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# Sums of the squares of terms of sequence $\{u_n\}$

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**Abstract.** In this paper, we consider generalized Fibonacci type second order linear recurrence $\{u_n\}$. We derive a generating matrix for both the sums of squares, $\sum_{i=0}^{n} u_i^2$ and the products of the form $u_n u_{n+2}$. We also derive explicit formulas for the sums and products by using matrix methods. Then we give a matrix method to generate the sums of product of two consecutive terms $u_n u_{n+1}$ as well as the product, $u_n u_{n+2}$. Further we give generating functions and combinatorial representations of the sums of squares of terms of $\{u_n\}$ and the product, $u_n u_{n+2}$.

**Keywords.** Second order recurrence; sums of squares; generating function; matrix method.

## 1. Introduction

For $n > 0$, the well-known Fibonacci sequence $\{F_n\}$ is defined as

$$F_{n+1} = F_n + F_{n-1},$$

where $F_0 = 0$ and $F_1 = 1$ and the Lucas sequence $\{L_n\}$ is defined as

$$L_{n+1} = L_n + L_{n-1},$$

where $L_0 = 2$ and $L_1 = 1$.

The Fibonacci and Lucas sequences can be generalized as follows: Let $A$ be nonzero integer such that $D = \sqrt{A^2 + 4} \neq 0$. The second order linear recurrences of the Fibonacci and Lucas types are defined by the following equations:

$$u_{n+1} = Au_n + u_{n-1},$$

$$v_{n+1} = Av_n + v_{n-1},$$

where $u_0 = 0, u_1 = 1$, and, $v_0 = 2, v_1 = A$, respectively.

When $A = 1, u_n = F_n$ (the $n$th Fibonacci number). When $A = 2, u_n = P_n$ (the $n$th Pell number). When $A = 1$, then $v_n = L_n$ (the $n$th Lucas number).

Let $\alpha$ and $\beta$ be the roots of the characteristic equation $x^2 - Ax - 1 = 0$. Then the Binet formulas of the sequences $\{u_n\}$ and $\{v_n\}$ have the form

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad v_n = \alpha^n + \beta^n.$$

Many authors have studied relationships between the Fibonacci sequence, its certain generalizations and matrix properties. Some authors derive the generating matrices for certain linear recurrences (for more details, see [1], [6–8], [11], [19], [21], [26], [29]). For example, the terms of generalized Fibonacci sequence $\{u_n\}$ can be generated by the powers of the following $2 \times 2$ companion matrix:

$$\begin{bmatrix} A & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} u_{n+1} & u_n \\ u_n & u_{n-1} \end{bmatrix}.$$

The sums of Fibonacci squares and the sums of products of consecutive Fibonacci numbers are given by the following formulas:

$$\sum_0^n F_t^2 = F_n F_{n+1} \tag{1.1}$$

and

$$\sum_1^n F_k F_{k-1} = \begin{cases} F_n^2, & \text{if } n \text{ is even,} \\ F_n^2 - 1, & \text{if } n \text{ is odd.} \end{cases} \tag{1.2}$$

Further, generating functions give us a powerful tool for solving linear homogeneous recurrence relations with constant coefficients. For example, V E Hoggatt Jr and D A Lind in 1967 gave the generating functions of the Fibonacci numbers and sums of the product of consecutive Fibonacci numbers as follows:

$$\frac{x}{1-x-x^2} = \sum_0^\infty F_n x^n,$$

$$\frac{x}{1-2x-2x^2+x^3} = \sum_0^\infty F_n F_{n+1} x^n$$

(see pp. 229 of [20]). One can find more about generating functions in [30].

Also some authors have derived interesting relationships between the determinant or permanent of certain matrices and the linear recurrence relations (see [2–4], [9–19], [21–28]). These relationships are valid for both second order linear recurrences and higher order linear recurrences.

For example, (see [25]),

$$\operatorname{per} T_n = \operatorname{per} \begin{bmatrix} 1 & 1 & & 0 \\ 1 & 1 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 1 \end{bmatrix}_{n \times n} = F_{n+1},$$

where $F_n$ is the $n$th Fibonacci number.

Matrix methods are useful tools for derivating some properties of linear recurrences. For example, some authors use matrix methods for derivation of sums, combinatorial representations and generalized Binet formula of the second order or higher order Fibonacci numbers and their certain generalizations [21], [5], [6], [19], [11]. In [11], the authors define a $(k \times k)$ matrix and use the $n$th power. Then they derive the generalized Binet formula, useful identities and combinatorial representations of the generalized order-$k$ Fibonacci numbers. In [19], the authors define the generalized order-$k$ Pell numbers and use the powers of certain $k \times k$ companion matrix for generating these numbers. Then using matrix methods, the authors obtain the sums, generalized Binet formula and combinatorial representation of the generalized Pell numbers.

In this paper, we define some generating matrices whose $n$th powers generate the sums of squares of the terms of sequence $\{u_n\}$, $\sum_{i=0}^{n} u_i^2$, the products $u_n u_{n+2}$ as well as the sums of products of consecutive terms of sequence $\{u_n\}$, $\sum_{i=0}^{n} u_i u_{i-1}$. Furthermore we obtain the generalizations of the identities (1.1) and (1.2) for the sequence $\{u_n\}$ by matrix methods. We also give the generating functions and combinatorial representations of the products $u_n u_{n+1}$ and $u_n u_{n+2}$.

## 2. Sums of the squares of terms of $\{u_n\}$

In this section we give a generating matrix for the sums of squares of the terms of sequence $\{u_n\}$, $\sum_{i=0}^{n} u_i^2$. Before this, we give the following lemma for an alternative representation of $\sum_{i=0}^{n} u_i^2$.

*Lemma* 1. *Let $u_n$ be the $n$-th term of the sequence $\{u_n\}$. Then*

$$\sum_{i=0}^{n} u_i^2 = \frac{u_n u_{n+1}}{A}.$$

*Proof* (*Induction on n*). If $n = 1$, then $\sum_{i=0}^{1} u_i^2 = u_1 u_2 / A = 1$. Suppose that the equation holds for $n$. Then we show that the equation holds for $n + 1$. Thus, by our assumption and the recurrence relation of $\{u_n\}$, we obtain

$$\sum_{i=0}^{n+1} u_i^2 = \sum_{i=0}^{n} u_i^2 + u_{n+1}^2 = u_n u_{n+1}/A + u_{n+1}^2$$

$$= u_{n+1}(A u_{n+1} + u_n)/A = u_{n+1} u_{n+2}/A.$$

Hence the proof is complete. □

Define two $3 \times 3$ matrices $T$ and $H_n$ as follows:

$$T = \begin{bmatrix} u_3 & u_3 & -u_1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{2.1}$$

and

$$H_n = \begin{bmatrix} \sum_{i=0}^{n+1} u_i^2 & u_n u_{n+2} & -\sum_{i=0}^{n} u_i^2 \\ \sum_{i=0}^{n} u_i^2 & u_{n-1} u_{n+1} & -\sum_{i=0}^{n-1} u_i^2 \\ \sum_{i=0}^{n-1} u_i^2 & u_{n-2} u_n & -\sum_{i=0}^{n-2} u_i^2 \end{bmatrix}, \tag{2.2}$$

where $u_n$ is the $n$th term of $\{u_n\}$.

Then we have the following theorem.

**Theorem 1.** *Let the matrices $T$ and $H_n$ have the forms* (2.1) *and* (2.2), *respectively. Then for $n > 1$,*

$$T^n = H_n.$$

*Proof* (*Induction on $n$*). If $n = 2$, then by $u_3 = A^2 + 1, u_2 = A, u_1 = 1, u_0 = 0, u_{-1} = 1, u_{-2} = -A$, we have

$$T^2 = \begin{bmatrix} A^4 + 3A^2 + 2 & A^2(A^2 + 2) & -A^2 - 1 \\ A^2 + 1 & A^2 + 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=0}^{3} u_i^2 & u_2 u_4 & -\sum_{i=0}^{2} u_i^2 \\ \sum_{i=0}^{2} u_i^2 & u_1 u_3 & -\sum_{i=0}^{1} u_i^2 \\ \sum_{i=0}^{1} u_i^2 & u_0 u_2 & -\sum_{i=0}^{0} u_i^2 \end{bmatrix} = H_2.$$

Suppose that the equation holds for $n$, $n > 2$. Then we show that the equation holds for $n + 1$. Thus, by the inductive hypothesis,

$$T^{n+1} = TT^n = TH_n$$

and so

$$T^{n+1} = \begin{bmatrix} \sum_{i=0}^{n+1} u_i^2 & u_n u_{n+2} & -\sum_{i=0}^{n} u_i^2 \\ \sum_{i=0}^{n} u_i^2 & u_{n-1} u_{n+1} & -\sum_{i=0}^{n-1} u_i^2 \\ \sum_{i=0}^{n-1} u_i^2 & u_{n-2} u_n & -\sum_{i=0}^{n-2} u_i^2 \end{bmatrix} \begin{bmatrix} A^2 + 1 & A^2 + 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (A^2+1)\sum_{i=0}^{n+1} u_i^2 + u_n u_{n+2} & (A^2+1)\sum_{i=0}^{n+1} u_i^2 - \sum_{i=0}^{n} u_i^2 & -\sum_{i=0}^{n+1} u_i^2 \\ (A^2+1)\sum_{i=0}^{n} u_i^2 + u_{n-1} u_{n+1} & (A^2+1)\sum_{i=0}^{n} u_i^2 - \sum_{i=0}^{n-1} u_i^2 & -\sum_{i=0}^{n} u_i^2 \\ (A^2+1)\sum_{i=0}^{n-1} u_i^2 + u_{n-2} u_n & (A^2+1)\sum_{i=0}^{n-1} u_i^2 - \sum_{i=0}^{n-2} u_i^2 & -\sum_{i=0}^{n-1} u_i^2 \end{bmatrix}.$$

By Lemma 1, we write $(A^2+1)\frac{u_n u_{n+1}}{A}+u_{n-1}u_{n+1}$ instead of $(A^2+1)\sum_{i=0}^{n}u_i^2+u_{n-1}u_{n+1}$. Thus, by the recurrence relation of $\{u_n\}$ and Lemma 1,

$$\begin{aligned}(A^2+1)u_n u_{n+1}/A+u_{n-1}u_{n+1} &= ((A^2+1)u_n u_{n+1}+u_{n-1}u_{n+1})/A \\ &= (Au_{n+1}(Au_n+u_{n-1})+u_n u_{n+1})/A \\ &= u_{n+1}(Au_{n+1}+u_n)/A = u_{n+1}u_{n+2}/A \\ &= \sum_{i=0}^{n+1} u_i^2. \end{aligned} \tag{2.3}$$

Also by Lemma 1, we write $(A^2+1)\frac{u_n u_{n+1}}{A}-\frac{u_{n-1}u_n}{A}$ instead of $(A^2+1)\sum_{i=0}^{n}u_i^2-\sum_{i=0}^{n-1}u_i^2$. Thus

$$\begin{aligned}(A^2+1)\frac{u_n u_{n+1}}{A}-\frac{u_{n-1}u_n}{A} &= u_n(A^2u_{n+1}+u_{n+1}-u_{n-1})/A \\ &= u_n(Au_{n+1}+u_n)=u_n u_{n+2}. \end{aligned} \tag{2.4}$$

Combining the formulas (2.3) and (2.4), we write

$$T^{n+1}=\begin{bmatrix} \sum_{i=0}^{n+2}u_i^2 & u_{n+1}u_{n+3} & -\sum_{i=0}^{n+1}u_i^2 \\ \sum_{i=0}^{n+1}u_i^2 & u_n u_{n+2} & -\sum_{i=0}^{n}u_i^2 \\ \sum_{i=0}^{n}u_i^2 & u_{n-1}u_{n+1} & -\sum_{i=0}^{n-1}u_i^2 \end{bmatrix}=H_{n+1}.$$

So the proof is complete. □

Considering the results of Lemma 1 and Theorem 1, we have the following result without proof.

COROLLARY 1

*Let the matrix* $T$ *have the form* (2.1). *Then*

$$T^n=\begin{bmatrix} u_{n+1}u_{n+2}/A & u_n u_{n+2} & -u_n u_{n+1}/A \\ u_n u_{n+1}/A & u_{n-1}u_{n+1} & -u_{n-1}u_n/A \\ u_{n-1}u_n/A & u_{n-2}u_n & -u_{n-2}u_{n-1}/A \end{bmatrix}.$$

By a simple calculation, we obtain $\det T=-1$ and the characteristic equation of the matrix $T$ is $x^3+(-A^2-1)x^2+(-A^2-1)x+1=0$. Thus the eigenvalues of the matrix $T$ are $\gamma_1=\frac{1}{2}A^2+\frac{1}{2}A\sqrt{A^2+4}+1,\ \gamma_2=\frac{1}{2}A^2-\frac{1}{2}A\sqrt{A^2+4}+1$ and $\gamma_3=-1$.

Since the roots of the characteristic equation of $\{u_n\}$, $x^2-Ax-1=0$, are $\alpha=\frac{1}{2}A+\sqrt{A^2+4}$ and $\beta=\frac{1}{2}A-\sqrt{A^2+4}$, we can easily obtain $\gamma_1=\alpha^2,\ \gamma_2=\beta^2$.

Define the $3 \times 3$ Vandermonde matrix $\Lambda$ as follows:

$$\Lambda = \begin{bmatrix} \gamma_1^2 & \gamma_2^2 & \gamma_3^2 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ 1 & 1 & 1 \end{bmatrix}.$$

Since $\alpha, \beta$ are distinct and $\Delta = \sqrt{A^2+4} \neq 0$, we obtain $\det \Lambda = A(A^2+4)^{3/2}$. Note that $\det \Lambda \neq 0$. Denote $\Lambda^T$ by $V$. Let $V_j^{(i)}$ be a $3 \times 3$ matrix obtained from $V$ by replacing the $j$th column of $V$ by $w_i$ where

$$w_i = \begin{bmatrix} \gamma_1^{n-i+3} \\ \gamma_2^{n-i+3} \\ \gamma_3^{n-i+3} \end{bmatrix}.$$

Then we have the following theorem.

**Theorem 2.** *Let the matrix $H_n = [h_{ij}]$ have the form* (2.2). *Then for all $i, j$ such that $1 \leq i, j \leq 3$,*

$$h_{ij} = \frac{\det(V_j^{(i)})}{\det V}.$$

*Proof.* We can verify that

$$T\Lambda = \Lambda D,$$

where $D$ is a diagonal matrix such that $D = \operatorname{diag}(\gamma_1, \gamma_2, \gamma_3)$. Since $\det \Lambda \neq 0$, the matrix $\Lambda$ is invertible. So we write $\Lambda^{-1} T \Lambda = D$. Thus the matrix $W$ is similar to the matrix $D$. Therefore, we obtain $T^n \Lambda = \Lambda D^n$. By Theorem 1, we write $H_n \Lambda = \Lambda D^n$. Then we have the following linear equation system:

$$h_{i1}\gamma_1^2 + h_{i2}\gamma_1 + h_{i3} = \gamma_1^{n-i+3},$$

$$h_{i1}\gamma_2^2 + h_{i2}\gamma_2 + h_{i3} = \gamma_2^{n-i+3},$$

$$h_{i1}\gamma_3^2 + h_{i2}\gamma_3 + h_{i3} = \gamma_3^{n-i+3},$$

where $H_n = [h_{ij}]$. By the Cramer solution of the above linear equation system, we obtain $h_{ij} = \frac{\det(V_j^{(i)})}{\det V}$. So the proof is complete. □

We now derive some formulas for the sums of squares of the terms of $\{u_n\}$, $\sum_{i=0}^n u_i^2$, and the product of two terms of $\{u_n\}$, $u_n u_{n+2}$.

COROLLARY 2

*For $n > 0$,*

$$\sum_{i=0}^n u_i^2 = \frac{v_{2n+1} + u_2(-1)^{n+1}}{A(A^2+4)},$$

*where $u_n$ and $v_n$ are the nth terms of sequences $\{u_n\}$ and $\{v_n\}$, respectively.*

*Proof.* By Theorem 2, we have $h_{i,1} = \sum_{i=1}^{n-i+2} u_i^2$ for $1 \leq i \leq 3$ and

$$h_{21} = \sum_{i=0}^{n} u_i^2 = \frac{\det(V_1^{(2)})}{\det V}.$$

Computing the det $V_1^{(2)}$ by considering $\gamma_1 = \alpha^2, \gamma_2 = \beta^2,\ \gamma_3 = -1$ and using the Binet formulas of the sequences $\{u_n\}$ and $\{v_n\}$ gives us

$$\det V_1^{(2)} = \begin{vmatrix} \gamma_1^{n+1} & \gamma_1 & 1 \\ \gamma_2^{n+1} & \gamma_2 & 1 \\ \gamma_3^{n+1} & \gamma_3 & 1 \end{vmatrix} = \begin{vmatrix} \alpha^{2n+2} & \alpha^2 & 1 \\ \beta^{2n+2} & \beta^2 & 1 \\ (-1)^{n+1} & -1 & 1 \end{vmatrix}$$

$$= ((\alpha^2 - \beta^2)(-1)^{n+1} + (\alpha^{2n+2} - \beta^{2n+2}) + (\alpha^{2n+2}\beta^2 - \alpha^2\beta^{2n+2}))$$

$$= \sqrt{A^2+4}(v_{2n+1} + u_2(-1)^{n+1}).$$

By det $V = A(A^2+4)^{3/2}$, the proof is complete. □

From Corollary 2, by taking $A = 1$, we have the following result:

$$\sum_{i=0}^{n} F_i^2 = \frac{L_{2n+1} + (-1)^{n+1}}{5},$$

where $F_n$ and $L_n$ are the $n$th Fibonacci and Lucas numbers, respectively.

COROLLARY 3

*For* $n > 0$,

$$u_n u_{n+2} = \frac{v_{2n+2} + u_4(-1)^{n+1}}{A(A^2+4)}.$$

*Proof.* By Theorem 2, we have $h_{12} = u_n u_{n+2}$ and

$$h_{12} = u_n u_{n+2} = \frac{\det(V_2^{(1)})}{\det V}.$$

By considering $\gamma_1 = \alpha^2, \gamma_2 = \beta^2, \gamma_3 = -1$ and using the Binet forms of $\{u_n\}$ and $\{v_n\}$, we obtain

$$\det V_2^{(1)} = \begin{vmatrix} \gamma_1^2 & \gamma_1^{n+1} & 1 \\ \gamma_2^2 & \gamma_2^{n+1} & 1 \\ \gamma_3^2 & \gamma_3^{n+1} & 1 \end{vmatrix} = \begin{vmatrix} \alpha^4 & \alpha^{2n+4} & 1 \\ \beta^4 & \beta^{2n+4} & 1 \\ 1 & (-1)^{n+2} & 1 \end{vmatrix}$$

$$= ((\alpha^{2n+4} - \beta^{2n+4}) + (-1)^{n+1}(\alpha^4 - \beta^4) + \alpha^4\beta^4(\beta^{2n} - \alpha^{2n}))$$

$$= \sqrt{A^2+4}(v_{2n+2} + u_4(-1)^{n+1}).$$

Since det $V = A(A^2+4)^{3/2}$, the proof is complete. □

When $A = 1$ in Corollary 3, we obtain

$$F_n F_{n+2} = \frac{L_{2n+2} + 3(-1)^{n+1}}{5},$$

where $F_n$ and $L_n$ are the $n$th Fibonacci and Lucas numbers, respectively.

Consequently, we have the following Corollary since the $n$th power of matrix $T$ generates the products $u_n u_{n+1}$ and $u_n u_{n+2}$.

COROLLARY 4

*For $n > 0$, the sequence $\{x_n\}$ satisfies the following recursion*

$$x_{n+2} = u_3 x_{n+1} + u_3 x_n - x_{n-1},$$

*where $x_n$ is $u_n u_{n+1}/A = \sum_{i=0}^{n} u_i^2$ or $u_n u_{n+2}$.*

*Proof.* From Theorem 1, we have $T^n = H_n$ where the matrix $T$ is a companion matrix. From companion matrices, it is well-known that the characteristic equations of matrix $T$ and sequences $\{u_n u_{n+1}\}$, $\{u_n u_{n+2}\}$ are the same. Thus the proof is complete. □

## 3. Sums of the products of consecutive terms of $\{u_n\}$

In this section, we consider the earlier results given in §2. Then we define a new matrix and show that its $n$th power generate sums of the products of two consecutive terms of $\{u_n\}$.

Expanding the matrix $T$, define the $4 \times 4$ matrix $G$ as follows:

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & u_3 & u_3 & -u_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{3.1}$$

Also $C_n$ denote the sums of products of two consecutive terms of $\{u_n\}$, that is, for $n > 0$,

$$C_n = \frac{1}{A} \sum_{k=0}^{n} u_k u_{k+1}. \tag{3.2}$$

We define the matrix $Q_n$ as follows:

$$Q_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ C_n & u_{n+1}u_{n+2}/A & u_n u_{n+2} & -u_n u_{n+1}/A \\ C_{n-1} & u_n u_{n+1}/A & u_{n-1}u_{n+1} & -u_{n-1}u_n/A \\ C_{n-2} & u_{n-1}u_n/A & u_{n-2}u_n & -u_{n-2}u_{n-1}/A \end{bmatrix}. \tag{3.3}$$

Then we have the following theorem.

**Theorem 3.** *Let the matrices $G$ and $Q_n$ are as in* (3.1) *and* (3.3), *respectively. Then for $n > 2$,*

$$G^n = Q_n. \tag{3.4}$$

*Proof* (*Induction on n*). Since $u_0 = 0, u_1 = 1, u_2 = A, C_0 = 0$ and $C_1 = A$, by Theorem 1, the conclusion is seen for $n = 1$. If $n = 2$, then by a simple calculation, one can obtain $G^2 = Q_2$. Suppose the claim is true for $n$. Then we show that the claim is true for $n + 1$. Thus, by our assumption, we write

$$G^{n+1} = G^n G = Q_n G$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ C_n & u_{n+1}u_{n+2}/A & u_n u_{n+2} & -u_n u_{n+1}/A \\ C_{n-1} & u_n u_{n+1}/A & u_{n-1}u_{n+1} & -u_{n-1}u_n/A \\ C_{n-2} & u_{n-1}u_n/A & u_{n-2}u_n & -u_{n-2}u_{n-1}/A \end{bmatrix}$$
$$\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & A^2+1 & A^2+1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

By the definition of $C_n$, we have $C_n = C_{n-1} + u_n u_{n+1}/A$. From matrix multiplication and Theorem 1, the proof is complete. □

Then we have the following Corollary.

COROLLARY 5

*For* $n > 1$, *the* $C_n$*'s satisfy the following nonhomogeneous recurrence relation*

$$C_{n+1} = (A^2+1)C_n + (A^2+1)C_{n-1} - C_{n-2} + 1,$$

*where* $C_n$ *is given by* (3.2).

*Proof.* From eq. (3.4), we write that $Q_{n+1} = G^{n+1} = G^n G = GG^n = Q_n Q = QQ_n$. Considering $Q_{n+1} = QQ_n$ and by a matrix multiplication, the proof is easily seen. □

We derive an explicit formula for the sums of products of two consecutive terms of $\{u_n\}$, $C_n = \frac{1}{A}\sum_{k=0}^{n} u_k u_{k+1}$, by matrix methods. Define two $4 \times 4$ matrices $E$ and $D_1$ as follows:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{-1}{2A^2} & \gamma_1^2 & \gamma_2^2 & \gamma_3^2 \\ \frac{-1}{2A^2} & \gamma_1 & \gamma_2 & \gamma_3 \\ \frac{-1}{2A^2} & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_1 & 0 & 0 \\ 0 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & \gamma_3 \end{bmatrix},$$

where $\gamma_i$'s are the eigenvalues of $T$ given by (2.1) for $1 \le i \le 3$. If we compute the det $T$ by expanding to the first row, we obtain $\det E = \det \Lambda = A(A^2+4)^{3/2}$.

**Theorem 4.** *For* $n > 0$,

$$\sum_{k=0}^{n} u_k u_{k+1} = \frac{u_{n+1}^2 + u_n u_{n+2} - 1}{2A^2}.$$

*Proof.* We can verify the equation $GE = ED_1$. Since $\det E \neq 0$, we write $E^{-1}GE = D_1$. Thus the matrix $G$ is similar to the matrix $D_1$. We write $E^{-1}G^n E = D_1^n$ and so $G^n E = ED_1^n$. Since $G^n = Q_n$, $Q_n E = ED_1^n$, we have the following equation by $C_n = (Q_n)_{2,1}$,

$$C_n - \frac{u_{n+1}u_{n+2}/A + u_n u_{n+2} - u_n u_{n+1}/A}{2A^2} = \frac{-1}{2A^2}$$

or

$$C_n = \frac{u_{n+1}u_{n+2}/A + u_n u_{n+2} - u_n u_{n+1}/A - 1}{2A^2}.$$

If we arrange the above equation, we obtain

$$C_n = \frac{u_{n+1}^2 + u_n u_{n+2} - 1}{2A^2}.$$

Since $C_n = \sum_{k=0}^n u_k u_{k+1}$, then the proof is complete. □

If we choose $A = 1$ in Theorem 4, then we have the following result:

$$\sum_{k=0}^n F_k F_{k+1} = \frac{F_{n+1}^2 + F_n F_{n+2} - 1}{2},$$

where $F_n$ is the $n$th Fibonacci number.

## 4. Generating functions for the products $u_k u_{k+1}$ and $u_k u_{k+2}$

In this section, we give generating functions for the products $u_n u_{n+1}$ and $u_n u_{n+2}$. Recall that the generating function of the product of consecutive Fibonacci numbers is given by

$$\frac{x}{1 - 2x - 2x^2 + x^3} = \sum_0^\infty F_n F_{n+1} x^n.$$

Now we generalize the above equation for the sequence $\{u_n\}$. Let

$$\begin{aligned}
g(x) &= u_0u_1 + u_1u_2x + u_2u_3x^2 + u_3u_4x^3 + \cdots, \\
u_3xg(x) &= u_0u_1u_3x + u_1u_2u_3x^2 + u_2u_3u_3x^3 + u_3u_4u_3x^4 + \cdots, \\
u_3x^2g(x) &= u_0u_1u_3x^2 + u_1u_2u_3x^3 + u_2u_3u_3x^4 + u_3u_4u_3x^5 + \cdots, \\
x^3g(x) &= u_0u_1x^3 + u_1u_2x^4 + u_2u_3x^5 + u_3u_4x^6 + \cdots, \\
(1 - u_3x - u_3x^2 + x^3)g(x) &= u_0u_1 + (u_1u_2 - u_0u_1u_3)x \\
&\quad + (u_2u_3 - u_1u_2u_3 - u_0u_1u_3)x^2 \\
&\quad + (u_3u_4 - u_2u_3u_3 - u_1u_2u_3)x^3 + \cdots \\
&\quad + (u_nu_{n+1} - u_{n-1}u_nu_3 - u_{n-2}u_{n-1}u_3 \\
&\quad + u_{n-3}u_{n-2})x^n + \cdots.
\end{aligned}$$

By $u_3 = A^2 + 1$,

$$\begin{aligned} & u_n u_{n+1} - u_{n-1} u_n u_3 - u_{n-2} u_{n-1} u_3 + u_{n-3} u_{n-2} \\ & \quad = u_n u_{n+1} - u_{n-1} u_n u_3 - A u_{n-2} u_n \\ & \quad = u_n (u_{n+1} - A u_{n-2}) - u_{n-1} u_n u_3 \\ & \quad = A u_n^2 + u_n u_{n-3} - u_{n-1} u_n u_3 \\ & \quad = 0 \end{aligned}$$

and $u_1 u_2 - u_0 u_1 u_3 = u_2$,

$$(1 - u_3 x - u_3 x^2 + x^3) g(x) = u_2 x.$$

Thus we obtain

$$g(x) = \sum_0^\infty u_n u_{n+1} x^n = \frac{u_2 x}{1 - u_3 x - u_3 x^2 + x^3}.$$

We now consider the product $u_n u_{n+2}$ and then give the generating function of the product. For ease, let

$$h(x) = \sum_0^\infty u_n u_{n+2} x^n.$$

Then by the definition of sequence $\{u_n\}$, we obtain

$$(1 - u_2 x - u_2 x^2 + x^3) h(x) = u_3 x - x^2.$$

Thus we have

$$\sum_0^\infty u_n u_{n+2} x^n \frac{u_3 x - x^2}{1 - u_3 x - u_3 x^2 + x^3}.$$

As a result of the above result, one can see that

$$\sum_0^\infty F_n F_{n+2} x^n \frac{2x - x^2}{1 - 2x - 2x^2 + x^3}.$$

## 5. A combinatorial representation for the products $u_k u_{k+1}$ and $u_k u_{k+2}$

In this section we give a combinatorial representation of $u_n u_{n+1} = A \sum_{i=1}^n u_i^2$. In [5], the authors obtain an explicit formula for the $n$th power of the companion matrix. Let the $k \times k$ companion matrix be as follows:

$$A_k = \begin{bmatrix} c_1 & c_2 & \dots & c_{k-1} & c_k \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}. \tag{5.1}$$

Then we find the following Theorem in [5].

**Theorem 5.** *The* $(i, j)$ *entry* $a_{ij}^{(n)}(c_1, c_2, \ldots, c_k)$ *in the matrix* $A_k^n(c_1, c_2, \ldots, c_k)$ *is given by the following formula*:

$$a_{ij}^{(n)}(c_1, c_2, \ldots, c_k) = \sum_{(t_1, t_2, \ldots, t_k)} \frac{t_j + t_{j+1} + \cdots + t_k}{t_1 + t_2 + \cdots + t_k} \times \binom{t_1 + t_2 + \cdots + t_k}{t_1, t_2, \ldots, t_k} c_1^{t_1} \ldots c_k^{t_k}, \tag{5.2}$$

*where the summation is over nonnegative integers satisfying* $t_1 + 2t_2 + \cdots + kt_k = n - i + j$, *and the coefficients in* (5.2) *is defined to be* 1 *if* $n = i - j$.

Then we have the following Corollaries.

COROLLARY 6

*Let* $u_n$ *be the nth term of* $\{u_n\}$. *Then*

$$\sum_{i=0}^{n} u_i^2 = u_n u_{n+1}/A = \sum_{(r_1, r_2, r_3)} \binom{r_1 + r_2 + r_3}{r_1, r_2, r_3} u_3^{r_1 + r_2} (-1)^{r_3},$$

*where the summation is over nonnegative integers satisfying* $r_1 + 2r_2 + 3r_3 = n - 1$.

*Proof.* From §1, we have

$$\begin{bmatrix} u_3 & u_3 & -u_1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} u_{n+1}u_{n+2}/A & u_n u_{n+2} & -u_n u_{n+1}/A \\ u_n u_{n+1}/A & u_{n-1}u_{n+1} & -u_{n-1}u_n/A \\ u_{n-1}u_n/A & u_{n-2}u_n & -u_{n-2}u_{n-1}/A \end{bmatrix}.$$

When $i = 2, j = 1, c_1 = c_2 = u_3$ and $c_3 = -u_1 = -1$ in Theorem 5, then the proof is immediately seen from (5.1). □

COROLLARY 7

*Let* $u_n$ *be the nth term of* $\{u_n\}$. *Then*

$$u_n u_{n+2} = \sum_{(r_1, r_2, r_3)} \frac{r_2 + r_3}{r_1 + r_2 + r_3} \binom{r_1 + r_2 + r_3}{r_1, r_2, r_3} u_3^{r_1 + r_2} (-1)^{r_3},$$

*where the summation is over nonnegative integers satisfying* $r_1 + 2r_2 + 3r_3 = n + 1$.

*Proof.* From §1, we have $T^n = H_n$. If we take $i = 1, j = 2, c_1 = c_2 = u_3$ and $c_3 = -u_1 = -1$ in Theorem 5, the proof is immediately seen from (5.1). □

## 6. Determinantal representations

In this section, we construct relationships between determinants of certain matrices and the two product types $u_n u_{n+1}$, $u_n u_{n+2}$.

For these relationships we define two matrices.

Define the $(n \times n)$ super-diagonal matrix $W_n$ in the following compact form:

$$W_n = \begin{bmatrix} u_3 & u_3 & -u_1 & & & \\ -1 & u_3 & u_3 & -u_1 & & \\ & -1 & u_3 & u_3 & \ddots & \\ & & \ddots & \ddots & \ddots & -u_1 \\ & & & -1 & u_3 & u_3 \\ & & & & -1 & u_3 \end{bmatrix}.$$

Then we have the following result.

**Theorem 6.** *For* $n > 1$,

$$\det W_n = \sum_{i=0}^{n+1} u_i^2,$$

*where* $\det W_1 = \sum_{i=0}^{n+1} u_i^2 = u_3$.

*Proof.* Expanding the $\det W_n$ according to the first row by the Laplace expansion of a determinant gives us

$$\det W_n = u_3 \det W_{n-1} + u_3 \det W_{n-2} - \det W_{n-3}.$$

After some simple calculations, we obtain $\det W_1 = \sum_{i=0}^{2} u_i^2$, $\det W_2 = \sum_{i=0}^{3} u_i^2$ and $\det W_3 = \sum_{i=0}^{4} u_i^2$. Since the recurrence relations (and initial conditions) of $\det W_n$ and the sequence $\left\{ \sum_{i=0}^{n+1} u_i^2 \right\}$ are the same the conclusion follows from Corollary 4 by taking $x_n \equiv \sum_{i=0}^{n+1} u_i^2$. □

Now we modify the matrix $W_n$ by deleting $(n, n)$th entry and denote this new matrix by $\hat{W}_n$ as in the following compact form:

$$\hat{W}_n = \begin{bmatrix} u_3 & u_3 & -u_1 & & & \\ -1 & u_3 & u_3 & -u_1 & & \\ & -1 & u_3 & u_3 & \ddots & \\ & & \ddots & \ddots & \ddots & -u_1 \\ & & & -1 & u_3 & u_3 \\ & & & & -1 & 0 \end{bmatrix}.$$

**Theorem 7.** *For* $n > 1$,

$$\det \hat{W}_n = u_{n-1} u_{n+1},$$

*where* $\det \hat{W}_2 = u_1 u_3$.

*Proof.* Similar to det $W_n$, if we expand det $\hat{W}_n$ with respect to the first row, then we obtain

$$\det \hat{W}_n = u_3 \det \hat{W}_{n-1} + u_3 \det \hat{W}_{n-2} - \det \hat{W}_{n-3}. \tag{6.3}$$

One can see that det $\hat{W}_2 = u_1u_3$, det $\hat{W}_3 = u_2u_4$ and det $\hat{W}_4 = u_3u_5$. Since the recurrence relations (and initial conditions) of det $\hat{W}_n$ and the sequence $\{u_{n-1}u_{n+1}\}$ are the same the conclusion follows from Corollary 4 by taking $x_n \equiv u_{n-1}u_{n+1}$. □

## References


[1] Bong N H, Fibonacci matrices and matrix representation of Fibonacci numbers, *Southeast Asian Bull. Math.* **23(3)** (1999) 357–374

[2] Byrd P F, Problem B-12: A Lucas Determinant, *Fibonacci Quart.* **1(4)** (1963) 78

[3] Cahill N D and Narayan D A, Fibonacci and Lucas numbers as tridiagonal matrix determinants, *Fibonacci Quart.* **42(3)** (2004) 216–221

[4] Cahill N D, D'Errica J R, Narayan D A and Narayan J Y, Fibonacci Determinants, *College Math. J.* **3(3)** (2002) 221–225

[5] Chen W Y C and Louck J D, The combinatorial power of the companion matrix, *Linear Algebra Appl.* **232** (1996) 261–278

[6] Er M C, Sums of Fibonacci numbers by matrix methods, *Fibonacci Quart.* **22(3)** (1984) 204–207

[7] Ercolano J, Matrix generators of Pell sequences, *Fibonacci Quart.* **17(1)** (1979) 71–77

[8] Kalman D, Generalized Fibonacci numbers by matrix methods, *Fibonacci Quart.* **20(1)** (1982) 73–76

[9] Kilic E and Tasci D, On families of Bipartite graphs associated with sums of Fibonacci and Lucas numbers, *Ars Combin.*, to appear

[10] Kilic E and Tasci D, On families of Bipartite graphs associated with sums of generalized order-$k$ Fibonacci and Lucas numbers, *Ars Combin.*, to appear

[11] Kilic E and Tasci D, On the generalized order-$k$ Fibonacci and Lucas numbers, *Rocky Mountain J. Math.* **36(6)** (2006) 1915–1926

[12] Kilic E and Tasci D, On the second order linear recurrence satisfied by the permanent of generalized doubly stochastic matrices, *Ars Combin.*, to appear

[13] Kilic E and Tasci D, On the generalized Fibonacci and Pell sequences by Hessenberg matrices, *Ars Combin.*, to appear

[14] Kilic E and Tasci D and Haukkanen P, On the generalized Lucas sequences by Hessenberg matrices, *Ars Combin.*, to appear

[15] Kilic E and Tasci D, On sums of second order linear recurrences by Hessenberg matrices, *Rocky Mountain J. Math.*, to appear

[16] Kilic E and Tasci D, On sums of second order linear recurrences by Hessenberg matrices, *Rocky Mountain J. Math.*, to appear

[17] Kilic E and Tasci D, On the second order linear recurrences by tridiagonal matrices, *Ars Combin.*, to appear

[18] Kilic E and Tasci D, On the permanents of some tridiagonal matrices with applications to the Fibonacci and Lucas numbers, *Rocky Mountain J. Math.* **37(6)** (2007) 203–219

[19] Kilic E and Tasci D, The generalized Binet formula, representation and sums of the generalized order-$k$ Pell numbers, *Taiwanese J. Math.* **10(6)** (2006) 1661–1670

[20] Koshy T, Fibonacci and Lucas numbers with applications. Pure and Applied Mathematics (2001) (New York: Wiley-Interscience) pp. 215–238

[21] Lee G-Y, Lee S-G, Kim J-S and Shin H K, The Binet formula and representations of $k$-generalized Fibonacci numbers, *Fibonacci Quart.* **39(2)** (2001) 158–164

[22] Lee G-Y and Lee S-G, A note on generalized Fibonacci numbers, *Fibonacci Quart.* **33** (1995) 273–278

[23] Lee G-Y, *k*-Lucas numbers and associated bipartite graphs, *Linear Algebra Appl.* **320** (2000) 51–61
[24] Lehmer D, Fibonacci and related sequences in periodic tridiagonal matrices, *Fibonacci Quart.* **13** (1975) 150–158
[25] Minc H, Permanents of (0, 1)-circulants, *Canad. Math. Bull.* **7(2)** (1964) 253–263
[26] Miles Jr E P, Generalized Fibonacci numbers and associated matrices, *Am. Math. Monthly* **67** (1960) 745–752
[27] Strang G, Introduction to linear algebra, 3rd ed. (2003) (Wellesley-Cambridge: Wellesley, MA)
[28] Strang G and Borre K, Linear algebra. Geodesy and GPS (1997) (Wellesley-Cambridge: Wellesley, MA) pp. 555–557
[29] Tasci D and Kilic E, On the order-*k* generalized Lucas numbers, *Appl. Math. Comput.* **155(3)** (2004) 63–641
[30] Wilf H S, Generating functionology, third edition (2006) (Wellesley, MA: A K Peters, Ltd)