write

$$\operatorname{per} D_{n+1} = \operatorname{per} B_n + \operatorname{per} D_n.$$

Combining the result of Theorem 4 and our assumption, we obtain

$$\operatorname{per} D_{n+1} = v_n + \sum_{i=0}^{n-1} v_i = \sum_{i=0}^{n} v_i.$$

So the proof is complete. □

When $\alpha = 1 + \sqrt{5}/2$ and $\beta = 1 - \sqrt{5}/2$, the sequence $\{v_n\}$ is reduced to the Lucas sequence $\{L_n\}$, and by Theorem 5, we have

$$\operatorname{per}\begin{bmatrix} 1 & 1 & 1 & 1 & \dots 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & 1 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 1 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & \dots & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & \dots & \dots & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & \dots & \dots & 0 & 1 & 1 \end{bmatrix}_{n\times n} = \sum_{i=0}^{n-1} L_i$$

where L_n is the nth Lucas number. This result can also be found in [**5**].

Let the $n \times n$ $(1,-1)$-matrix S be as before, and denote the matrix $D_n \circ S$ by W_n, that is,

(3.2)

$$W_n = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 2 \\ -1 & \alpha+\beta & -\alpha\beta & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \alpha+\beta & -\alpha\beta & 0 & \cdots & \cdots & 0 \\ 0 & 0 & -1 & \alpha+\beta & -\alpha\beta & \ddots & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -1 & \alpha+\beta & -\alpha\beta & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & \alpha+\beta & -2\alpha\beta \\ 0 & 0 & 0 & \cdots & \cdots & 0 & -1 & \alpha+\beta \end{bmatrix}$$

Then we have the following theorem without proof.

Theorem 6. *Let α, β be the roots of the equation $t^2 - At + B = 0$, and let the upper Hessenberg matrix W_n have the form* (3.2)*. Then, for $n \geq 3$,*

$$\det W_n = \sum_{i=0}^{n-1} v_i$$

where v_n is the nth term of the sequence $\{v_n\}$.

In [**5**], the authors derived some relationships between sums of Fibonacci and Lucas numbers and permanents or determinants of certain Hessenberg matrices. Regarding all second order linear recurrences, in this paper we give more general results including the results of [**5**].

Acknowledgments. The authors would like to thank an anonymous referee for a number of helpful suggestions.

REFERENCES

1. R.A. Brualdi and P.M. Gibson, *Convex polyhedra of doubly stochastic matrices I: Applications of the permanent function*, J. Combin. Theory **22** (1977), 194–230.

2. P.F. Byrd, *Problem* B-12: *A Lucas determinant*, Fibonacci Quart. **1** (1963), 78.

3. N.D. Cahill, J.R. D'Errica, D.A. Narayan and J.Y. Narayan, *Fibonacci determinants*, College Math. J. **3** (2002), 221–225.

4. N.D. Cahill and D.A. Narayan, *Fibonacci and Lucas numbers as tridiagonal matrix determinants*, Fibonacci Quart. **42** (2004), 216–221.

5. E. Kilic and D. Tasci, *On families of bipartite graphs associated with sums of Fibonacci and Lucas numbers*, ARS Combin., to appear.

6. ———, *On the second order linear recurrences by tridiagonal matrices*, ARS Combin., to appear.

7. ———, *On the generalized order-k Fibonacci and Lucas numbers*, Rocky Mountain J. Math. **36** (2006), 1915–1926.

8. ———, *On the permanents of some tridiagonal matrices with applications to the Fibonacci and Lucas numbers*, Rocky Mountain J. Math. **37** (2007), 203–219.

9. G.-Y. Lee, *$k-$ Lucas numbers and associated bipartite graphs*, Linear Algebra Appl. **320** (2000), 51–61.

10. G.-Y. Lee and S.-G. Lee, *A note on generalized Fibonacci numbers*, Fibonacci Quart. **33** (1995), 273–278.

11. D.H. Lehmer, *Fibonacci and related sequences in periodic tridiagonal matrices*, Fibonacci Quart. **13** (1975), 150–158.

12. H. Minc, *Permanents of* $(0,1)$*-circulants*, Canad. Math. Bull. **7** (1964), 253–263.

13. ———, *Permanents*, in *Encyclopedia of mathematics and its applications*, Addison-Wesley, New York, 1978.

14. N. Robbins, *Beginning number theory*, William C. Brown Publishers, Dubuque, Iowa, 1993.

15. G. Strang, *Introduction to linear algebra*, 2nd edition, Wellesley-Cambridge, Wellesley MA, 1998.

16. G. Strang and K. Borre, *Linear algebra*, *Geodesy and* GPS, Wellesley-Cambridge, Wellesley MA, 1997.

17. D. Tasci and E. Kilic, *On the order-k generalized Lucas numbers*, Appl. Math. Comput. **155** (2004), 637–641.

18. F. Zhang, *Matrix theory*, Springer-Verlag, New York, 1999.

Department of Mathematics, TOBB University of Economics and Technology, TR-06560, Sogutozu, Ankara, Turkey
Email address: ekilic@etu.edu.tr

Gazi University, Mathematics Department, 06500 Teknikokullar, Ankara Turkey
Email address: dtasci@gazi.edu.tr