

Explicit formula for the inverse of a tridiagonal matrix by backward continued fractions

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Abstract

In this paper, we consider a general tridiagonal matrix and give the explicit formula for the elements of its inverse. For this purpose, considering usual continued fraction, we define backward continued fraction for a real number and give some basic results on backward continued fraction. We give the relationships between the usual and backward continued fractions. Then we reobtain the LU factorization and determinant of a tridiagonal matrix. Furthermore, we give an efficient and fast computing method to obtain the elements of the inverse of a tridiagonal matrix by backward continued fractions. Comparing the earlier result and our result on the elements of the inverse of a tridiagonal matrix, it is seen that our method is more convenient, efficient and fast.

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1. Introduction

There has been increasing interest in tridiagonal matrices in many different theoretical fields, especially in applicative fields such as numerical analysis, orthogonal polynomials, engineering, telecommunication system analysis, system identification, signal processing (e.g., speech decoding, deconvolution), special functions, partial differential equations and naturally linear algebra (see [3,5,10,13,14,18]). In many of these areas, inversions of tridiagonal matrices are necessary. Parallel computations or efficient algorithms [7,15], indirect formulas [1,2,14,17,19], direct formulas of some certain cases [5,13] for such inversions are known. Some authors consider a general tridiagonal matrix of finite order and then describe the LU factorizations, determine the determinant and inverse of a tridiagonal matrix under certain conditions (see [4,6,8,11,16]).

Recently explicit formula for the elements of the inverse of a general tridiagonal matrix inverse is given by Mallik [12]. The author's approach is based on linear difference equations. Then the author extends the results on the explicit solution of a second-order linear homogenous difference equation with variable coefficients to the nonhomogeneous case. Thus the author obtains the explicit formula by applying these extended results to a boundary value problem.

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In [4], the authors give two types LU factorization of the $n \times n$ tridiagonal matrix A

$$A = \begin{bmatrix} d_1 & a_1 & & & \\ b_2 & d_2 & \ddots & & \\ & \ddots & \ddots & a_{n-1} & \\ & & & b_n & d_n \end{bmatrix}$$

by setting

$$\beta_i = \begin{cases} d_1 & \text{if } i = 1, \\ d_i - \frac{b_i}{\beta_{i-1}} a_{i-1} & \text{if } i = 2, \dots, n, \end{cases}$$

as $A = L_1 U_1$ and $A = L_2 U_2$ where

$$L_1 = \begin{bmatrix} 1 & & & & \\ \frac{b_2}{\beta_1} & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & 1 & \\ & & & \frac{b_n}{\beta_{n-1}} & 1 \end{bmatrix} \quad \text{and} \quad U_1 = \begin{bmatrix} \beta_1 & a_1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \beta_{n-1} & a_{n-1} \\ & & & & \beta_n \end{bmatrix},$$

and

$$L_2 = \begin{bmatrix} \beta_1 & & & & \\ b_2 & \beta_2 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & b_{n-1} & \beta_n \end{bmatrix} \quad \text{and} \quad U_2 = \begin{bmatrix} 1 & \frac{a_1}{\beta_1} & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & \frac{a_{n-1}}{\beta_{n-1}} \\ & & & & 1 \end{bmatrix}$$

called the Doolittle and Crout factorizations, respectively.

A general continued fraction representation of a real number x is one of the form

$$x = a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{\ddots + \frac{b_n}{a_n}}}}, \quad (1)$$

where a_0, a_1, \dots and b_1, b_2, \dots are integers. Such representations are sometimes in the form $x = a_0 + \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \dots \frac{b_n}{a_n}$ for compactness.

Definition 1. The fraction

$$C_k = \left[a_0 + \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \dots \frac{b_k}{a_k} \right]$$

with $k = 0, 1, \dots$, where $k \leq n$, is called the k th forward convergent of the continued fraction (1).

Convergents are usually denoted $C_n = \frac{P_n}{Q_n}$ where the sequences $\{P_n\}$ and $\{Q_n\}$ are defined by the recurrence relations for $n > 0$ [9]

$$\begin{aligned} P_n &= a_n P_{n-1} + b_n P_{n-2}, & P_{-1} &= 1, & P_0 &= a_0, \\ Q_n &= a_n Q_{n-1} + b_n Q_{n-2}, & Q_{-1} &= 0, & Q_0 &= 1. \end{aligned} \quad (2)$$

In this paper, we define *backward continued fraction* (BCF) for a real number and derive some elementary convergent properties of it. Also using the backward continued fractions, we give some results about LU factorization and determinant of a tridiagonal matrix. The notion backward continued fraction was firstly used in [20] for vector valued continued fractions. Further, we present an explicit formula for the elements of the inverse of a general tridiagonal matrix. Our approach is based on using the backward continued fractions.

Further note that in the next studies, we will again use backward continued fractions for LU factorizations, determinants and inverses of some certain Toeplitz matrices.

2. Backward continued fraction

In this section, we give some new definitions and results. Let a_0, a_1, \dots, a_n be real numbers. Now we define simple backward continued fraction in the following inductive manner.

Definition 2 (*Backward Continued Fraction*)

$$[a_0]_b = a_0 \quad \text{and} \quad [a_0, a_1, \dots, a_n]_b = a_n + \frac{1}{[a_0, a_1, \dots, a_{n-1}]_b} \quad \text{for } n \geq 1.$$

For example,

$$\begin{aligned} [2]_b &= 2, \\ [2, 3]_b &= 3 + \frac{1}{[2]_b} = 3 + \frac{1}{2} = \frac{7}{2}, \\ [2, 3, 8]_b &= 8 + \frac{1}{[2, 3]_b} = 8 + \frac{1}{\frac{7}{2}} = 8 + \frac{2}{7} = \frac{58}{7}. \end{aligned}$$

Note that, in general

$$\begin{aligned} [a_0, a_1]_b &= a_1 + \frac{1}{[a_0]_b}, \\ [a_0, a_1, a_2]_b &= a_2 + \frac{1}{[a_0, a_1]_b} = a_2 + \frac{1}{a_1 + \frac{1}{a_0}}, \end{aligned}$$

and

$$[a_0, a_1, \dots, a_n]_b = [a_n, a_{n-1}, \dots, a_0].$$

The following theorem shows that a given backward fraction can be represented by another backward fraction with one less term.

Theorem 3. *Let a_i real numbers. Then $[a_0, a_1, \dots, a_n]_b = [a_1 + \frac{1}{a_0}, a_2, \dots, a_n]_b$.*

Proof. The proof is easy.

For example, $[1, 3, 8, 2]_b = [4, 8, 2]_b$.

In generally, the a_i are called the *terms*, or, partial quotients, of the backward continued fraction. Clearly, if all the a_i are rational numbers (or integers), then $[a_0, a_1, \dots, a_n]_b$ is rational. \square

Definition 4. Let $A = [a_0, a_1, \dots, a_n]_b$ be a backward continued fraction and let $C_k^b = [a_0, a_1, \dots, a_k]_b$. C_k^b is called the k th convergent of backward continued fraction A .

For example, let $A = [8, 4, 2, 3]_b$. Then

$$\begin{aligned} C_0^b &= [8]_b = 8, \\ C_1^b &= [8, 4]_b = 4 + \frac{1}{8} = \frac{33}{8}, \\ C_2^b &= [8, 4, 2]_b = 2 + \frac{1}{4 + \frac{1}{8}} = 2 + \frac{8}{33} = \frac{74}{33}, \\ C_3^b &= A = [8, 4, 2, 3]_b = 3 + \frac{1}{2 + \frac{1}{4 + \frac{1}{8}}} = 3 + \frac{33}{74} = \frac{255}{74}. \end{aligned}$$

Next we develop an iterative method to generate the convergents of a backward continued fraction, assuming that we have the terms a_k . Define sequence $\{p_n\}$ by the recurrence relation for $n > 0$

$$p_n = a_n p_{n-1} + p_{n-2} \quad \text{for } n \geq 0,$$

where $p_{-1} = 1, p_0 = a_0$. The sequence $\{p_n\}$ is obtained from the sequence $\{P_n\}$ by taking $b_n = 1$ for all n in (2).

Theorem 5. Given a backward continued fraction $A = [a_0, a_1, \dots, a_n]_b$. If $0 \leq k \leq n$ and C_k^b is the k th backward convergent to A , $C_k^b = [a_0, a_1, \dots, a_k]_b$, then $C_k^b = \frac{p_k}{p_{k-1}}$.

Proof (Induction on n). If $k = 0$, then by the definition of the sequence $\{p_n\}$, $\frac{p_0}{p_{-1}} = a_0 = [a_0]_b = C_0^b$. If $k = 1$, then

$$\frac{p_1}{p_0} = \frac{a_1 a_0 + 1}{a_0} = a_1 + \frac{1}{a_0} = [a_0, a_1]_b = C_1^b.$$

Suppose that the claim is true for $k - 1$. Then we show that the claim is true for k . Thus, by induction hypothesis

$$\frac{p_k}{p_{k-1}} = \frac{a_k p_{k-1} + p_{k-2}}{p_{k-1}} = a_k + \frac{1}{\frac{p_{k-1}}{p_{k-2}}} = a_k + \frac{1}{[a_0, a_1, \dots, a_{k-1}]_b}.$$

Therefore, by the definition of the backward continued fraction

$$\frac{p_k}{p_{k-1}} = [a_0, a_1, \dots, a_{k-1}, a_k]_b = C_k^b.$$

So the proof is complete. \square

For example, we compute $[8, 4, 2, 3]_b$ by Theorem 5. For $p_{-1} = 1, p_0 = 8$, we have $p_1 = 33, p_2 = 74, p_3 = 255$ and so $[8, 4, 2, 3]_b = \frac{p_3}{p_2} = \frac{255}{74}$.

From the above results, we have the following result: Let $A = [a_0, a_1, \dots, a_n]$ be a finite continued fraction, in usual manner, with k th convergent C_k and let $B = [a_0, a_1, \dots, a_n]_b$ be a backward continued fraction with k th convergent C_k^b . Then the numerators of C_k^b and C_k are the same.

Let a_0, a_1, \dots and b_1, b_2, \dots, b_n be integers. Normally, a general continued fraction representation of a real number x is one of the form

$$x = a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \dots + \frac{b_n}{a_n}}}$$

Now we define a general backward continued fraction. Let a_0, a_1, \dots, a_n and b_1, b_2, \dots, b_n be integers.

Definition 6. The general backward continued fraction have the form

$$a_n + \frac{b_n}{a_{n-1} + \frac{b_{n-1}}{\dots + \frac{b_1}{a_0}}}$$

We denote such representations by in the form

$$C^b = \left[a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \dots + \frac{b_n}{a_n}} \right]_b$$

for compactness. We denote the k th convergent of general backward fraction by

$$C_k^b = \left[a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \dots + \frac{b_k}{a_k}} \right]_b.$$

For example, we compute $[3, \frac{2}{5}, \frac{7}{4}]_b$. Thus $a_0 = 3, a_1 = 5, a_2 = 4$ and $b_1 = 2, b_2 = 7$, and

$$a_2 + \frac{b_2}{a_1 + \frac{b_1}{a_0}} = 4 + \frac{7}{5 + \frac{2}{3}} = \frac{89}{17}.$$

The backward continued fraction $[a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \dots \frac{b_n}{a_n}}]_b$ is equal to the usual continued fraction $[a_n + \frac{b_n}{a_{n-1} + \frac{b_{n-2}}{a_{n-2} + \dots \frac{b_1}{a_0}}]_b$. Indeed, $[3, \frac{2}{5}, \frac{7}{4}]_b = [4, \frac{7}{5}, \frac{2}{3}]$.

We give a ratio for a general backward fraction. Recall that the sequence $\{P_n\}$ be as in (2), that is, for $n > 0$

$$P_n = a_n P_{n-1} + b_n P_{n-2},$$

where $P_{-1} = 1, P_0 = a_0$.

Then we have the following the theorem as a generalization of Theorem 5.

Theorem 7. Given a general backward continued fraction $A = [a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \dots \frac{b_n}{a_n}}]_b$. If $0 \leq k \leq n$ and C_k^b is the k th backward convergent to A , that is, $C_k^b = [a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \dots \frac{b_k}{a_k}}]_b$, then $C_k^b = \frac{P_k}{P_{k-1}}$.

Proof (Induction on n). If $n = 0$, then we have $C_0^b = [a_0]_b$ and $\frac{P_0}{P_{-1}} = a_0$. If $n = 1$, then we have $C_1^b = [a_0 + \frac{b_1}{a_1}]_b = a_1 + \frac{b_1}{a_0} = \frac{a_1 a_0 + b_1}{a_0}$ and by definition of the sequence $\{P_n\}$, $\frac{P_1}{P_0} = \frac{a_1 a_0 + b_1}{a_0}$. Suppose that the equation holds for k . Then we show that the equation holds for $k + 1$. Thus, by the definition of fraction and the inductive hypothesis

$$C_{k+1}^b = a_{k+1} + \frac{b_{k+1}}{C_k^b} = a_{k+1} + \frac{b_{k+1}}{\frac{P_k}{P_{k-1}}} = \frac{a_{k+1} P_k + b_{k+1} P_{k-1}}{P_k} = \frac{P_{k+1}}{P_k}.$$

So the proof is complete. \square

By the well known results and the above results, we have the following Corollary.

Corollary 8. Given a usual continued fraction $[a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \dots \frac{b_n}{a_n}}]_b$ and a backward continued fraction $[a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \dots \frac{b_n}{a_n}}]_b$. Then the numerators of these fractions are the same, P_n .

Now using backward continued fractions, we reobtain LU factorization of a tridiagonal matrix.

3. LU Factorization of a tridiagonal matrix

In this section, we give the LU decomposition of a general tridiagonal matrix by backward continued fractions.

Let $G_n = [g_{ij}]$ be a $n \times n$ tridiagonal matrix with $g_{k,k} = x_k$ for $1 \leq k \leq n$ and $g_{k+1,k} = z_k$ and $g_{k,k+1} = y_k$ for $1 \leq k \leq n - 1$. That is,

$$G_n = \begin{bmatrix} x_1 & y_1 & & & \\ z_1 & x_2 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & y_{n-1} \\ & & & z_{n-1} & x_n \end{bmatrix}. \tag{3}$$

Suppose that the entries of G_n are given, that is, we have $x_1, \dots, x_n, y_1, \dots, y_{n-1}$ and z_1, \dots, z_{n-1} .

We define a general backward continued fraction C_n^b by the entries of the matrix G_n as follows:

$$C_n^b = \left[x_1 + \frac{-y_1 z_1}{x_2 +} \frac{-y_2 z_2}{x_3 +} \dots \frac{-y_{n-1} z_{n-1}}{x_n} \right]_b = x_n + \frac{-y_{n-1} z_{n-1}}{x_{n-1} + \frac{-y_{n-2} z_{n-2}}{x_{n-2} + \dots \frac{-y_1 z_1}{x_2 +} \frac{-y_1 z_1}{x_1}}}. \tag{4}$$

If we rearrange the sequence $\{P_n\}$ with $a_i = x_{i+1}$ for $0 \leq i \leq n - 1, b_i = -y_i z_i$ for $1 \leq i \leq n$, then

$$P_n = x_{n+1} P_{n-1} - y_n z_n P_{n-2}, \tag{5}$$

where $P_{-1} = 1, P_0 = x_1$.

Thus few convergents of C_n^b are

$$C_1^b = [x_1]_b = \frac{P_0}{P_{-1}} = x_1,$$

$$C_2^b = \left[x_1 + \frac{-y_1 z_1}{x_2} \right]_b = \frac{P_1}{P_0} = \frac{x_1 x_2 - y_1 z_1}{x_1},$$

$$C_3^b = \left[x_1 + \frac{-y_1 z_1}{x_2 + \frac{-y_2 z_2}{x_3}} \right]_b = \frac{P_2}{P_1} = \frac{x_3 x_2 x_1 - x_3 y_1 z_1 - x_1 y_2 z_2}{x_1 x_2 - y_1 z_1}.$$

We define two $n \times n$ matrices as follows:

$$L_1 = [l_{ij}] = \begin{bmatrix} 1 & & & 0 \\ \frac{z_1}{C_1^b} & 1 & & \\ & \frac{z_2}{C_2^b} & 1 & \\ & & \ddots & \ddots & 0 \\ 0 & & & \frac{z_{n-1}}{C_{n-1}^b} & 1 \end{bmatrix} \tag{6}$$

and

$$U_1 = [u_{ij}] = \begin{bmatrix} C_1^b & y_1 & & & 0 \\ & C_2^b & y_2 & & \\ & & C_3^b & \ddots & \\ & & & \ddots & y_{n-1} \\ 0 & & & & C_n^b \end{bmatrix}, \tag{7}$$

where C_i^b is the i th convergent of the backward continued fraction given by (4) for $1 \leq i \leq n$.

Now we give the Doolittle type LU decomposition of matrix G_n with the following Theorem.

Theorem 9. *Let the matrix G_n and the general backward fraction C_n^b be as in (3) and (4), respectively. Then the LU decomposition of the matrix G_n have the form*

$$G_n = L_1 U_1,$$

where L_1 and U_1 given by (6) and (7), respectively.

Proof. We consider three cases. First, we consider the case $i = j$. By the definitions of matrices U_1 and L_1 , we have $l_{ij} = 0$ for $i > j + 1$ and $j > i$, $l_{ii} = 1$ for $1 \leq i \leq n$, and $u_{ij} = 0$ for $i > j$ and $j > i + 1$, $u_{i,i+1} = y_i$ for $1 \leq i \leq n - 1$. Then, by matrix multiplication, we have for $1 \leq i \leq n$

$$g_{ii} = \sum_{k=1}^n l_{i,k} u_{k,i} = l_{i,i-1} u_{i-1,i} + l_{i,i} u_{i,i} = y_{i-1} l_{i,i-1} + C_i^b = y_{i-1} \frac{z_{i-1}}{C_{i-2}^b} + C_i^b.$$

Clearly we may write the last equation as, for $1 \leq i \leq n$

$$g_{ii} = y_{i-1} \left(\frac{z_{i-1}}{x_{i-1} + \frac{-y_{i-2} z_{i-2}}{x_{i-2} + \frac{-y_{i-3} z_{i-3}}{x_{i-3} + \dots}}} \right) + \left(x_i + \frac{-y_{i-1} z_{i-1}}{x_{i-1} + \frac{-y_{i-2} z_{i-2}}{x_{i-2} + \frac{-y_{i-3} z_{i-3}}{x_{i-3} + \dots}}} \right),$$

which provides, for $1 \leq i \leq n$

$$g_{ii} = x_i.$$

Now we consider the case $i + 1 = j$. Thus, for $1 \leq i \leq n - 1$

$$g_{i,i+1} = \sum_{k=1}^n l_{i,k} u_{k,i+1} = l_{i,i} u_{i,i+1} = u_{i,i+1} = y_i.$$

Finally we consider the case $i = j + 1$. Then we have, for $1 \leq i \leq n - 1$

$$g_{i+1,i} = \sum_{k=1}^n l_{i+1,k} u_{k,i} = l_{i+1,i} u_{i,i} = \frac{z_i}{C_i^b} C_i^b = z_i.$$

Thus the proof is complete for all cases. \square

Therefore we have the following Corollary.

Corollary 10. *Let the matrix G_n and the general backward continued fraction C_n^b be as in (3) and (4), respectively. Then*

$$\det G_n = \prod_{i=1}^n C_i^b,$$

where C_i^b is the i th convergent of (4).

Proof. From Theorem 9, we have $G_n = L_1 U_1$. Thus $\det G_n = \det(L_1 U_1)$. Since L_1 is the unit bidiagonal matrix and U_1 is the upper bidiagonal matrix with diagonal entries are the convergents of (4), the conclusion is immediately seen. \square

Then we can express the $\det G_n$ by the terms of sequence $\{P_n\}$.

Corollary 11. *Let the matrix G_n is as in (3). Then for $n > 1$*

$$\det G_n = P_{n-1},$$

where P_n given by (5).

Proof. From Theorem 9 and Corollary 10, we have $\det G_n = \prod_{i=0}^{n-1} C_i^b$ and $C_i^b = \frac{P_{i-1}}{P_{i-2}}$. Then we write

$$\det G_n = \frac{P_0}{P_{-1}} \frac{P_1}{P_0} \dots \frac{P_{n-2}}{P_{n-3}} \frac{P_{n-1}}{P_{n-2}} = \frac{P_{n-1}}{P_{-1}}$$

and since $P_{-1} = 1$, $\det G_n = P_{n-1}$. So the proof is complete. \square

Now we consider the second kind LU factorization of matrix G_n called the Crout factorization. Define two $n \times n$ matrices as follows:

$$L_2 = [t_{ij}] = \begin{bmatrix} C_1^b & & & & 0 \\ z_1 & C_2^b & & & \\ & z_2 & \ddots & & \\ & & \ddots & C_{n-1}^b & \\ 0 & & & z_{n-1} & C_n^b \end{bmatrix} \tag{8}$$

and

$$U_2 = [h_{ij}] = \begin{bmatrix} 1 & \frac{y_1}{C_1^b} & & & 0 \\ & 1 & \frac{y_2}{C_2^b} & & \\ & & 1 & \ddots & \\ & & & \ddots & \frac{y_{n-1}}{C_{n-1}^b} \\ 0 & & & & 1 \end{bmatrix}, \tag{9}$$

where C_i^b is the i th convergent of the backward continued fraction given by (4) for $1 \leq i \leq n$.

Then we have the following Theorem.

Theorem 12. Let the matrix G_n and the general backward fraction C_n^b be as in (3) and (4), respectively. Then the Crout type LU decomposition of matrix G_n have the form

$$G_n = L_2 U_2,$$

where L_2 and U_2 given by (8) and (9), respectively.

Proof. By the definitions of matrices L_2 and U_2 , $t_{ii} = C_i^b$ for all i , $t_{i+1,i} = z_i$ for $1 \leq i \leq n - 1$, $h_{ii} = 1$ for all i , $h_{i,i+1} = \frac{y_i}{C_i^b}$ for $1 \leq i \leq n - 1$ and 0 otherwise. Thus we start with the case $i = j$. Thus, by the definitions of C_i^b 's for $1 \leq i \leq n$

$$g_{ii} = \sum_{k=1}^n t_{i,k} h_{k,i} = t_{ii} h_{ii} + t_{i,i-1} h_{i-1,i} = C_i^b + z_{i-1} \left(\frac{y_{i-1}}{C_{i-1}^b} \right) = x_i.$$

Consider the case $i + 1 = j$. Thus for $1 \leq i \leq n - 1$

$$g_{i,i+1} = \sum_{k=1}^n t_{i,k} h_{k,i+1} = t_{ii} h_{i,i+1} = C_i^b \left(\frac{y_i}{C_i^b} \right) = y_i.$$

Finally consider the case $i = j + 1$. Thus for $1 \leq i \leq n - 1$

$$g_{i+1,i} = \sum_{k=1}^n t_{i+1,k} h_{k,i} = t_{i+1,i} h_{i,i} = z_i.$$

So the proof is complete. \square

By Theorem 12, we have that $\det G_n = \det(L_2 U_2)$. Since $\det L_2 = C_1^b \cdot C_2^b \cdots C_n^b$ and $\det U_2 = 1$, $\det G_n = C_1^b \cdot C_2^b \cdots C_n^b$. By the definitions of C_i^b 's, we reobtain the result of Corollary 11.

4. Inversion of a tridiagonal matrix by BCF

In this section, we use the advantage of backward continued fractions and give inversion of a tridiagonal matrix. Before this, we give some results about the inversion of matrices L_1 and U_1 . Then we have the following Lemma.

Lemma 13. Let the $n \times n$ unit bidiagonal matrix L_1 have the form (6) and let $Q_1 = (q_{ij}) = L_1^{-1}$. Then

$$q_{ij} = \begin{cases} (-1)^{i+j} \prod_{k=j}^{i-1} \frac{z_k}{C_k^b} & \text{if } i > j, \\ 1 & \text{if } i = j, \\ 0 & \text{if } i < j. \end{cases}$$

Proof. Denote $L_1 Q_1$ by $R = (r_{ij})$. If $i = j$, then $r_{ii} = \sum_{k=1}^n l_{ik} q_{ki} = l_{ii} q_{ii} = 1$. If $i + 1 > j \geq 1$, then by the definitions of L_1 and Q_1 , for $1 < i \leq n$

$$\begin{aligned} r_{ij} &= \sum_{k=1}^n l_{ik} q_{kj} = l_{i,i-1} q_{i-1,j} + l_{ii} q_{ij} = \frac{z_{i-1}}{C_{i-1}^b} q_{i-1,j} + q_{ij} = (-1)^{i+j-1} \left(\frac{z_{i-1}}{C_{i-1}^b} \right) \left(\prod_{k=j}^{i-2} \frac{z_k}{C_k^b} \right) + (-1)^{i+j} \left(\prod_{k=j}^{i-1} \frac{z_k}{C_k^b} \right) \\ &= (-1)^{i+j-1} \left(\prod_{k=j}^{i-1} \frac{z_k}{C_k^b} \right) + (-1)^{i+j} \left(\prod_{k=j}^{i-1} \frac{z_k}{C_k^b} \right) = 0. \end{aligned}$$

If $i + 1 = j \geq 1$, then we immediately obtain $r_{i,i+1} = 0$. It is clear that $r_{ij} = 0$ for the case $i < j$. So we obtain $R = I_n$ where I_n is the $n \times n$ unit matrix. Similarly it can be shown that $Q_1 L_1 = I_n$. Thus the proof is complete. \square

Lemma 14. Let the $n \times n$ bidiagonal matrix U_1 have the form (7) and let $H_1 = (h_{ij}) = U_1^{-1}$. Then

$$h_{ij} = \begin{cases} \frac{(-1)^{i+j}}{C_j^b} \prod_{k=i}^{j-1} \frac{y_k}{C_k^b} & \text{if } i < j, \\ \frac{1}{C_i^b} & \text{if } i = j, \\ 0 & \text{if } i > j. \end{cases}$$

Proof. Denote $U_1 H_1$ by $E = (e_{ij})$. If $i = j$, then, by the definitions of U_1 and H_1 , we obtain $e_{ii} = \sum_{k=1}^n u_{ik} h_{kj} = u_{ii} h_{ii} = C_i^b \left(\frac{1}{C_i^b}\right) = 1$. It is clear that $e_{ij} = 0$ for the case $i > j$. Finally we consider the case $i < j$. Thus by the definitions of U_1 and H_1 ,

$$\begin{aligned} e_{ij} &= \sum_{k=1}^n u_{ik} h_{kj} = u_{ii} h_{ij} + u_{i,i+1} h_{i+1,j} = (-1)^{i+j} C_i^b \left(\frac{\prod_{k=i}^{j-1} y_k}{\prod_{k=i}^j C_k^b}\right) + (-1)^{i+j+1} y_i \left(\frac{\prod_{k=i+1}^{j-1} y_k}{\prod_{k=i+1}^j C_k^b}\right) \\ &= (-1)^{i+j} \left(\frac{\prod_{k=i}^{j-1} y_k}{\prod_{k=i+1}^j C_k^b}\right) + (-1)^{i+j+1} \left(\frac{\prod_{k=i}^{j-1} y_k}{\prod_{k=i+1}^j C_k^b}\right) = 0. \end{aligned}$$

Thus we see that $E = I_n$ ($n \times n$ unit matrix). Also it can be shown that $H_1 U_1 = I_n$. So we have the conclusion. \square

Suppose that $y_i z_i \neq 0$ for $1 \leq i \leq n - 1$. Then we have the following theorem for the inversion of a tridiagonal matrix.

Theorem 15. Let the $n \times n$ tridiagonal matrix G_n have the form (3). Let $G_n^{-1} = [w_{ij}]$ denote the inverse of G_n . Then

$$w_{ij} = \begin{cases} \frac{1}{C_i^b} + \sum_{k=i+1}^n \left(\frac{1}{C_k^b} \prod_{t=i}^{k-1} \frac{y_t z_t}{(C_t^b)^2}\right) & \text{if } i = j, \\ (-1)^{i+j} \left(\prod_{t=i}^{j-1} \frac{y_t}{C_t^b}\right) \left(\frac{1}{C_j^b} + \sum_{k=j+1}^n \left(\frac{1}{C_k^b} \prod_{t=j}^{k-1} \frac{y_t z_t}{(C_t^b)^2}\right)\right) & \text{if } i < j, \\ (-1)^{i+j} \left(\prod_{t=j}^{i-1} \frac{z_t}{C_t^b}\right) \left(\frac{1}{C_i^b} + \sum_{k=i+1}^n \left(\frac{1}{C_k^b} \prod_{t=i}^{k-1} \frac{y_t z_t}{(C_t^b)^2}\right)\right) & \text{if } i > j. \end{cases}$$

Proof. We consider first case $i = j$. By matrix multiplication and since $h_{ij} = 0$ for $i > j$ and $G_n^{-1} = (L_1 U_1)^{-1} = H_1 Q_1$, we have

$$\begin{aligned} w_{ii} &= \sum_{k=1}^n h_{ik} q_{ki} = \sum_{k=i}^n h_{ik} q_{ki} = h_{ii} q_{ii} + h_{i,i+1} q_{i+1,i} + \dots + h_{in} q_{ni} = h_{ii} q_{ii} + \sum_{k=i+1}^n h_{ik} q_{ki} \\ &= \frac{1}{C_i^b} + \sum_{k=i+1}^n \left[\left((-1)^{i+k} \left(\prod_{t=i}^{k-1} y_t\right) \left(\prod_{t=i}^k (C_t^b)^{-1}\right) \right) \left((-1)^{i+k} \left(\prod_{t=i}^{k-1} \frac{z_t}{C_t^b}\right) \right) \right] = \frac{1}{C_i^b} + \sum_{k=i+1}^n \left(\frac{1}{C_k^b} \prod_{t=i}^{k-1} \frac{y_t z_t}{(C_t^b)^2}\right). \end{aligned}$$

If $i < j$, then by the definition Q_1 , we write

$$\begin{aligned}
 w_{ij} &= \sum_{k=1}^n h_{ik}q_{kj} = \sum_{k=j}^n h_{ik}q_{kj} = h_{ij}q_{jj} + h_{i,j+1}q_{j+1,j} + h_{i,j+2}q_{j+2,j} + \cdots + h_{in}q_{nj} = h_{ij}q_{jj} + \sum_{k=j+1}^n h_{ik}q_{kj} \\
 &= (-1)^{i+j} \left(\prod_{k=i}^{j-1} y_k \right) \left(\prod_{k=i}^j (C_k^b)^{-1} \right) + \sum_{k=j+1}^n \left[\left((-1)^{i+k} \prod_{t=i}^{k-1} y_t \prod_{t=i}^k (C_t^b)^{-1} \right) \left((-1)^{k+j} \prod_{t=j}^{k-1} \frac{z_t}{C_t^b} \right) \right] \\
 &= (-1)^{i+j} \frac{1}{C_j^b} \left(\prod_{k=i}^{j-1} \frac{y_k}{C_k^b} \right) + \sum_{k=j+1}^n \left[(-1)^{i+j} \frac{\left(\prod_{t=i}^{k-1} y_t \right) \left(\prod_{t=j}^{k-1} z_t \right)}{\left(\prod_{t=i}^k C_t^b \right) \left(\prod_{t=j}^{k-1} C_t^b \right)} \right] \\
 &= (-1)^{i+j} \frac{1}{C_j^b} \left(\prod_{k=i}^{j-1} \frac{y_k}{C_k^b} \right) + (-1)^{i+j} \sum_{k=j+1}^n \left[\frac{\left(\prod_{t=i}^{j-1} y_t \right) \left(\prod_{t=j}^{k-1} y_t z_t \right)}{C_k^b \left(\prod_{t=i}^{j-1} C_t^b \right) \left(\prod_{t=j}^{k-1} (C_t^b)^2 \right)} \right] \\
 &= (-1)^{i+j} \frac{1}{C_j^b} \left(\prod_{t=i}^{j-1} \frac{y_t}{C_t^b} \right) + (-1)^{i+j} \left(\prod_{t=i}^{j-1} \frac{y_t}{C_t^b} \right) \sum_{k=j+1}^n \left[\frac{1}{C_t^b} \left(\prod_{t=j}^{k-1} \frac{y_t z_t}{(C_t^b)^2} \right) \right] \\
 &= (-1)^{i+j} \left(\prod_{t=i}^{j-1} \frac{y_t}{C_t^b} \right) \left[\frac{1}{C_j^b} + \sum_{k=j+1}^n \left[\frac{1}{C_k^b} \left(\prod_{t=j}^{k-1} \frac{y_t z_t}{(C_t^b)^2} \right) \right] \right].
 \end{aligned}$$

Finally, we consider last case $i > j$. Thus, by the definition of H_1 ,

$$\begin{aligned}
 w_{ij} &= \sum_{k=1}^n h_{ik}q_{kj} = \sum_{k=i}^n h_{ik}q_{kj} = h_{ii}q_{ij} + h_{i,i+1}q_{i+1,j} + h_{i,i+2}q_{i+2,j} + \cdots + h_{in}q_{nj} = h_{ii}q_{ij} + \sum_{k=i+1}^n h_{ik}q_{kj} \\
 &= \frac{1}{C_i^b} (-1)^{i+j} \left(\prod_{t=j}^{i-1} \frac{z_t}{C_t^b} \right) + \sum_{k=i+1}^n \left[\left((-1)^{i+k} \frac{\prod_{t=i}^{k-1} y_t}{\prod_{t=i}^k C_t^b} \right) \left((-1)^{k+j} \left(\prod_{t=j}^{k-1} \frac{z_t}{C_t^b} \right) \right) \right] \\
 &= \frac{(-1)^{i+j}}{C_i^b} \left(\prod_{t=j}^{i-1} \frac{z_t}{C_t^b} \right) + \sum_{k=i+1}^n \left[\frac{(-1)^{i+j}}{C_k^b} \frac{\left(\prod_{t=i}^{k-1} y_t \right) \left(\prod_{t=j}^{k-1} z_t \right)}{\left(\prod_{t=i}^{k-1} C_t^b \right) \left(\prod_{t=j}^{k-1} C_t^b \right)} \right] \\
 &= \frac{(-1)^{i+j}}{C_i^b} \left(\prod_{t=j}^{i-1} \frac{z_t}{C_t^b} \right) + (-1)^{i+j} \sum_{k=i+1}^n \left[\frac{\left(\prod_{t=i}^{k-1} y_t z_t \right) \left(\prod_{t=j}^{i-1} z_t \right)}{C_k^b \left(\prod_{t=i}^{k-1} (C_t^b)^2 \right) \left(\prod_{t=j}^{i-1} C_t^b \right)} \right] \\
 &= \frac{(-1)^{i+j}}{C_i^b} \left(\prod_{t=j}^{i-1} \frac{z_t}{C_t^b} \right) + (-1)^{i+j} \left(\prod_{t=j}^{i-1} \frac{z_t}{C_t^b} \right) \sum_{k=i+1}^n \left(\frac{1}{C_k^b} \prod_{t=i}^{k-1} \frac{y_t z_t}{(C_t^b)^2} \right) \\
 &= (-1)^{i+j} \left(\prod_{t=j}^{i-1} \frac{z_t}{C_t^b} \right) \left[\frac{1}{C_i^b} + \sum_{k=i+1}^n \left(\frac{1}{C_k^b} \prod_{t=i}^{k-1} \frac{y_t z_t}{(C_t^b)^2} \right) \right].
 \end{aligned}$$

We have the conclusion for all cases. \square

Considering the statement of [Theorem 15](#), we can give the following Corollary for fast computing of the inverse elements of a tridiagonal matrix.

Corollary 16. Let w_{ij} denote the (i, j) th element of the inverse of matrix G_n given by (3). Then

$$w_{ij} = \begin{cases} \frac{1}{C_i^b} + \sum_{k=i+1}^n \left(\frac{1}{C_k^b} \prod_{t=i}^{k-1} \frac{y_t z_t}{(C_t^b)^2} \right) & \text{if } i = j, \\ (-1)^{i+j} w_{jj} \prod_{t=i}^{j-1} \frac{y_t}{C_t^b} & \text{if } i < j, \\ (-1)^{i+j} w_{ii} \prod_{t=j}^{i-1} \frac{z_t}{C_t^b} & \text{if } i > j. \end{cases} \tag{10}$$

Consequently while computing the nonmain-diagonal elements of the inverse of a tridiagonal matrix, we reduce the required levels by the main diagonal elements of its inverse.

If we compare the earlier some methods and our results about the computing of the elements of a tridiagonal matrix, then we see that our process is more convenient, efficient and fast.

As an example of Theorem 15 and Corollary 16, now we consider a tridiagonal matrix T of order 3 as shown

$$T = [t_{ij}] = \begin{bmatrix} \alpha_1 & \beta_1 & 0 \\ \delta_1 & \alpha_2 & \beta_2 \\ 0 & \delta_2 & \alpha_3 \end{bmatrix}.$$

Thus according to our process, we construct the following backward continued fraction C_3^b :

$$C_3^b = \alpha_3 + \frac{-\beta_2\delta_2}{\alpha_2 + \frac{-\beta_1\delta_1}{\alpha_1}}.$$

Then its convergents are as follows: $C_1^b = \alpha_1$, $C_2^b = \alpha_2 + \frac{-\beta_1\delta_1}{\alpha_1} = \frac{\alpha_1\alpha_2 - \beta_1\delta_1}{\alpha_1}$ and $C_3^b = \alpha_3 + \frac{-\beta_2\delta_2}{\alpha_2 + \frac{-\beta_1\delta_1}{\alpha_1}} = \frac{\alpha_1\alpha_2\alpha_3 - \beta_1\delta_1\alpha_3 - \beta_2\delta_2\alpha_1}{\alpha_1\alpha_2 - \beta_1\delta_1}$.

For $i = j$, by Theorem 15, the main diagonal elements of the inverse of matrix $T = (t_{ij})$ are given by

$$t_{11}^{-1} = \frac{1}{C_1^b} + \frac{\beta_1\delta_1}{C_2^b(C_1^b)^2} + \frac{\beta_1\delta_1\beta_2\delta_2}{(C_1^b)^2(C_2^b)^2C_3^b} = -\frac{\alpha_2\alpha_3 - \beta_2\delta_2}{\alpha_1\beta_2\delta_2 - \alpha_1\alpha_2\alpha_3 + \beta_1\alpha_3\delta_1},$$

$$t_{22}^{-1} = \frac{1}{C_2^b} + \frac{\beta_2\delta_2}{C_3^b(C_2^b)^2} = \frac{-\alpha_1\alpha_3}{\alpha_1\beta_2\delta_2 - \alpha_1\alpha_2\alpha_3 + \beta_1\alpha_3\delta_1},$$

$$t_{33}^{-1} = \frac{1}{C_3^b} = \frac{-(\alpha_1\alpha_2 - \beta_1\delta_1)}{\alpha_1\beta_2\delta_2 - \alpha_1\alpha_2\alpha_3 + \beta_1\alpha_3\delta_1}.$$

The upper diagonal elements of T^{-1} are then given by

$$t_{12}^{-1} = -\frac{\beta_1}{C_1^b} t_{22}^{-1} = \frac{\beta_1\alpha_3}{\alpha_1\beta_2\delta_2 - \alpha_1\alpha_2\alpha_3 + \beta_1\alpha_3\delta_1},$$

$$t_{13}^{-1} = \prod_{t=1}^2 \frac{\beta_t}{C_t^b} t_{33}^{-1} = \frac{-\beta_1\beta_2}{\alpha_1\beta_2\delta_2 - \alpha_1\alpha_2\alpha_3 + \beta_1\alpha_3\delta_1},$$

$$t_{23}^{-1} = -\frac{\beta_2}{C_2^b} t_{33}^{-1} = \frac{-\alpha_1\beta_2}{\alpha_1\beta_2\delta_2 - \alpha_1\alpha_2\alpha_3 + \beta_1\alpha_3\delta_1}.$$

The lower diagonal elements of T^{-1} are given by

$$t_{21}^{-1} = -\frac{\delta_1}{C_1^b} t_{22}^{-1} = \frac{\alpha_3\delta_1}{\alpha_1\beta_2\delta_2 - \alpha_1\alpha_2\alpha_3 + \beta_1\alpha_3\delta_1},$$

$$t_{31}^{-1} = \frac{\delta_1\delta_2}{C_1^b C_2^b} t_{33}^{-1} = \frac{-\delta_1\delta_2}{\alpha_1\beta_2\delta_2 - \alpha_1\alpha_2\alpha_3 + \beta_1\alpha_3\delta_1},$$

$$t_{32}^{-1} = \frac{-\delta_2}{C_2^b} t_{33}^{-1} = \frac{\delta_2\alpha_1}{\alpha_1\beta_2\delta_2 - \alpha_1\alpha_2\alpha_3 + \beta_1\alpha_3\delta_1}.$$

From (10), it is clear that if the matrix G_n is a symmetric tridiagonal matrix, that is,

$$G_n = \begin{bmatrix} x_1 & y_1 & & & \\ y_1 & x_2 & \ddots & & \\ & \ddots & \ddots & & \\ & & \ddots & y_{n-1} & \\ & & & y_{n-1} & x_n \end{bmatrix}$$

then the inverse of the matrix $G_n, G_n^{-1} = [w_{ij}]$ is given by

$$w_{ij} = \begin{cases} \frac{1}{C_i^b} + \sum_{k=i+1}^n \left(\frac{1}{C_k^b} \prod_{t=i}^{k-1} \frac{y_t^2}{(C_t^b)^2} \right) & \text{if } i = j, \\ (-1)^{i+j} \prod_{t=j}^{i-1} \frac{y_t}{C_t^b} w_{ii} & \text{otherwise.} \end{cases}$$

Finally, in order to see that how susceptible the process to loss of accuracy through subtraction is, we give a numerical example where the matrix is near singularity.

Let

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 3 & 2 & 3 \\ 0 & 0 & 4 & 1.999 \end{bmatrix}$$

then the $\det A = 0.018$. Thus according to the our method, for the matrix $A, y_1 = 1, y_2 = 2, y_3 = 3, z_1 = 1, z_2 = 3, z_3 = 4$ and so we easily obtain

$$C_1^b = 2, \quad C_2^b = -\frac{3}{2}, \quad C_3^b = 6, \quad C_4^b = -0.001.$$

From Corollary 16, we have the diagonal entries of matrix A

$$w_{11} = \frac{1}{C_1^b} + \sum_{k=2}^4 \left(\frac{1}{C_k^b} \prod_{t=1}^{k-1} \frac{y_t z_t}{(C_t^b)^2} \right), \quad w_{22} = \frac{1}{C_2^b} + \sum_{k=3}^4 \left(\frac{1}{C_k^b} \prod_{t=2}^{k-1} \frac{y_t z_t}{(C_t^b)^2} \right),$$

$$w_{33} = \frac{1}{C_3^b} + \frac{y_3 z_3}{C_4^b (C_3^b)^2}, \quad w_{44} = \frac{1}{C_4^b}$$

and so

$$w_{11} = -\frac{1996}{9}, \quad w_{22} = \frac{-8002}{9}, \quad w_{33} = \frac{-1999}{6}, \quad w_{44} = -1000.$$

Since the nondiagonal entries of A^{-1} can be computed by the diagonal entries of it, we note that

$$w_{12} = -\frac{y_1 w_{22}}{C_1^b} = \frac{4001}{9}, \quad w_{13} = \frac{y_1 y_2 w_{33}}{C_1^b C_2^b} = -\frac{1999}{9}, \quad w_{14} = -\frac{y_1 y_2 y_3 w_{44}}{C_1^b C_2^b C_3^b} = -\frac{1000}{3},$$

$$w_{21} = -\frac{z_1 w_{11}}{C_1^b} = \frac{4001}{9}, \quad w_{23} = -\frac{y_2 w_{33}}{C_2^b} = -\frac{3998}{9}, \quad w_{24} = \frac{y_2 y_3 w_{44}}{C_2^b C_3^b} = \frac{2000}{3},$$

$$w_{31} = \frac{z_1 z_2 w_{33}}{C_1^b C_2^b} = \frac{1999}{6}, \quad w_{32} = -\frac{z_2 w_{33}}{C_2^b} = -\frac{1999}{3}, \quad w_{34} = -\frac{y_3 w_{44}}{C_3^b} = 500,$$

$$w_{41} = -\frac{z_1 z_2 z_3 w_{44}}{C_1^b C_2^b C_3^b} = -\frac{2000}{3}, \quad w_{42} = \frac{z_1 z_2 z_3 w_{44}}{C_1^b C_2^b C_3^b} = \frac{4000}{3}, \quad w_{43} = -\frac{z_3 w_{44}}{C_3^b} = \frac{2000}{3}.$$

Clearly the matrix A^{-1} takes the form

$$A^{-1} = \begin{bmatrix} -\frac{1996}{9} & \frac{4001}{9} & -\frac{1999}{9} & -\frac{1000}{3} \\ \frac{4001}{9} & \frac{-8002}{9} & -\frac{3998}{9} & \frac{2000}{3} \\ \frac{1999}{6} & -\frac{1999}{3} & -\frac{1999}{6} & 500 \\ -\frac{2000}{3} & \frac{4000}{3} & \frac{2000}{3} & -1000 \end{bmatrix}.$$

5. Remarks and conclusions

In this present paper, we consider the Doolittle type LU factorization of a tridiagonal matrix by BCF. After this, we easily obtain the inverses of factor matrices L and U . Thus we find an explicit formula for the elements of the inverse of a general tridiagonal matrix. Comparing some old results and our result on the computing of the elements of the inverse of a tridiagonal matrix, one can see that the number of required computations in our method are less than the number of required computations of earlier methods. Also the same result on the inverse of a tridiagonal matrix by the Crout type LU factorization of a tridiagonal matrix by BCF can easily be obtained.

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