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On a constant-diagonals matrix

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ABSTRACT

In this paper, we consider a constant-diagonals matrix. The matrix was discussed in Wituła and Słota [R. Wituła, D. Słota, On computing the determinants and inverses of some special type of tridiagonal and constant-diagonals matrices, Appl. Math. Comput. 189 (1) (2007) 514–527]. The authors gave some results on determinant and the inverse of the matrix for some special cases. We give LU factorization and then compute determinant of the matrix. We determine eigenvalues of the matrix. Further we obtain some relationships between permanent of the matrix and terms of a certain recurrence relation.

1. Introduction

A Toeplitz matrix or diagonal-constant matrix is a matrix in which each descending diagonal from left to right is constant. For $r \leqslant n$, a $(n \times n)$ r-Toeplitz matrix is determined by given $2r-1$ numbers $\alpha_i \in \mathbb{C},\ i=-r+1,\ldots,-1,0,1,\ldots,r-1$. Those numbers are then placed as matrix elements constant along the (upper-left to lower-right) diagonals of the matrix:

$$A=[a_{ij}]=\begin{bmatrix} \alpha_0 & \alpha_{-1} & \cdots & \alpha_{-(r-1)} & & 0 \\ \alpha_1 & \alpha_0 & \alpha_{-1} & \ddots & \ddots & \\ \vdots & \alpha_1 & \alpha_0 & \ddots & \ddots & \alpha_{-(r-1)} \\ \alpha_{r-1} & \ddots & \ddots & \ddots & \alpha_{-1} & \vdots \\ & \ddots & \ddots & \alpha_1 & \alpha_0 & \alpha_{-1} \\ 0 & & \alpha_{r-1} & \cdots & \alpha_1 & \alpha_0 \end{bmatrix}.$$

The matrix A is reduced to a tridiagonal Toeplitz matrix by taking $r=2$. These types of matrices arise not only in different theoretical fields (in linear algebra or numerical analysis), but also in applicative fields. For example, these matrices appear in time series analysis, in signal processing and in solving differential equations (see [3,4,19]). For properties of these matrices, we refer to [1,2,5–13,15–18,21]. In [22], the authors considered a constant-diagonals matrix and they gave many examples on determinant and the inverse of the matrix for some special cases.

In this paper, we consider the same matrix and compute determinant of the matrix by its LU factorization. We determine the eigenvalues of the matrix. Finally, some trigonometric representations for a recurrence relation are derived as applications of our results.

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2. A constant-diagonals matrix $H_n(a,b,c)$

We will compute determinant of the matrix. For this purpose, firstly we use elementary operations.

Definition 1. For $n>2$, let $H_n(a,b,c)=[h_{kj}]_{n\times n}$ denote the constant-diagonals matrix with $h_{ii}=a$ for $1\leqslant i\leqslant n$, $h_{i,i+2}=b$, $h_{i+2,i}=c$ for $1\leqslant i\leqslant n-2$ and 0 otherwise.

Clearly, the matrix $H_n(a,b,c)$ takes the form:

$$H_n(a,b,c)=\begin{bmatrix} a & 0 & b & 0 & & 0 \\ 0 & a & 0 & b & \ddots & \\ c & 0 & a & 0 & \ddots & 0 \\ 0 & c & 0 & a & \ddots & b \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & 0 & c & 0 & a \end{bmatrix}.$$

Let A and B be nonzero integers. Define the second order linear recurrence $\{u_n\}$ by for all $n\geqslant 2$:

$$u_n=Au_{n-1}-Bu_{n-2}, \tag{1}$$

where $u_0=0,\ u_1=1$.

Let $\{u_n\}$ be as in (1) by taking $A=a$ and $B=bc$.

Theorem 2. *For $n=2k,\ k\geqslant 1$,*

$$\det H_n(a,b,c)=u_{k+1}^2,$$

and for $n=2k+1,\ k\geqslant 1$,

$$\det H_n(a,b,c)=u_{k+1}u_{k+2}.$$

Proof. If we multiply the first row by $\frac{c}{a}=\frac{c}{u_2}$ and subtract this from the third row, then using the notation $a-\frac{bc}{u_2}=\frac{u_3}{u_2}$, we obtain:

$$\det H_n(a,b,c)=\begin{vmatrix} a & 0 & b & & & 0 \\ 0 & a & 0 & b & & \\ 0 & 0 & \frac{u_3}{u_2} & 0 & \ddots & \\ & c & 0 & a & \ddots & b \\ & & \ddots & \ddots & \ddots & 0 \\ 0 & & & c & 0 & a \end{vmatrix}.$$

If we multiply the second row by $\frac{c}{a}=\frac{c}{u_2}$ and subtract this from the fourth row, then using the notation $\frac{u_2}{u_1}=a$, we obtain:

$$\det H_n(a,b,c)=\begin{vmatrix} \frac{u_2}{u_1} & 0 & b & & & & 0 \\ 0 & \frac{u_2}{u_1} & 0 & b & & & \\ 0 & 0 & \frac{u_3}{u_2} & 0 & b & & \\ & 0 & 0 & \frac{u_3}{u_2} & 0 & \ddots & \\ & & c & 0 & a & \ddots & b \\ & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & & & c & 0 & a \end{vmatrix}.$$

If we multiply the third row by $\frac{cu_2}{u_3}$ and subtract this from the fifth row, then using the identity $a-\frac{u_2bc}{u_3}=\frac{u_4}{u_3}$, we obtain:

$$\det H_n(a,b,c)=\begin{vmatrix} \frac{u_2}{u_1} & 0 & b & & & & & 0 \\ 0 & \frac{u_2}{u_1} & 0 & b & & & & \\ 0 & 0 & \frac{u_3}{u_2} & 0 & b & & & \\ & 0 & 0 & \frac{u_3}{u_2} & 0 & b & & \\ & & 0 & 0 & \frac{u_4}{u_3} & 0 & \ddots & \\ & & & c & 0 & a & \ddots & b \\ & & & & \ddots & & \ddots & 0 \\ 0 & & & & & c & 0 & a \end{vmatrix}.$$

For $n=2k,\ k>1$, continuing this process, subtracting the $(n-3)$th row from the $(n-1)$th row and the $(n-2)$th row from the nth row multiplied by $\frac{cu_{k-1}}{u_k}$ for $n\geqslant 6$ and using by the identity $a-\frac{bcu_k}{u_{k+1}}=\frac{u_{k+2}}{u_{k+1}}$, we get:

$$\det H_{2k}(a,b,c)=\begin{vmatrix} \frac{u_2}{u_1} & 0 & b & & & & 0\\ & \frac{u_2}{u_1} & 0 & b & & & \\ & & \frac{u_3}{u_2} & 0 & \ddots & & \\ & & & \frac{u_3}{u_2} & \ddots & b & \\ & & & & \ddots & 0 & b\\ & & & & & \frac{u_{k+1}}{u_k} & 0\\ 0 & & & & & & \frac{u_{k+1}}{u_k} \end{vmatrix}.$$

Thus,

$$\det H_n(a,b,c)=\det H_{2k}(a,b,c)=\frac{u_2}{u_1}\frac{u_2}{u_1}\frac{u_3}{u_2}\frac{u_3}{u_2}\cdots\frac{u_k}{u_{k-1}}\frac{u_k}{u_{k-1}}\frac{u_{k+1}}{u_k}\frac{u_{k+1}}{u_k}=u_{k+1}^2.$$

In the rest of the proof, we consider the case $n=2k+1,\ k\geqslant 1$. In addition to the above processes, if we multiply the $(n-2)$th row by $\frac{cu_k}{u_{k+1}}$ and subtract this from the nth row, then

$$\det H_{2k+1}(a,b,c)=\begin{vmatrix} \frac{u_2}{u_1} & 0 & b & & & & 0\\ 0 & \frac{u_2}{u_1} & 0 & b & & & \\ 0 & 0 & \frac{u_3}{u_2} & 0 & \ddots & & \\ & 0 & 0 & \ddots & \ddots & b & \\ & & \ddots & \ddots & \frac{u_{k+1}}{u_k} & 0 & b\\ & & & 0 & 0 & \frac{u_{k+1}}{u_k} & 0\\ 0 & & & & c & 0 & a \end{vmatrix}=\begin{vmatrix} \frac{u_2}{u_1} & 0 & b & & & & 0\\ 0 & \frac{u_2}{u_1} & 0 & b & & & \\ 0 & 0 & \frac{u_3}{u_2} & 0 & \ddots & & \\ & 0 & 0 & \ddots & \ddots & b & \\ & & \ddots & \ddots & \frac{u_{k+1}}{u_k} & 0 & b\\ & & & 0 & 0 & \frac{u_{k+1}}{u_k} & 0\\ 0 & & & & 0 & 0 & \frac{u_{k+2}}{u_{k+1}} \end{vmatrix}.$$

Therefore, we have:

$$\det H_{2k+1}(a,b,c)=\frac{u_{k+2}}{u_{k+1}}\det H_{2k}(a,b,c)=u_{k+1}^2\frac{u_{k+2}}{u_{k+1}}=u_{k+1}u_{k+2}.$$

Thus, the proof is complete. $\square$

As an example, when $a=1,\ b=c=i$ in Theorem 2, where $i^2=-1$, then

$$\begin{vmatrix} 1 & 0 & i & & 0\\ 0 & 1 & 0 & \ddots & \\ i & 0 & 1 & \ddots & i\\ & \ddots & \ddots & \ddots & 0\\ 0 & & i & 0 & 1 \end{vmatrix}_{2n\times 2n}=F_{n+1}^2$$

where F_n is the nth Fibonacci number.

3. LU factorization of $H_n(a,b,c)$

In this section, we give the Doolittle type LU factorization of $H_n(a,b,c)$.

We define a $n\times n$ unit lower triangular matrix $L_0=[l_{ij}]$ with $l_{2k+1,2k-1}=l_{2k+2,2k}=\frac{cu_k}{u_{k+1}}$ for $1\leqslant k\leqslant\lfloor(n-1)/2\rfloor,\ l_{ii}=1$ for $1\leqslant i\leqslant n$ and 0 otherwise. Define the $n\times n$ upper triangular matrix $U_0=[q_{kj}]$ with $q_{2k-1,2k-1}=q_{2k,2k}=\frac{u_{k+1}}{u_k}$ for $1\leqslant k\leqslant\lfloor(n+1)/2\rfloor,\ q_{i,i+2}=b$ for $1\leqslant i\leqslant n-2$ and 0 otherwise. Clearly, the matrices L_0 and U_0 take the forms for $n=2k,\ k>1$,

$$L_0=\begin{bmatrix} 1 &&&&&&& \\ 0 & 1 &&&&&& \\ \frac{cu_1}{u_2} & 0 & \ddots &&&&& \\ & \frac{cu_1}{u_2} & \ddots & 1 &&&& \\ && \ddots & 0 & 1 &&& \\ &&& \frac{cu_{k-1}}{u_k} & 0 & 1 && \\ &&&& \frac{cu_{k-1}}{u_k} & 0 & 1 \end{bmatrix} \quad \text{and} \quad U_0=\begin{bmatrix} \frac{u_2}{u_1} & 0 & b &&&& \\ & \frac{u_2}{u_1} & 0 & b &&& \\ && \frac{u_3}{u_2} & 0 & \ddots && \\ &&& \ddots && b & \\ &&&& \frac{u_{k+1}}{u_k} & 0 & \\ &&&&& \frac{u_{k+1}}{u_k} & \end{bmatrix}. \tag{2}$$

where u_n is the nth term of $\{u_n\}$ with $A=a$ and $B=bc$.

Theorem 3. *The LU factorization of $H_n(a,b,c)$ has the form:*

$$H_n(a,b,c)=L_0U_0,$$

where L_0 and U_0 be as before.

Proof. By the definitions of L_0 and U_0, we have $l_{ij}=0$ for $j>i,\ l_{i+1,i}=0$ for $1\leqslant i\leqslant n-1$ and $l_{ij}=0$ for $i\geqslant j+3$, and, $q_{ij}=0$ for $j<i,\ q_{i,i+1}=0$ for $1\leqslant i\leqslant n-1,\ q_{ij}=0$ for $j\geqslant i+3$. First consider the case $i=j$. From matrix multiplication, we write:

$$h_{ii}=\sum_{k=1}^{n} l_{ik}q_{ki}=l_{ii}q_{ii}+l_{i,i-2}q_{i-2,i}.$$

If we suppose that $i=2t,\ t>0$, then we write the above equation as follows:

$$h_{2t,2t}=l_{2t,2t}q_{2t,2t}+l_{2t,2t-2}q_{2t-2,2t}=\frac{u_{t+1}+bcu_{t-1}}{u_t},$$

which, by the recurrence relation of $\{u_n\}$, satisfies

$$h_{2t,2t}=\frac{au_t-bcu_{t-1}+bcu_{t-1}}{u_t}=a.$$

Second, for $i=2t+1, t\geqslant 1$, we consider:

$$h_{2t+1,2t+1}=l_{2t+1,2t+1}q_{2t+1,2t+1}+l_{2t+1,2t-1}q_{2t-1,2t+1}=q_{2t+1,2t+1}+l_{2t+1,2t-1}q_{2t-1,2t+1}=\frac{u_{t+2}+bcu_t}{u_{t+1}}=\frac{au_{t+1}-bcu_t+bcu_t}{u_{t+1}}=a,$$

where $h_{11}=a$. Now consider the second case $j=i+2$. From matrix multiplication and the definitions of L_0 and U_0,

$$h_{i,i+2}=\sum_{k=1}^{n} l_{ik}q_{k,i+2}=l_{ii}q_{i,i+2}=q_{i,i+2}=b.$$

Finally, we consider the case $i=j+2$. Then, by $l_{i+2,i}=c(q_{ii})^{-1}$, we have:

$$h_{i+2,i}=\sum_{k=1}^{n} l_{i+2,k}q_{k,i}=l_{i+2,i}q_{ii}=c.$$

So the proof is complete. □

Thus, we can obtain the proof of Theorem 2 as a consequence of Theorem 3. Since $H_n(a,b,c)=L_0U_0$ where L_0 and U_0 given by (2). Thus, $\det H_n(a,b,c)=\det(L_0)\det(U_0)=\det(U_0)$ since L_0 is the unit lower triangular matrix. Then for $n=2k,\ k>1$,

$$\det H_n(a,b,c)=\det U_0=\frac{u_2}{u_1}\frac{u_2}{u_1}\cdots\frac{u_k}{u_{k-1}}\frac{u_k}{u_{k-1}}\frac{u_{k+1}}{u_k}\frac{u_{k+1}}{u_k}=u_{k+1}^2.$$

If $n=2k+1,\ k>1$, then we get:

$$\det H_n(a,b,c)=\det U_0=\frac{u_2}{u_1}\frac{u_2}{u_1}\cdots\frac{u_k}{u_{k-1}}\frac{u_k}{u_{k-1}}\frac{u_{k+1}}{u_k}\frac{u_{k+1}}{u_k}\frac{u_{k+2}}{u_{k+1}}=u_{k+1}u_{k+2}.$$

For example, let,

$$H_5(2,1,-1)=\begin{bmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ -1 & 0 & 2 & 0 & 1 \\ 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}.$$

According to the Theorem 3, the sequence $\{u_n\}$ takes the following form:

$$u_n = 2u_{n-1} + u_{n-2},$$

where $u_0 = 0,\ u_1 = 1$. Then the first few terms of $\{u_n\}$ are $u_2 = 2,\ u_3 = 5,\ u_4 = 12$. Then,

$$L_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & -\frac{2}{5} & 0 & 1 \end{bmatrix} \quad \text{and} \quad U_0 = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & \frac{5}{2} & 0 & 1 \\ 0 & 0 & 0 & \frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{12}{5} \end{bmatrix}.$$

Thus, $\det H_5(2, 1, -1) = 60$.

4. Eigenvalues of $H_n(a, b, c)$

In this section, we give an explicit form for the eigenvalues of $H_n(a, b, c)$. Recall that the sequence $\{u_n\}$ is defined by for $n \geqslant 2$;

$$u_n = Au_{n-1} - Bu_{n-2},$$

where $u_0 = 0,\ u_1 = 1$. Throughout this section, we consider the sequence $\{u_n\}$ by taking $A = a$ and $B = bc$.

We construct an $(n \times n)$ tridiagonal Toeplitz matrix T_n as follows:

$$T_n = \begin{bmatrix} a & \sqrt{bc} & & \\ \sqrt{bc} & a & \ddots & \\ & \ddots & \ddots & \sqrt{bc} \\ & & \sqrt{bc} & a \end{bmatrix}. \tag{3}$$

If we expand the $|T_n|$ with respect to first row by the Laplace expansion, then we get:

$$\det T_n = u_{n+1}. \tag{4}$$

Let D_n denote the characteristic polynomial of T_n, that is,

$$D_n = \det(T_n - \lambda I_n),$$

where I_n is the unit matrix of order n.

From [7,20,14], we have that the zeros of D_n are given by

$$\lambda_k = a - 2\sqrt{bc}\cos\frac{\pi k}{n+1}, \quad k = 1, 2, \ldots, n. \tag{5}$$

To find the eigenvalues of $H_k(a, b, c)$, we consider two cases. First we begin with the case where k is even number.

Theorem 4. *The matrix $H_{2n}(a, b, c)$ has n double eigenvalues with the form:*

$$\lambda_k = a - 2\sqrt{bc}\cos\frac{\pi k}{n+1}, \quad k = 1, 2, \ldots, n.$$

Proof. Denote the characteristic polynomial of matrix $H_{2n}(a, b, c)$ by C_n, that is, $C_n = \det(H_{2n}(a, b, c) - \lambda I_{2n})$. If we replace a with $a - \lambda$ in Theorem 2, then by considering (4) and Theorem 2, we easily get:

$$C_k = D_{k+1}^2.$$

From (5), we have the zeros of D_n. So we derive the zeros of C_n. Thus, the proof is complete. □

Second, we consider the eigenvalues of $H_{2n+1}(a, b, c)$ by the following theorem.

Theorem 5. *The eigenvalues of $H_{2n-1}(a, b, c)$ are given by*

$$\begin{cases} a - 2\sqrt{bc}\cos\frac{k\pi}{n+1} & \text{for } k = 1, 2, \ldots n, \\ a - 2\sqrt{bc}\cos\frac{k\pi}{n} & \text{for } k = 1, 2, \ldots, n-1. \end{cases}$$

Proof. Denote the characteristic polynomial of matrix $H_{2-1}(a, b, c)$ by E_n, that is, $E_n = \det(H_{2n-1}(a, b, c) - \lambda I_{2n-1})$. Replacing a with $a - \lambda$ in Theorem 2, we get by considering (4) and Theorem 2:

$$E_k = D_{k-1}D_k.$$

From (5), we have the zeros of D_n. Thus, the proof is easily seen. □

For example, the eigenvalues of $H_5(2,1,-1)$ are

$$2-i\sqrt{2},\quad 2,\quad i\sqrt{2}+2,\quad 2-i\quad \text{and}\quad 2+i,$$

where $i=\sqrt{-1}$.

5. Some special cases

Now we give some special results about determinant and permanent of the matrix $H_n(a,b,c)$.

Definition 6. A matrix A is called convertible if there is an $n\times n$ $(1,-1)$-matrix C such that $\operatorname{per}A=\det(A\circ C)$, where $A\circ C$ denotes the Hadamard product of A and C. Such a matrix C is called a converter of A.

Let S be a $(1,-1)$-matrix of order n, defined by

$$S=\begin{bmatrix} 1 & 1 & -1 & 1 & \dots & 1 \\ 1 & 1 & 1 & -1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 1 & 1 & \dots & 1 & -1 \\ 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}.$$

Let $\{u_n\}$ be as in (1) with $A=a$ and $B=bc$.

Corollary 7. *For $n=2k,\ k>1$,*

$$\operatorname{per}H_n(a,b,-c)=\operatorname{per}H_n(a,-b,c)=u_{k+1}^2,$$

and for $n=2k+1,\ k>1$,

$$\operatorname{per}H_n(a,b,-c)=\operatorname{per}H_n(a,-b,c)=u_{k+1}u_{k+2}.$$

Proof. Since the matrices S and S^{T} are the converter of $H_n(a,b,c)$, the proof is readily seen. □

Thus, we give a relationship between the determinants of $H_n(a,b,c)$ and $H_n(-a,b,c)$.

Theorem 8. *Let the matrix $H_n(a,b,c)$ be as in Definition 1. Then for $n>1$,*

$$\det H_n(-a,b,c)=(-1)^n\det H_n(a,b,c).$$

We obtain a result involving the negatively and positively subscripted Fibonacci numbers as follows:

$$\det H_n(-1,i,i)=\begin{vmatrix} -1 & 0 & i & & 0 \\ 0 & -1 & 0 & \ddots & \\ i & 0 & -1 & \ddots & i \\ & \ddots & \ddots & \ddots & 0 \\ 0 & & i & 0 & -1 \end{vmatrix}=\begin{cases} F_{-(k+1)}^2 & \text{if } n=2k, \\ F_{-(k+1)}F_{-(k+2)} & \text{if } n=2k+1, \end{cases}$$

where $i=\sqrt{-1}$.

We have that $\det H_{2n}(a,b,c)=u_{n+1}^2$ and by the eigenvalues of $H_n(a,b,c)$, we can obtain:

$$u_{n+1}=\prod_{k=1}^{n}\left[a-2\sqrt{bc}\cos(k\pi/n+1)\right].$$

Similarly, considering $\det H_{2n+1}=u_{n+1}u_{n+2}$, we have:

$$F_{-(n+1)}=\prod_{k=1}^{n}[-1-2i\cos(k\pi/n+1)].$$

The Chebysev polynomials of second kind $\{U_n(x)\}$ are defined in terms of trigonometric polynomials in $\cos\theta$ as

$$U_n(x)=\frac{\sin(n+1)\theta}{\sin\theta},\quad x=\cos\theta. \tag{6}$$

The family of Chebysev polynomials of second kind $\{U_n(x)\}$ satisfies the recurrence relation, for $n\geqslant 1$,

$$U_{n+1}(x)=2xU_n(x)-U_{n-1}(x)$$

with $U_0(x) = 1,\ U_1(x) = 2x$. The family can be obtained by successive determinants of:

$$K_n(x) = \begin{bmatrix} 2x & 1 & & \\ 1 & 2x & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2x \end{bmatrix}_{n\times n}.$$

Thus, $u_{n+1} = \left(\sqrt{bc}\right)^n \det K_n\left(\frac{a}{2\sqrt{bc}}\right)$. Since $\det K_n(x) = U_n(x)$, we write $u_{n+1} = \left(\sqrt{bc}\right)^n U_n\left(\frac{a}{2\sqrt{bc}}\right)$ and then we have the following corollary.

Corollary 9. *For $n \geqslant 1$,*

$$\det H_{2n}(a,b,c) = (bc)^n\left(U_n\left(\frac{a}{2\sqrt{bc}}\right)\right)^2$$

and

$$\det H_{2n+1}(a,b,c) = \left(\sqrt{bc}\right)^{2n+1} U_n\left(\frac{a}{2\sqrt{bc}}\right)U_{n+1}\left(\frac{a}{2\sqrt{bc}}\right),$$

where $U_n(x)$ is the nth term of the $\{U_n(x)\}$.

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