# FORMULAS FOR SUMS OF GENERALIZED ORDER-$k$ FIBONACCI TYPE SEQUENCES BY MATRIX METHODS

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ABSTRACT. In this paper, we give formulas for the sums of generalized order-$k$ Fibonacci, Pell and similar other sequences which we obtain using matrix methods. As applications, we give explicit formulas for the Tribonacci and Tetranacci numbers.

## 1. Introduction

The well-known Fibonacci sequence $\{F_n\}$ is defined for $n > 2$ by

$$F_{n+1} = F_n + F_{n-1}$$

where $F_1 = F_2 = 1$.

The well-known Pell sequence $\{P_n\}$ is defined for $n > 2$ by

$$P_{n+1} = 2P_n + P_{n-1}$$

where $P_1 = 1, P_2 = 2$.

Also, in [2], the author defined $k$ sequences of generalized order-$k$ Fibonacci numbers for $n > 0$ and $1 \le i \le k$ by

$$g_n^i = \sum_{j=1}^{k} g_{n-j}^i \tag{1.1}$$

with initial conditions

$$g_n^i = \begin{cases} 1 & n = 1 - i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 - k \le n \le 0$$

where $g_n^i$ is the $n$th term of the $i$th sequence. For example, when $k = 2$, then the sequence $\{g_n^2\}$ is reduced to the usual Fibonacci sequence $\{F_n\}$.

When $k = 3$ in (1.1), the sequence $\{g_n^3\}$ is reduced to the usual Tribonacci sequence $\{t_n\}$. The Tribonacci sequence is $0, 1, 1, 2, 4, 7, 13, 24, 44, \ldots.$

When $k = 4$ in (1.1), the sequence $\{g_n^4\}$ is reduced to the Tetranacci sequence $\{T_n\}$. The Tetranacci sequence is $0, 1, 1, 2, 4, 8, 15, 29, 56, 108, 208, \ldots.$

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Er [2] showed that $G_n = A^n$ where the $k \times k$ matrices $A$ and $G_n$ is given by

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \; G_n = \begin{bmatrix} g_n^1 & g_n^2 & \cdots & g_n^k \\ g_{n-1}^1 & g_{n-1}^2 & \cdots & g_{n-1}^k \\ \vdots & \vdots & & \vdots \\ g_{n-k+1}^1 & g_{n-k+1}^2 & \cdots & g_{n-k+1}^k \end{bmatrix}, \tag{1.2}$$

respectively. The matrix $A$ is said to be the generalized order-$k$ Fibonacci matrix. Furthermore, he gave the following identities:

$$g_{n+1}^i = g_n^1 + g_n^{i+1}, \text{ for } 1 \le i \le k-1 \tag{1.3}$$

$$g_{n+1}^k = g_n^1. \tag{1.4}$$

In [4], the authors gave the generalized Binet formula and the combinatorial representations of the generalized order-$k$ Fibonacci numbers $g_n^i$ and Lucas numbers $l_n^i$ (for more details see [5]).

In [2], the author defined the $(k+1) \times (k+1)$ matrices $C$ and $U_n$ as follows

$$C = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 1 & & & \\ 0 & & A & \\ \vdots & & & \\ 0 & & & \end{bmatrix} \text{ and } U_n = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ S_n & & & \\ S_{n-1} & & G_n & \\ \vdots & & & \\ S_{n-k+1} & & & \end{bmatrix} \tag{1.5}$$

where the matrices $A$, $G_n$ are given by (1.2) and $S_n = \sum_{j=1}^{n} g_j^k$. Then the author showed that $C^n = U_n$.

The authors defined the $k$ sequences of the generalized order-$k$ Pell numbers as follows: for $n > 0$ and $1 \le i \le k$

$$P_n^i = 2P_{n-1}^i + \sum_{j=2}^{k} P_{n-j}^i$$

with initial conditions

$$P_n^i = \begin{cases} 1 & n = 1-i, \\ 0 & \text{otherwise}, \end{cases} \quad \text{for } 1-k \le n \le 0$$

where $P_n^i$ is the $n$th term of the $i$th generalized Pell sequence. When $k = i = 2$, then the generalized order-$k$ Pell sequence $\{P_n^i\}$ is reduced to the well-known Pell sequence $\{P_n\}$. Also the authors defined the $k \times k$ companion matrix $R$ and the $k \times k$ matrix $E_n$ as follows

$$R = \begin{bmatrix} 2 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & \vdots & \vdots \\ \vdots & & \ddots & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}, \; E_n = \begin{bmatrix} P_n^1 & P_n^2 & \dots & P_n^k \\ P_{n-1}^1 & P_{n-1}^2 & \dots & P_{n-1}^k \\ \vdots & \vdots & & \vdots \\ P_{n-k+1}^1 & P_{n-k+1}^2 & \dots & P_{n-k+1}^k \end{bmatrix}, \tag{1.6}$$

respectively. The $k \times k$ companion matrix $R$ is said to be the generalized order-$k$ Pell matrix. Then the authors showed that $R^n = E_n$. Also they defined two $(k+1) \times (k+1)$ matrices $T$ and $H_n$ as follows

$$T = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & & & \\ 0 & & R & \\ \vdots & & & \\ 0 & & & \end{bmatrix} \text{ and } H_{n=} \begin{bmatrix} 1 & 0 & \dots & 0 \\ S_n^P & & & \\ \vdots & & E_n & \\ S_{n-k+1}^P & & & \end{bmatrix} \tag{1.7}$$

where $S_n^P = \sum_{j=1}^n P_j^k$ and the matrices $R,\ E_n$ are given by (1.6).

Then they showed that $T^n = H_n$. Also authors gave some useful formulas, the generalized Binet formula and the combinatorial representation of the generalized order-$k$ Pell numbers.

In this paper, we use the matrix methods to give explicit formulas for the sums of the generalized order-$k$ Fibonacci and Pell numbers from 1 to $n$, $S_n = \sum_{j=1}^n g_j^k$ and $S_n^P = \sum_{j=1}^n P_j^k$, respectively.

## 2. Formula for the sums of the generalized Fibonacci numbers

Let $f(\lambda)$ be the characteristic polynomial of the generalized order-$k$ Fibonacci matrix $A$, then we have $f(\lambda) = \lambda^k - \lambda^{k-1} - \ldots - \lambda - 1$ which is a well-known fact. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the roots of equation $\lambda^k - \lambda^{k-1} - \ldots - \lambda - 1 = 0$. From [6, 7, 8], we know that $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct and $\lambda_i \neq 1$ for all $i$.

**Lemma 1.** *Then the characteristic equation of matrix $C$ is $x^{k+1} - 2x^k + 1 = 0$.*

*Proof.* If we compute the $|C - \lambda I|$ by the Laplace expansion of determinant with respect to the first row and by (1.5), we obtain $|C - \lambda I| = (1-\lambda)|A - \lambda I| = -\lambda^{k+1} + 2\lambda^k - 1$ which is as desired. □

By Lemma 1, (1.5), we have the following Corollary without proof.

**Corollary 1.** *Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the eigenvalues of $A$. Then the eigenvalues of $C$ are $\lambda_1, \lambda_2, \ldots, \lambda_k, 1$ and all of them are distinct.*

Let $V$ be the $k\times k$ Vandermonde matrix as follows:

$$V=\begin{bmatrix} \lambda_1^{k-1} & \lambda_2^{k-1} & \dots & \lambda_k^{k-1} \\ \lambda_1^{k-2} & \lambda_2^{k-2} & \dots & \lambda_k^{k-2} \\ \vdots & \vdots & & \vdots \\ \lambda_1 & \lambda_2 & \dots & \lambda_k \\ 1 & 1 & \dots & 1 \end{bmatrix}.$$

Then we have [4]:

$$G_nV=VD^n$$

where $\lambda_1,\lambda_2,\dots,\lambda_k$ are the eigenvalues of the matrix $A$, $D$=$diag(\lambda_1,\lambda_2,\dots,\lambda_k)$ and $G_n$ given by (1.2).

Now we define the $(k+1)\times(k+1)$ matrix $\Lambda$ as follows:

$$\Lambda=\begin{bmatrix} 1 & 0 & \dots & 0 \\ \frac{-1}{k-1} & & & \\ \vdots & & V & \\ \frac{-1}{k-1} & & & \end{bmatrix} \tag{2.1}$$

where the $k\times k$ Vandermonde matrix $V$ is as before.

**Lemma 2.** *Let the matrix $\Lambda$ have the form (2.1). Then the matrix $\Lambda$ is an invertible matrix.*

*Proof.* If we compute the $\det\Lambda$ by the Laplace expansion of determinant with respect to the first row, then it is readily seen that the values of $\det V$ and $\det\Lambda$ are the same. Since $V$ is the Vandermonde matrix and $\lambda_1,\lambda_2,\dots,\lambda_k$'s are different, $\det V\neq 0$ and so $\det\Lambda\neq 0$ which is as desired. □

By a simple calculation, we have the following Lemma without proof.

**Lemma 3.** *Let the matrices $\Lambda$ and $C$ have the form (2.1) and (1.5), respectively. Then*

$$C\Lambda=\Lambda D$$

*where* $D=diag\,(1,\lambda_1,\lambda_2,\dots,\lambda_k)$.

**Theorem 1.** *Then for $n>k\geq 2$*

$$S_n=\sum_{j=1}^{n}g_j^k=\left(g_n^1+g_n^2+\dots+g_n^k-1\right)/\left(k-1\right).$$

*Proof.* From Lemma 3, we have $C\Lambda=\Lambda D$, where $D=diag\,(1,\lambda_1,\lambda_2,\dots,\lambda_k)$. By Lemma 2, we know $\Lambda$ is invertible, so we write that $\Lambda^{-1}C\Lambda=D$. Hence, $C$ is similar to $D$. Thus we obtain $C^n\Lambda=\Lambda D^n$. Since $C^n=U_n$, we write $U_n\Lambda=\Lambda D^n$. From a matrix multiplication by considering the first entry in the second row of the matrix products $U_n\Lambda=\Lambda D^n$, we have

$$S_n=\left(g_n^1+g_n^2+\dots+g_n^k-1\right)/\left(k-1\right)$$

So the proof is complete. □

When $k = 2$, $S_n$ is the sum of the usual Fibonacci numbers from 1 to $n$ and by Theorem 1, we have

$$S_n = g_n^1 + g_n^2 - 1.$$

Since $g_n^1 = g_{n+1}^2$ for all positive integer $n$ and $g_n^2 = F_n$, we write

$$S_n = \sum_{i=1}^{n} F_i = F_{n+1} + F_n - 1 = F_{n+2} - 1$$

which is the well-known fact from [1].

**Corollary 2.** *Let $t_n$ be the nth Tribonacci number. Then for $n > 1$,*

$$\sum_{j=1}^{n} t_j = \left(t_{n+2} + t_n - 1\right)/2.$$

*Proof.* When $k = 3$, the generalized order-$k$ Fibonacci sequence $\left\{g_n^3\right\}$ is reduced to the Tribonacci sequence $\{t_n\}$ and let $S_n$ denote the sums of the Tribonacci numbers from 1 to $n$. Then by Theorem 1, we write $S_n = \left(g_n^1 + g_n^2 + g_n^3 - 1\right)/2$. By (1.3) and (1.4), we write $g_n^1 = g_{n+1}^3$ and $g_n^2 = g_n^3 + g_{n-1}^3$. Since $g_n^3 = t_n$, we may write

$$S_n = \left(g_{n+1}^3 + g_n^3 + g_{n-1}^3 + g_n^3 - 1\right)/2 = \left(t_{n+1} + 2t_n + t_{n-1} - 1\right)/2.$$

By the recurrence relation of Tribonacci numbers, $t_{n+2} = t_{n+1} + t_n + t_{n-1}$ and so we have the conclusion

$$S_n = \left(t_{n+2} + t_n - 1\right)/2.$$

□

**Corollary 3.** *Let $T_n$ be the nth Tetranacci number. Then for $n > 1$,*

$$\sum_{j=1}^{n} T_j = \left(T_{n+1} + 3T_n + 2T_{n-1} + T_{n-2} - 1\right)/3.$$

*Proof.* From Theorem 1, the Eqs. (1.3), (1.4) and the recurrence relation of Tetranacci numbers, the proof is readily seen. □

From Theorem 1, (1.3), (1.4) and considering the above Corollaries, we have the following Theorem without proof.

**Theorem 2.** *For $n > 1$,*

$$\sum_{j=1}^{n} g_j^k = \frac{g_{n+1}^k + (k-1)\,g_n^k + (k-2)\,g_{n-1}^k + \ldots + 2g_{n-k+3}^k + g_{n-k+2}^k - 1}{k-1}.$$

Using Theorem 2, we now show that the sums of the generalized order-$k$ Fibonacci numbers from 1 to $n$ can be expressed as a linear combination of $k$ terms of the sequence.

Since $g_{n+1}^k = g_n^k + g_{n-1}^k + \ldots + g_{n-k+1}^k$ and by Theorem 2, we have

$$\begin{aligned}\sum_{j=1}^{n} g_j^k &= \frac{kg_n^k + (k-1)\, g_{n-1}^k + \ldots + 3g_{n-k+3}^k + 2g_{n-k+2}^k + g_{n-k+1}^k - 1}{k-1} \\ &= \left( \sum_{j=1}^{k} (k+1-j)\, g_{n-j}^k - 1 \right) / (k-1).\end{aligned}$$

## 3. Formula for the sums of the generalized Pell numbers

In this section we give a formula for the sums of the generalized order-$k$ Pell numbers by matrix methods. From the Companion matrix, it is a well-known fact that the characteristic equation of the generalized order-$k$ Pell matrix $R$ is $g(x) = x^k - 2x^{k-1} - x^{k-2} - \ldots - x - 1 = 0$. Let $\mu_1, \mu_2, \ldots, \mu_k$ be the roots of the equation $g(x) = 0$. From [9], we know that the eigenvalues $\mu_1, \mu_2, \ldots, \mu_k$ are distinct and $\mu_i \neq 1$ for all $i$.

We consider the $k \times k$ Vandermonde matrix

$$\hat{V} = \begin{bmatrix} \mu_1^{k-1} & \mu_2^{k-1} & \cdots & \mu_k^{k-1} \\ \mu_1^{k-2} & \mu_2^{k-2} & \cdots & \mu_k^{k-2} \\ \vdots & \vdots & & \vdots \\ \mu_1 & \mu_2 & \cdots & \mu_k \\ 1 & 1 & \cdots & 1 \end{bmatrix}. \tag{3.1}$$

Then we have $E_n \hat{V} = \hat{V} \breve{D}^n$ where $\breve{D} = diag(\mu_1, \mu_2, \ldots, \mu_k)$ [9].

Now we extend the $k \times k$ Vandermonde matrix $\hat{V}$ to the $(k+1) \times (k+1)$ matrix $W$ as follows

$$W = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ -k^{-1} & & & \\ \vdots & & \hat{V} & \\ -k^{-1} & & & \end{bmatrix}. \tag{3.2}$$

If the $\det W$ is expanded to the first row, then it is seen that $\det W = \det \hat{V}$. Since $\mu_1, \mu_2, \ldots, \mu_k$ are distinct, the matrix $\hat{V}$ is invertible and so $\det W \neq 0$.

Expanding the characteristic equation of the matrix $T$, $|T - \lambda I| = 0$, to the first row, by the characteristic equation of the matrix $R$, we have the following Lemma without proof.

**Lemma 4.** *Then the characteristic equation of matrix $T$ is $h(x) = x^{k+1} - 3x^k - x^{k-1} - 1 = 0$.*

Note that $h(x) = x^{k+1} - 3x^k - x^{k-1} - 1 = (x-1)g(x)$ where $g(x)$ is the characteristic polynomial of the matrix $R$. Since the roots of $g(x)$ are distinct and all of them are different from 1, thus the roots of the $h(x)$ are distinct.

**Lemma 5.** *Let the matrix $W$ have the form (3.2). Then the matrix $W$ is invertible.*

*Proof.* By the Laplace expansion of determinant with respect to the first row, we see that $\det W = \det \hat{V}$ where the matrix $\hat{V}$ given by (3.1). Since $\det \hat{V} \neq 0$, the proof is complete. □

Now by a simple calculation, we give the following Lemma.

**Lemma 6.** *Let the matrices $W$ and $T$ have the forms (3.2) and (1.7), respectively. Then*

$$TW = WQ$$

*where $Q = diag(1, \mu_1, \mu_2, \ldots, \mu_k)$ is the diagonal matrix of order $k+1$.*

Then we have the following Theorem for the explicit formula for the sums of the generalized Pell numbers.

**Theorem 3.** *Let $S_n^P$ denote the sums of the generalized Pell numbers from 1 to $n$. Then*

$$S_n^P = \sum_{j=1}^{n} P_j^k = \left(P_n^1 + P_n^2 + \ldots + P_n^k - 1\right)/k.$$

*Proof.* Since the matrix $W$ is invertible and by Lemma 6, we write $W^{-1}TW = Q$. Thus, $T$ is similar to $Q$. Then we obtain $T^nW = WQ^n$. Since $T^n = H_n$, we write $H_nW = WQ^n$. From a matrix multiplication by considering the first entry in the second row of the matrix products $T^nW = WQ^n$, we have

$$S_n^P = \textstyle\sum_{j=1}^{n} P_j^k = \left(P_n^1 + P_n^2 + \ldots + P_n^k - 1\right)/k.$$

So the theorem is proven. □

When $k = i = 2$, the generalized Pell sequence $\left\{P_n^i\right\}$ is reduced to the usual Pell sequence $\{P_n\}$. Since $P_n^1 = P_{n+1}^2$ for all $n$ and by Theorem 3, we obtain

$$\textstyle\sum_{j=1}^{n} P_j = (P_n^1 + P_n^2 - 1)/2 = (P_{n+1}^2 + P_n^2 - 1)/2$$

which is well-known result from [3].

Since $E_{n+1} = E_nE_1 = E_1E_n$, the matrix $E_1$ is commutative under matrix multiplication where $E_n$ is given by (1.6), we have

$$P_n^i = P_{n-1}^1 + P_{n-1}^{i+1} \text{ for } 2 \leq i \leq k, \tag{3.3}$$

$$P_n^1 = P_n^k. \tag{3.4}$$

Thus as an analogue of the Theorem 2, by Theorem 3, (3.3) and (3.4), for generalized order-$k$ Pell numbers, we have the following result:

$$\sum_{j=1}^{n} P_j = \frac{kP_n^k + (k-1)\,P_{n-1}^k + \ldots + 3P_{n-k+3}^k + 2P_{n-k+2}^k + P_{n-k+1}^k - 1}{k}.$$

## 4. Conclusions

For common generalization of the generalized order-$k$ Fibonacci, Pell numbers and similar other sequences, define the sequence $\left\{a_n^i\right\}$ as follows: for $n > 0,\ 1 \le i \le k$ and fixed constant $\alpha$,

$$a_n^i = \alpha a_{n-1}^i + a_{n-1}^i + \ldots + a_{n-k}^i$$

with initial conditions

$$a_n^i = \begin{cases} 1 & n = 1-i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1-k \le n \le 0.$$

If the characteristic equation of the sequence $\left\{a_n^i\right\}$, that is, $x^{k+1} - \alpha x^k - x^{k-1} - \ldots x - 1 = 0$, does not have multiple roots, then by considering Theorems 2 and 3, one can obtain the following result:

$$\sum_{j=1}^{n} a_j = \frac{ka_n^k + (k-1)\,a_{n-1}^k + \ldots + 3a_{n-k+3}^k + 2a_{n-k+2}^k + a_{n-k+1}^k - 1}{k-2+\alpha}.$$

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