TRIBONACCI SEQUENCES WITH CERTAIN INDICES AND THEIR SUMS

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Abstract. In this paper, we derive new recurrence relations and generating matrices for the sums of usual Tribonacci numbers and $4n$ subscripted Tribonacci sequences, $\{T_{4n}\}$, and their sums. We obtain explicit formulas and combinatorial representations for the sums of terms of these sequences. Finally we represent relationships between these sequences and permanents of certain matrices.

1. Introduction

The *Tribonacci sequence* is defined by for $n > 1$

$$T_{n+1} = T_n + T_{n-1} + T_{n-2}$$

where $T_0 = 0, T_1 = 1, T_2 = 1$. The few first terms are

$$0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, \ldots.$$

We define $T_n = 0$ for all $n \leq 0$. The Tribonacci sequence is a well known generalization of the Fibonacci sequence. In (see page 527-536, [3]), one can find some known properties of Tribonacci numbers. For example, the generating matrix of $\{T_n\}$ is given by

$$Q^n = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & T_{n-2} + T_{n-3} & T_{n-2} \end{bmatrix}.$$

For further properties of Tribonacci numbers, we refer to [1, 4, 5].

Let

$$S_n = \sum_{k=0}^{n} T_k. \tag{1.1}$$

In this paper, we obtain generating matrices for the sequences $\{T_n\}$,$\{T_{4n}\}$, $\{S_n\}$ and $\{S_{4n}\}$. (The second result follows from a third order recurrence for T_{4n}.) We also obtain Binet-type explicit and closed-form formulas for S_n and S_{4n}. Further on, we present relationships between permanents of certain matrices and all the above-mentioned sequences.

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2. On the Tribonacci sequence $\{T_n\}$

In this section, we give two new generating matrices for Tribonacci numbers and their sums. Then we derive an explicit formula for the sums. Considering the matrix Q, define the 4×4 matrices A and B_n as shown:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } B_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ S_n & T_{n+1} & T_n + T_{n-1} & T_n \\ S_{n-1} & T_n & T_{n-1} + T_{n-2} & T_{n-1} \\ S_{n-2} & T_{n-1} & T_{n-2} + T_{n-3} & T_{n-2} \end{bmatrix}$$

where S_n is given by (1.1).

Lemma 1. *If $n \geq 3$, then $S_n = 1 + S_{n-1} + S_{n-2} + S_{n-3}$*

Proof. Induction on n. □

Theorem 1. *If $n \geq 3$, then $A^n = B_n$.*

Proof. Using Lemma 1 and direct computation, we have $B_n = AB_{n-1}$, from which it follows that $B_n = A^{n-3}B_3$. By direct computation, $B_3 = A^3$ from which the conclusion follows. □

By the definition of matrix B_n, we write $B_{n+m} = B_n B_m = B_m B_n$ for all $n, m \geq 3$. From a matrix multiplication, we have the following Corollary without proof.

Corollary 1. *For $n > 0$ and $m \geq 3$,*

$$S_{n+m} = S_n + T_{n+1} S_m + (T_n + T_{n-1}) S_{m-1} + T_n S_{m-2}.$$

The roots of characteristic equation of Tribonacci numbers, $x^3 - x^2 - x - 1 = 0$, are

$$\begin{aligned} \alpha &= \left(1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}\right)/3, \\ \beta &= \left(1 + \omega \sqrt[3]{19 + 3\sqrt{33}} + \omega^2 \sqrt[3]{19 - 3\sqrt{33}}\right)/3, \\ \gamma &= \left(1 + \omega^2 \sqrt[3]{19 + 3\sqrt{33}} + \omega \sqrt[3]{19 - 3\sqrt{33}}\right)/3 \end{aligned}$$

where $\omega = \left(1 + i\sqrt{3}\right)/2$ is the primitive cube root of unity.

The Binet formula of Tribonacci sequence is given by

$$T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}.$$

Computing the eigenvalues of matrix A, we obtain α, β, γ,1.

Define the diagonal matrix D and the matrix V as shown, respectively:

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \gamma \end{bmatrix} \text{ and } V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & \alpha^2 & \beta^2 & \gamma^2 \\ -1/2 & \alpha & \beta & \gamma \\ -1/2 & 1 & 1 & 1 \end{bmatrix}.$$

One can check that $AV = VD$. Since the roots α, β, γ are distinct, it follows that $\det V \neq 0$.

Theorem 2. *If $n > 0$, then $S_n = (T_{n+2} + T_n - 1)/2$.*

Proof. Since $AV = VD$ and $\det V \neq 0$, we write $V^{-1}AV = D$. Thus the matrix A is similar to the matrix D. Then $A^nV = VD^n$. By Theorem 1, we write $B_nV = VD^n$. Equating the $(2,1)$th elements of the equation and since $T_{n+1} + 2T_n + T_{n-1} = T_{n+2} + T_n$, the theorem is proven. □

Define the 4×4 matrices R and K as shown:

$$R = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, K_n = \begin{bmatrix} S_{n+1} & -S_{n-2} & -S_{n-1} & -S_n \\ S_n & -S_{n-3} & -S_{n-2} & -S_{n-1} \\ S_{n-1} & -S_{n-4} & -S_{n-3} & -S_{n-2} \\ S_{n-2} & -S_{n-5} & -S_{n-4} & -S_{n-3} \end{bmatrix}$$

where S_n is given by (1.1).

Theorem 3. *If $n > 4$, then $R^n = K_n$.*

Proof. Considering $2S_{n+1} - S_{n-2} = S_{n+1} + S_{n+1} - S_{n-2} = S_{n+1} + T_{n+1} + T_n + T_{n-1} = S_{n+2}$, we write $K_n = RK_{n-1}$. By a simple inductive argument, we write $K_n = R^{n-1}K_1$. By the definitions of matrices R and K_n, one can see that $K_1 = R$ and so we have the conclusion, $K_n = R^n$. □

Then the characteristic equations of matrix R and sequence $\{S_n\}$ is $x^4 - 2x^3 + 1 = 0$. Computing the roots of the equation, we obtain α, β, γ, 1.

Corollary 2. *The sequence $\{S_n\}$ satisfies the following recursion, for $n > 3$*

$$S_n = 2S_{n-1} - S_{n-4}$$

where $S_0 = 0$, $S_1 = 1$, $S_2 = 2$, $S_3 = 4$.

Define the Vandermonde matrix V_1 and diagonal matrix D_1 as follows:

$$V_1 = \begin{bmatrix} \alpha^3 & \beta^3 & \gamma^3 & 1 \\ \alpha^2 & \beta^2 & \gamma^2 & 1 \\ \alpha & \beta & \gamma & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ and } D_1 = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let w_i be a 4×1 matrix such that $w_i = \begin{bmatrix} \alpha^{n-i+4} & \beta^{n-i+4} & \gamma^{n-i+4} & 1 \end{bmatrix}^T$ and $V_j^{(i)}$ be a 4×4 matrix obtained from V_1 by replacing the jth column of V_1^T by w_i.

Theorem 4. *For $n > 4$, $k_{ij} = \det\left(V_j^{(i)}\right) / \det(V_1)$ where $K_n = [k_{ij}]$.*

Proof. One can see that $RV_1 = V_1 D_1$. Since $\alpha, \beta, \gamma, 1$ are different and V_1 is a Vandermonde matrix, V_1 is invertible. Thus we write $V_1^{-1} R V_1 = D_1$ and so $R^n V_1 = V_1 D_1^n$. By Theorem 3, $K_n V_1 = V_1 D_1^n$. Thus we have the following equations system:

$$\begin{aligned} \alpha^3 k_{i1} + \alpha^2 k_{i2} + \alpha k_{i3} + k_{i4} &= \alpha^{n-i+4} \\ \beta^3 k_{i1} + \beta^2 k_{i2} + \beta k_{i3} + k_{i4} &= \beta^{n-i+4} \\ \gamma^3 k_{i1} + \gamma^2 k_{i2} + \gamma k_{i3} + k_{i4} &= \gamma^{n-i+4} \\ k_{i1} + k_{i2} + k_{i3} + k_{i4} &= 1 \end{aligned}$$

where $K_n = [k_{ij}]$. By Cramer solution of the above system, the proof is seen. $\square$

Corollary 3. *Then for $n > 0$,*

$$S_n = \frac{\alpha^{n+2}}{(\alpha-1)(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+2}}{(\beta-1)(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+2}}{(\gamma-1)(\gamma-\alpha)(\gamma-\beta)}.$$

Proof. Taking $i = 2,\ j = 1$ in Theorem 4, $k_{21} = S_n$. Computing $\det V_1$ and $\det\left(V_1^{(2)}\right)$, we obtain $\det V_1 = (\alpha - 1)(\beta - 1)(\gamma - 1)(\alpha - \beta)(\alpha - \gamma)(\beta - \gamma)$ and $\det\left(V_1^{(2)}\right) = \alpha^{n+2}(\beta - \gamma)(1-(\beta+\gamma))+\beta\gamma-\beta^{n+2}(\alpha-\gamma)(1-(\alpha+\gamma)+\alpha\gamma) + \gamma^{n+2}(\alpha - \beta)(1 - (\alpha + \beta) + \alpha\beta)$, respectively. So the proof is complete. $\square$

From Corollary 3 and Theorem 2, we give the following result: For $n > 0$

$$\frac{T_{n+2}+T_n-1}{2} = \frac{\alpha^{n+2}}{(\alpha-1)(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+2}}{(\beta-1)(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+2}}{(\gamma-1)(\gamma-\alpha)(\gamma-\beta)}.$$

3. On the Tribonacci sequence $\{T_{4n}\}$

In this section, we consider the $4n$ subscripted Tribonacci numbers. First we define a new third-order linear recurrence relation for the $4n$ subscripted Tribonacci numbers. Then we give a new generating matrix for these terms, T_{4n}. We obtain new formulas for the sequence $\{T_{4n}\}$.

Lemma 2. *For $n > 1$,*

$$T_{4(n+1)} = 11T_{4n} + 5T_{4(n-1)} + T_{4(n-2)}$$

where $T_0 = 0,\ T_4 = 4,\ T_8 = 44$.

Proof. (Induction on n) . If $n=2$, then $11T_8+5T_4+T_0=11(44)+5(4)+0=504=T_{12}$. Suppose that the claim is true for $n>2$. Then we show that the claim is true for $n+1$. By the definition of $\{T_n\}$, we write

$$\begin{aligned} & 11T_{4(n+1)}+5T_{4n}+T_{4(n-1)} \\ = \; & 22T_{4n+2}+11T_{4n+1}+26T_{4n}+13T_{4n-1}+11T_{4n-2} \\ = \; & 44T_{4n+1}+37T_{4n}+24T_{4n-1} \\ = \; & T_{4n+8}. \end{aligned}$$

Thus the proof is complete. □

Define the 4×4 matrices F and G_n defined by

$$F=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 11 & 5 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; G_n=\frac{1}{T_4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ s_n & T_{4n+4} & 5T_{4n}+T_{4n-4} & T_{4n} \\ s_{n-1} & T_{4n} & 5T_{4n-4}+T_{4n-8} & T_{4n-4} \\ s_{n-2} & T_{4n-4} & 5T_{4n-8}+T_{4n-12} & T_{4n-8} \end{bmatrix}$$

where s_n is given by

$$s_n=\textstyle\sum_{k=0}^{n} T_{4k}. \tag{3.1}$$

Since $s_n=T_{4n}+s_{n-1}$ and considering Lemma 1, we have the following Corollary without proof.

Corollary 4. *If $n>0$, then $F^n=G_n$.*

After some computations, the eigenvalues of matrix F are $\alpha^4,\ \beta^4,\gamma^4$ and 1.

Define the matrices Λ and D_2 as shown:

$$\Lambda=\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/16 & \alpha^8 & \beta^8 & \gamma^8 \\ -1/16 & \alpha^4 & \beta^4 & \gamma^4 \\ -1/16 & 1 & 1 & 1 \end{bmatrix} \text{ and } D_2=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha^4 & 0 & 0 \\ 0 & 0 & \beta^4 & 0 \\ 0 & 0 & 0 & \gamma^4 \end{bmatrix}.$$

Theorem 5. *If $n>0$, then $s_n=\left(T_{4n+4}+6T_{4n}+T_{4n-4}-T_4\right)/T_4^2$.*

Proof. Since α,β and γ are different, and extending to the first row, we obtain $\det\Lambda\neq 0$. One can check that $F\Lambda=\Lambda D_2$ so that $F^n\Lambda=\Lambda D_2^n$. By Corollary 4, $G_n\Lambda=\Lambda D_2^n$. Equating the (2.1) elements of this matrix equation, the theorem is proven. □

In the above, we give the generating matrix for both the terms of $\{T_{4n}\}$ and their sums. Now we give a new matrix to generate only the sums.

Define the 4×4 matrices L and P as shown:

$$L=\begin{bmatrix} 12 & -6 & -4 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$P_n = \frac{1}{T_4}\left[\begin{array}{cccc} s_{n+1} & -(6s_n+4s_{n-1}+s_{n-2}) & -(4s_n+s_{n-1}) & -s_n \\ s_n & -(6s_{n-1}+4s_{n-2}+s_{n-3}) & -(4s_{n-1}+s_{n-2}) & -s_{n-1} \\ s_{n-1} & -(6s_{n-2}+4s_{n-3}+s_{n-4}) & -(4s_{n-2}+s_{n-3}) & -s_{n-2} \\ s_{n-2} & -(6s_{n-3}+4s_{n-4}+s_{n-5}) & -(4s_{n-3}+s_{n-4}) & -s_{n-3} \end{array}\right]$$

where s_n given by (3.1).

Theorem 6. *If $n > 4$, then $L^n = P_n$.*

Proof. The proof follows from the induction method. □

The characteristic equation of matrix L is $x^4 - 12x^3 + 6x^2 + 4x + 1 = 0$. Computing the roots of the equation, we obtain α, β, γ and 1. Define the 4×4 Vandermonde matrix Λ_1 and diagonal matrix D_3 as shown, respectively:

$$\Lambda_1 = \left[\begin{array}{cccc} \alpha^{12} & \beta^{12} & \gamma^{12} & 1 \\ \alpha^8 & \beta^8 & \gamma^8 & 1 \\ \alpha^4 & \beta^4 & \gamma^4 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right] \text{ and } D_3 = \left[\begin{array}{cccc} \alpha^4 & 0 & 0 & 0 \\ 0 & \beta^4 & 0 & 0 \\ 0 & 0 & \gamma^4 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Since α, β, γ,1 are different and Λ_1 is a Vandermonde matrix, $\det \Lambda_1 \neq 0$.

Theorem 7. *Then for $n > 4$,*

$$s_n = T_4\left(\tfrac{\alpha^{4n+8}}{(\alpha^4-1)(\alpha^4-\beta^4)(\alpha^4-\gamma^4)} + \tfrac{\beta^{4n+8}}{(\beta^4-1)(\beta^4-\alpha^4)(\beta^4-\gamma^4)} + \tfrac{\gamma^{4n+8}}{(\gamma^4-1)(\alpha^4-\gamma^4)(\beta^4-\gamma^4)}\right).$$

Proof. It can be shown that $L\Lambda_1 = \Lambda_1 D_3$. Since $\det \Lambda_1 \neq 0$, the matrix Λ_1 is invertible. Thus we write $\Lambda_1^{-1} L \Lambda_1 = D_3$ so that $L^n \Lambda_1 = \Lambda_1 D_3^n$. From Theorem 6, we know $L^n = P_n$. Thus $P_n \Lambda_1 = \Lambda_1 D_3^n$. Clearly we have the following linear equations system:

$$\begin{aligned} \alpha^{12} p_{i1} + \alpha^8 p_{i2} + \alpha^4 p_{i3} + p_{i4} &= \alpha^{4(n-i)+16} \\ \beta^{12} p_{i1} + \beta^8 p_{i2} + \beta^4 p_{i3} + p_{i4} &= \beta^{4(n-i)+16} \\ \gamma^{12} p_{i1} + \gamma^8 p_{i2} + \gamma^4 p_{i3} + p_{i4} &= \gamma^{4(n-i)+16} \\ p_{i1} + p_{i2} + p_{i3} + p_{i4} &= 1 \end{aligned}$$

where $P_n = [p_{ij}]$. Let u_i be a 4×1 matrix as follows: $u_i = \left[\begin{array}{cccc} \alpha^{4(n-i)+16} & \beta^{4(n-i)+16} & \gamma^{4(n-i)+16} & 1 \end{array}\right]^T$ and $\Lambda_{1,j}^{(i)}$ be a 4×4 matrix obtained from Λ_1 by replacing the jth column of Λ_1^T by u_i. By Cramer solution of the above system and since $p_{21} = s_n / T_4$,

$$p_{ij} = \det\left(\Lambda_{1,j}^{(i)}\right) / \det(\Lambda_1) \text{ and so } s_n = T_4 \det\left(\Lambda_{1,1}^{(2)}\right) / \det(\Lambda_1).$$

Also we obtain

$$\begin{aligned} \det\left(\Lambda_{1,1}^{(2)}\right) &= \alpha^{4n+8}\left(\beta^4-1\right)\left(\gamma^4-1\right)\left(\beta^4-\gamma^4\right) - \beta^{4n+8}\left(\alpha^4-1\right) \times \\ &\quad \left(\gamma^4-1\right)\left(\alpha^4-\gamma^4\right) + \gamma^{4n+8}\left(\alpha^4-1\right)\left(\beta^4-1\right)\left(\alpha^4-\beta^4\right) \end{aligned}$$

and

$$\det(\Lambda_1) = \left(\alpha^4 - 1\right)\left(\beta^4 - 1\right)\left(\gamma^4 - 1\right)\left(\alpha^4 - \beta^4\right)\left(\alpha^4 - \gamma^4\right)\left(\beta^4 - \gamma^4\right).$$

Thus the proof is easily seen. □

Corollary 5. *For $n > 3$, the sequence $\{s_n\}$ satisfies the following recursion*

$$s_n = 12s_{n-1} - 6s_{n-2} - 4s_{n-3} - s_{n-4}$$

where $s_0 = 0, s_1 = 4, s_2 = 48, s_3 = 552, s_4 = 6320.$

Since the recurrence relations of sequence $\{T_{4n}\}$ and their sums, we can give generating functions of them :

Let $G(x) = T_0 + T_4x + T_8x^2 + T_{12}x^3 + \ldots + T_{4n}x^n + \ldots$. Then

$$G(x) = \sum_{n=0}^{\infty} T_{4n}x^n = \tfrac{4x}{1-11x-5x^2-x^3}.$$

Let $W(x) = s_1x + s_2x^2 + s_3x^3 + \ldots + s_nx^n + \ldots$, where s_n is as before. Then

$$W(x) = \sum_{n=0}^{\infty} s_nx^n = \tfrac{4x}{1-12x+6x^2+4x^3+x^4}.$$

4. Determinantal Representations

In this section, we give relationships between the sequence $\{T_{4n}\}$, its sums and the permanents of certain matrices. In [6], Minc derived an interesting relation including the permanent of $(0,1)$-matrix $F(n,k)$ of order n and the generalized order-k Fibonacci numbers. According to the Minc's result, for $k = 3$, the $n \times n$ matrix $F(n,3)$ takes the following form

$$F(n,3) = \begin{bmatrix} 1 & 1 & 1 & & 0 \\ 1 & 1 & 1 & \ddots & \\ & 1 & \ddots & \ddots & 1 \\ & & \ddots & 1 & 1 \\ 0 & & & 1 & 1 \end{bmatrix},$$

then $perF(n,3) = T_{n+1}$ where T_n is the nth Tribonacci number.

For $n > 1$, define the $n \times n$ matrix $M_n = [m_{ij}]$ with $m_{4j} = m_{ii} = 1$ for all i, $m_{i+1,i} = m_{i,i+1} = 1$ for $1 \le i \le n-1$, $m_{i,i+2} = 1$ for $1 \le i \le n-2$ and 0 otherwise.

Theorem 8. *If $n > 1$, then $perM_n = \sum_{i=0}^{n} T_i$.*

Proof. (Induction on n) If $n = 2$, then per$M_2 = T_1 + T_2 = 2$. Suppose that the equation holds for n. Then we show that the equation holds for $n+1$. By the definitions of matrices $F(3,n)$ and M_n, expanding the perM_{n+1} with respect to the first column gives us perM_{n+1} =per$F(3,n)$ +perM_n.By our assumption and the result of Minc, per$M_{n+1} = T_{n+1} + \sum_{i=0}^{n} T_i = \sum_{i=0}^{n+1} T_i$. Thus the proof is complete. □

Define the $n \times n$ matrix $U_n = [u_{ij}]$ with $u_{ii} = 2$ for $1 \le i \le n,\ u_{i+1,i} = 1$ for $1 \le i \le n-1,\ u_{i,i+3} = -1$ for $1 \le i \le n-3$ and 0 otherwise.

Theorem 9. *Then for $n > 4$,*

$$perU_n = S_{n+1}$$

where S_n is as before and $perU_1 = 2$, $perU_2 = 4$, $perU_3 = 8$, $perU_4 = 15$

Proof. Expanding the perU_n according to the last column four times, we obtain

$$perU_n = 2perU_{n-1} - perU_{n-4}. \tag{4.1}$$

Since per$U_1 = S_2 = \sum_{i=0}^{2} T_i$, per$U_2 = S_3 = \sum_{i=0}^{3} T_i$, per$U_3 = S_4 = \sum_{i=0}^{4} T_i$, per$U_4 = S_5 = \sum_{i=0}^{5} T_i$, then, by Corollary 2, the recurrence relation in (4.1) generate the sums of Tribonacci numbers. Thus we have the conclusion. □

Now we derive a similar relation for terms of sequence $\{T_{4n}\}$. Define the $n \times n$ matrix $H_n = [h_{ij}]$ with $h_{ii} = 11$ for $1 \le i \le n,\ h_{i,i+1} = 5$ for $1 \le i \le n-1,\ h_{i,i+2} = 1$ for $1 \le i \le n-2,\ h_{i+1,i}$ for $1 \le i \le n-1$ and 0 otherwise.

Theorem 10. *Then for $n > 1$*

$$perH_n = T_{4(n+1)}/T_4$$

where $perH_1 = T_8/T_4$.

Proof. Expanding the perT_{n+1}according to the last column, by our assumption and the definition of H_n, we obtain

$$\mathrm{per}H_{n+1} = 11\mathrm{per}H_n + 5\mathrm{per}H_{n-1} + \mathrm{per}H_{n-2}. \tag{4.2}$$

Since per$H_1 = T_8/T_4$, per$H_2 = T_{12}/T_4$ and per$H_3 = T_{16}/T_4$, by Lemma 2, the recurrence relation in (4.2) generates the $T_{4(n+1)}/T_4$. The theorem is proven. □

For $n > 1$, we define the $n \times n$ matrix Z_n as in the compact form, by the definition of H_n,

$$Z_n = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & & & \\ 0 & & H_{n-1} & \\ \vdots & & & \\ 0 & & & \end{bmatrix}.$$

Theorem 11. *If $n > 1$, then $perZ_n = \left(\sum_{i=1}^{n} T_{4i}\right)/T_4$.*

Proof. (Induction on n) If $n = 2$, then per$Z_2 = \left(\sum_{i=1}^{2} T_{4i}\right)/T_4 = 12$. Suppose that the equation holds for n. We show that the equation holds for $n+1$. Thus, by the definitions of H_n and Z_n, expanding perZ_{n+1} according to the first column gives us perZ_{n+1} =perZ_n+perH_n. By our assumption and Theorem 10, we have the conclusion. □

Finally, define the 4×4 matrix $V_n = [v_{ij}]$ with $v_{ii} = 12$ for $1 \le i \le n$, $v_{i,i+1} = -6$ for $1 \le i \le n-1$, $v_{i,i+2} = -4$ for $1 \le i \le n-2$, $v_{i,i+3} = -1$ for $1 \le i \le n-3$, $v_{i+1,i} = 1$ for $1 \le i \le n-1$ and 0 otherwise.

Theorem 12. *Then for $n > 1$,*

$$perY_n = s_n/T_4.$$

where $perY_1 = s_2/T_4$, $perY_2 = s_3/T_4$,$perY_3 = s_4/T_4$, $perY_4 = s_4/T_4$.

Proof. Expanding the perY_n according to the last column gives

$$\text{per}Y_n = 12\text{per}Y_{n-1} - 6\text{per}Y_{n-2} - 4\text{per}Y_{n-3} - \text{per}Y_{n-4}. \tag{4.3}$$

Since per$Y_1 = s_2/T_4 = 12$, per$Y_2 = s_3/T_4 = 138$, per$Y_3 = s_4/T_4 = 1580$, per$Y_4 = s_4/T_4 = 18083$ and by Corollary 5, the recurrence relation in (4.3) generate the terms of sequence $\{s_n\}$. Thus the proof is complete. □

5. Combinatorial Representations

In this section, we consider the result of Chen about the nth power of a companion matrix, we give some combinatorial representations.

Let A_k be a $k \times k$ companion matrix as follows:

$$A_k(c_1, c_2, \ldots, c_k) = \begin{bmatrix} c_1 & c_2 & \ldots & c_k \\ 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then one can find the following result in [2]:

Theorem 13. *The (i,j) entry $a_{ij}^{(n)}(c_1, c_2, \ldots, c_k)$ in the matrix $A_k^n(c_1,c_2,\ldots,c_k)$ is given by the following formula:*

$$a_{ij}^{(n)}(c_1, c_2, \ldots, c_k) = \sum_{(t_1,t_2,\ldots t_k)} \tfrac{t_j+t_{j+1}+\ldots+t_k}{t_1+t_2+\ldots+t_k} \times \tbinom{t_1+t_2+\ldots+t_k}{t_1,t_2,\ldots,t_k} c_1^{t_1} \ldots c_k^{t_k} \tag{5.1}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \ldots + kt_k = n - i + j$, and the coefficients in (5.1) is defined to be 1 if $n = i - j$.

Corollary 6. *Let S_n be the sums of Tribonacci numbers. Then*

$$S_n = \sum_{(r_1,r_2,r_3,r_4)} \tbinom{r_1+r_2+r_3+r_4}{r_1,r_2,r_3,r_4} 2^{r_1} (-1)^{r_4}$$

where the summation is over nonnegative integers satisfying $r_1+2r_2+3r_3+4r_4 = n-1$.

Proof. In Theorem 13, if $j = 1, i = 2, c_1 = 2, c_2 = c_3 = 0$ and $c_4 = -1$, the proof follows from Theorem 3 by considering the matrices R and K_n. $\square$

Corollary 7. *Let T_n be the nth Tribonacci number. Then*

$$T_{4n} = \sum_{(t_1,t_2,t_3)} \binom{t_1+t_2+t_3}{t_1,t_2,t_3} 11^{t_1} 5^{t_2}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + 3t_3 = n - 1$.

Proof. When $j = 1, i = 2, c_1 = 11, c_2 = 4, c_3 = 1$ in Theorem 13, the proof follows from Corollary 4 by ignoring the first columns and rows of matrices F and G_n. $\square$

Corollary 8. *Let s_n be as before. Then*

$$s_n = \sum_{(r_1,r_2,r_3,r_4)} \binom{r_1+r_2+r_3+r_4}{r_1,r_2,r_3,r_4} 12^{r_1} 6^{r_2} 4^{r_3} (-1)^{r_2+r_3+r_4}$$

where the summation is over nonnegative integers satisfying $r_1 + 2r_2 + 3r_3 + 4r_4 = n - 1$.

Proof. When $j = 1, i = 2, c_1 = 12, c_2 = -6, c_3 = -4, c_4 = -1$ in Theorem 13, the proof follows from Theorem 6 by considering the matrices L and P_n. $\square$

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