ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 38, Number 6, 2008

GENERALIZED ORDER-k FIBONACCI AND LUCAS FUNCTIONS

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ABSTRACT. In this paper, we consider the usual Lucas numbers and the generalized order-k Fibonacci numbers. Then we give a new definition for generalization of the Lucas numbers. Therefore, we give the generalized order-k Fibonacci and Lucas functions. Further, we derive new relationships between these functions.

1. Introduction. In [**2**], Er defined k sequences of the generalized order-k Fibonacci numbers as shown:

$$g_n^i = \sum_{j=1}^{k} g_{n-j}^i, \quad \text{for } n > 0 \text{ and } 1 \le i \le k,$$

with initial conditions for $1 - k \le n \le 0$,

$$g_n^i = \begin{cases} 1 & \text{if } i = 1 - n, \\ 0 & \text{otherwise,} \end{cases}$$

where g_n^i is the nth term of the ith sequence. For example, when $i = 2$, then $\{g_n^2\}$ is the Fibonacci sequence $\{F_n\}$ and $k = 4$, then the generalized order-4 Fibonacci sequence is

$$1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, \dots .$$

Also Er showed that

$$G_n = A^n \tag{1.1}$$

2000 AMS *Mathematics subject classification.* Primary 11B39, 11B37, 33E, 15A36.
Keywords and phrases. Generalized order-k Fibonaccci and Lucas numbers, generalized Fibonaccci and Lucas functions.
Received by the editors on March 28, 2005, and in revised form on December 20, 2005.

DOI:10.1216/RMJ-2008-38-6-1991

where

$$A=\begin{bmatrix}1&1&\cdots&1&1\\1&0&\cdots&0&0\\0&1&\cdots&0&0\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\cdots&1&0\end{bmatrix}_{k\times k}\tag{1.2}$$

and

$$G_n=\begin{bmatrix}g_n^1&g_n^2&\cdots&g_n^k\\g_{n-1}^1&g_{n-1}^2&\cdots&g_{n-1}^k\\\vdots&\vdots&&\vdots\\g_{n-k+1}^1&g_{n-k+1}^2&\cdots&g_{n-k+1}^k\end{bmatrix}.$$

The matrix A is said to be a generalized order-k Fibonacci matrix. Also, the following identities can be found in [**2**]:

$$g_{n+1}^i=g_n^1+g_n^{i+1}\quad\text{for } 1\le i\le k-1\tag{1.3}$$

$$g_{n+1}^k=g_n^1.\tag{1.4}$$

In [**6**], the authors defined k sequences of the generalized order-k Lucas numbers as shown:

$$l_n^i=\sum_{j=1}^{k}l_{n-j}^i,\quad\text{for } n>0 \text{ and } 1\le i\le k,$$

with initial conditions for $1-k\le n\le 0$,

$$l_n^i=\begin{cases}-1&\text{if } i=1-n,\\2&\text{if } i=2-n,\\0&\text{otherwise,}\end{cases}$$

where l_n^i is the nth term of the ith sequence. For example, if $i=k=2$, then $\{l_n^2\}$ is the usual Lucas sequence.

Further, in [**3**], the following formulas can be found for all m,n and $p>0$,

$$g_{n+m+p}^i=\sum_{j=1}^{k}g_n^jg_{m+p+1-j}^i \text{ and } g_{n+m}^i=\sum_{j=1}^{k}g_{n-p}^jg_{m+p-j}^i.$$

In [**1**], Elmore introduced the Fibonacci function as follows:

$$f_0\,(x) = \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{\sqrt{5}}, \quad f_n\,(x) = f_0^{(n)}\,(x) = \frac{\lambda_1^n e^{\lambda_1 x} - \lambda_2^n e^{\lambda_2 x}}{\sqrt{5}},$$

and hence $f_{n+1}(x) = f_n(x) + f_{n-1}(x)$, where

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

In [**4**], the authors gave a generalization of the Fibonacci function for k-Fibonacci numbers.

For positive integers k and n with $k \leq n$, let $Q_{k,n}$ denote the set of all strictly increasing k-sequences from $\{1, 2, \dots, n\}$. For an $n \times n$ matrix V and for $\alpha, \beta \in Q_{k,n}$, let $V[\alpha \mid \beta]$ denote the matrix lying in rows α and columns β, and let $V(\alpha \mid \beta)$ denote the matrix complementary to $V[\alpha \mid \beta]$ in V.

Note that the generalized order-k Fibonacci numbers can be expressed by powers of 2 for some n. For $1 \leq i < k$, we see that $g_1^i = 2^0 = 1$, $g_2^i = 2^1 = 2$, $g_3^i = 2^2 = 4, \dots, g_{k-i+1}^i = 2^{k-i}$. In general, for $1 \leq i < k$ and $1 \leq n \leq k-i+1$, $g_n^i = 2^{n-1}$. When $i = k$, $g_1^k = g_2^k = 2^0$ and $g_n^k = 2^{n-2}$ for $3 \leq n \leq k+1$.

2. Generalized order-k Fibonacci functions. In this section, we define generalized Fibonacci functions and then we investigate some properties of these functions.

We define a function $F(i,k,x)$ by, for $1 \leq i \leq k$,

$$F\,(i,k,x) = \sum_{t=0}^{\infty} \frac{g_t^i}{t!}\, x^t.$$

Since

$$\lim_{n\to\infty} \frac{g_n^i\,(n+1)}{g_{n+1}^i} \to \infty,$$

the function $F(i,k,x)$ is convergent for any real number x.

The power series $F(i,k,x)$ satisfies the differential equation

$$\text{(2.1)} \quad F^{(k)}\,(i,k,x) - F^{(k-1)}\,(i,k,x) - \cdots - F''\,(i,k,x) - F'\,(i,k,x) - F\,(i,k,x) = 0.$$

From [**5, 7**], we have that the characteristic equation, $x^k - x^{k-1} - \cdots - x - 1 = 0$, of matrix A does not have multiple roots. If $\lambda_1, \lambda_2, \dots, \lambda_k$ are the roots of $x^k - x^{k-1} - \cdots - x - 1 = 0$, then $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct.

Define V to be a $k \times k$ Vandermonde matrix as

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1} \end{bmatrix}. \tag{2.2}$$

Theorem 1. *Then, the initial-value problem* $\sum_{r=0}^{k-1} F^{(r)}(i,k,x) = F^{(k)}(i,k,x)$, *where* $F^{(r)}(i,k,0) = g_r^i$ *for* $r = 0, 1, 2, \dots, k-1$ *has the unique solution* $F(i,k,x) = \sum_{r=1}^{k} c_r e^{\lambda_r x}$, *where*

$$c_r = (-1)^{k+r} \frac{\det V\,(k \mid r)}{\det V}, \quad r = 1, 2, \dots, k \tag{2.3}$$

and λ_i*'s are as before.*

Proof. Since the characteristic equation of A is $x^k - x^{k-1} - \cdots - x - 1 = 0$, it is clear that $c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_k e^{\lambda_k x}$ is a solution of (2.1). Since $F(i,k,x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_k e^{\lambda_k x}$ and $F^{(r)}(i,k,0) = g_r^i$ for $r = 1, 2, \dots, k-1$ and $1 \le i \le k$, we have

$$\begin{aligned} F\,(i,k,0) &= c_1 + c_2 + \cdots + c_k = g_0^i \\ F'\,(i,k,0) &= c_1\lambda_1 + c_2\lambda_2 + \cdots + c_k\lambda_k = g_1^i \\ F''\,(i,k,0) &= c_1\lambda_1^2 + c_2\lambda_2^2 + \cdots + c_k\lambda_k^2 = g_2^i \\ &\vdots \\ F^{(k-1)}\,(i,k,0) &= c_1\lambda_1^{k-1} + c_2\lambda_2^{k-1} + \cdots + c_k\lambda_k^{k-1} = g_{k-1}^i. \end{aligned}$$

Let $c = (c_1, c_2, \dots, c_k)^T$ and $u = (g_0^i, g_1^i, \dots, g_{k-1}^i)^T$. We have that $Vc = u$. Since the matrix V is a Vandermonde matrix and $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct, V is nonsingular. The matrix $V(k \mid r)$

is a Vandermonde matrix and nonsingular for $r = 1, 2, \ldots, k$. Thus, we obtain by Cramer's rule

$$c_r = (-1)^{k+r} \frac{\det V\,(k \mid r)}{\det V}.$$

So the proof is complete. □

We can rewrite (2.1) as in the form

$$F^{(k)}(i,k,x) = F^{(k-1)}(i,k,x) + F^{(k-2)}(i,k,x) + \cdots + F''(i,k,x) + F'(i,k,x) + F(i,k,x).$$

Here we use the notation $F_0(i,k,x) = F(i,k,x)$ and, for $t \geq 1$, $F_t(i,k,x) = F^{(t)}(i,k,x)$. Thus,

$$F_n(i,k,x) = F^{(n)}(i,k,x) = c_1\lambda_1^n e^{\lambda_1 x} + c_2\lambda_2^n e^{\lambda_2 x} + \cdots + c_k\lambda_k^n e^{\lambda_k x}$$

gives us the sequence of functions $\{F_n(i,k,x)\}$ with

$$F_n(i,k,x) = F_{n-1}(i,k,x) + F_{n-2}(i,k,x) + \cdots + F_{n-k}(i,k,x) \tag{2.4}$$

where c_i is as in (2.3). We refer to the above functions as generalized order-k Fibonacci functions. Also, when $i = k$, we denote $F(i,k,x)$ by $F(k,x)$. If $k = 2$, $F(2,x) = f_0(x)$ is the Fibonacci function as in [**1**].

Theorem 2. *For the generalized order-k Fibonacci function $F(i,k,x)$,*

$$\begin{aligned}
&F_0(i,k,0) = g_0^i,\ \ F_1(i,k,0) = g_1^i,\\
&F_2(i,k,0) = g_2^i, \ldots, F_{k-1}(i,k,0) = g_{k-1}^i,\\
&F_k(i,k,0) = F_1(i,k,0) + F_2(i,k,0) + \cdots + F_{k-1}(i,k,0) = g_k^i,\\
&\qquad g_n^i = F_n(i,k,0) = c_1\lambda_1^n + c_2\lambda_2^n + \cdots + c_k\lambda_k^n\\
&\qquad\quad = g_{n-1}^i + g_{n-2}^i + \cdots + g_{n-k}^i, \quad n > k,
\end{aligned}$$

where each c_i is given by (2.3).

Let $\mathcal{F}_n(i,k,x) = (F_{n+k-1}(i,k,x), F_{n+k-2}(i,k,x), \ldots, F_n(i,k,x))^T$. By (2.4), we can write that $\mathcal{F}_{n+1}(i,k,x) = A\mathcal{F}_n(i,k,x)$. Generalizing, we derive

$$T_{n+1} = AT_n.$$

Since $A^n = G_n$, inductively we get

$$T_{n+1} = A^n T_1 = G_n T_1$$

where A is given by (1.2) and

$$T_n = \begin{bmatrix} F_{n+k-1}(1,k,x) & F_{n+k-1}(2,k,x) & \cdots & F_{n+k-1}(k,k,x) \\ F_{n+k-2}(1,k,x) & F_{n+k-2}(2,k,x) & \cdots & F_{n+k-2}(k,k,x) \\ \vdots & \vdots & & \vdots \\ F_n(1,k,x) & F_n(2,k,x) & \cdots & F_n(k,k,x) \end{bmatrix}.$$

From the definition of $T_n = [t_{ij}]$, we have $t_{ij} = F_{n+k-i}(j,k,x)$.

Theorem 3. *For all $m, n, p > 0$ and $1 \leq i \leq k$,*

$$F_{n+m+p}(i,k,x) = \sum_{j=1}^{k} g_{n-k+1}^{j} F_{m+p+k-j}(i,k,x).$$

Proof. Since $T_{n+m+p} = G_{n+m+p-1}T_1$, $T_{n+m+p} = G_n T_{m+p}$ and $F_{n+m+p}(i,k,x) = (T_{n+m+p})_{p+1,i}$, so the proof is complete. □

Theorem 4. *For all $m, n > 0$ and $1 \leq i \leq k$,*

$$F_{n+m}(i,k,x) = \sum_{j=1}^{k} g_{n-p-k+1}^{j} F_{m+p+k-j}(i,k,x).$$

In particular,

$$F_k(i,k,x) = \sum_{t=0}^{\infty} \frac{g_{k+t}^{i}}{t!} x^t.$$

Proof. Since $T_{n+m} = G_{n+m-1}T_1$ and $T_{n+m} = G_{n-p}G_{m+p-1}T_1 = G_{n-p}T_{m+p}$, we have the conclusion. Especially since $\sum_{t=0}^{k-1} F_t(i,k,x) = F_k(i,k,x)$ and

$$\sum_{t=0}^{k-1} F_t(i,k,x) = g_k^i + g_{k+1}^i x + \frac{g_{k+2}^i}{2!}x^2 + \cdots + \frac{g_{k+n}^i}{n!}x^n + \cdots,$$

we obtain

$$F_k\left(i,k,x\right)=\sum_{t=0}^{\infty}\frac{g_{k+t}^{i}}{t!}x^t.$$

The theorem is proved. □

Lemma 1. *For $n \geq k > 0$,*

$$\lambda^n=\sum_{j=1}^{k} g_{n-k+1}^{j}\lambda^{k-j}$$

where $G_n=[t_{ij}]=g_{n-i+1}^{j}$ and λ_i's are as before.

Proof. (Induction on n). First we assume that $n = k$. Since $t_{ij}=g_{n-i+1}^{j}$ and $g_1^j=1$ for all j, we write

$$\begin{aligned}\lambda^k&=\sum_{j=1}^{k} g_1^j\lambda^{k-j}=g_1^1\lambda^{k-1}+g_1^2\lambda^{k-2}+\cdots+g_1^{k-1}\lambda+g_1^k\\&=\lambda^{k-1}+\lambda^{k-2}+\cdots+\lambda+1.\end{aligned}$$

Now we suppose that $n > k$. Thus

(2.5)
$$\begin{aligned}\lambda^{n+1}&=\lambda^n\lambda\\&=\left(\sum_{j=1}^{k} g_{n-k+1}^{j}\lambda^{k-j}\right)\lambda\\&=\left(g_{n-k+1}^{1}\lambda^{k-1}+g_{n-k+1}^{2}\lambda^{k-2}+\cdots+g_{n-k+1}^{k-1}\lambda+g_{n-k+1}^{k}\right)\lambda\\&=g_{n-k+1}^{1}\lambda^{k}+g_{n-k+1}^{2}\lambda^{k-1}+\cdots+g_{n-k+1}^{k-1}\lambda^2+g_{n-k+1}^{k}\lambda.\end{aligned}$$

Since $\lambda^k=\lambda^{k-1}+\lambda^{k-2}+\cdots+\lambda+1$, we rewrite (2.5) as

$$\begin{aligned}\lambda^{n+1}&=g_{n-k+1}^{1}\left(\lambda^{k-1}+\lambda^{k-2}+\cdots+\lambda+1\right)+\\&\qquad g_{n-k+1}^{2}\lambda^{k-1}+\cdots+g_{n-k+1}^{k-1}\lambda^2+g_{n-k+1}^{k}\lambda\\&=\left(g_{n-k+1}^{1}+g_{n-k+1}^{2}\right)\lambda^{k-1}+\left(g_{n-k+1}^{1}+g_{n-k+1}^{3}\right)\lambda^{k-2}+\cdots\\&\quad+\left(g_{n-k+1}^{1}+g_{n-k+1}^{k-1}\right)\lambda^2+\left(g_{n-k+1}^{1}+g_{n-k+1}^{k}\right)\lambda+g_{n-k+1}^{1}.\end{aligned}$$

From (1.3), we have

$$\begin{aligned} g^1_{n-k+1} + g^2_{n-k+1} &= g^1_{n-k+2},\\ g^1_{n-k+1} + g^3_{n-k+1} &= g^2_{n-k+2},\\ &\vdots\\ g^1_{n-k+1} + g^k_{n-k+1} &= g^{k-1}_{n-k+2}. \end{aligned}$$

Thus,

$$\lambda^{n+1} = g^1_{n-k+2}\lambda^{k-1} + g^2_{n-k+2}\lambda^{k-2} + \cdots + g^{k-1}_{n-k+2}\lambda + g^1_{n-k+1}. \tag{2.6}$$

By (1.4), we have that $g^1_{n-k+1} = g^k_{n-k+2}$; thus, we write (2.6) as

$$\lambda^{n+1} = g^1_{n-k+2}\lambda^{k-1} + g^2_{n-k+2}\lambda^{k-2} + \cdots + g^{k-1}_{n-k+2}\lambda + g^k_{n-k+2}.$$

So the proof is complete. □

Theorem 5. *For $n > 0$,*

$$F_n(i,k,\lambda) = \sum_{j=1}^{k} \gamma_{j_n}\lambda^{j-1}$$

where

$$\gamma_{j_n} = \frac{g^i_{n-1+j}}{(j-1)!} + \sum_{t=1}^{\infty} g_t^{k+1-j}\,\frac{g^i_{n+k-1+t}}{(k-1+t)!}.$$

Proof. Since $\lambda^k = \lambda^{k-1} + \lambda^{k-2} + \cdots + \lambda + 1$ and by Lemma 1, we have

$$\begin{aligned} F_n(i,k,\lambda) &= g^i_n + \frac{g^i_{n+1}}{1!}\lambda + \frac{g^i_{n+2}}{2!}\lambda^2 + \frac{g^i_{n+3}}{3!}\lambda^3 + \cdots\\ &\quad + \frac{g^i_{n+k-1}}{(k-1)!}\lambda^{k-1} + \frac{g^i_{n+k}}{k!}\lambda^k + \cdots + \frac{g^i_{2n}}{n!}\lambda^n + \cdots\\ &= \left(g^i_n + g^k_1\frac{g^i_{n+k}}{k!} + g^k_2\frac{g^i_{n+k+1}}{(k+1)!} + \cdots + g^k_{n-k+1}\frac{g^i_{2n}}{n!} + \cdots \right) \end{aligned}$$

$$\begin{aligned}
&+\left(\frac{g^i_{n+1}}{1!}+g_1^{k-1}\frac{g^i_{n+k}}{k!}+g_2^{k-1}\frac{g^i_{n+k+1}}{(k+1)!}+\cdots\right.\\
&\qquad\qquad\qquad\left.+\,g^{k-1}_{n-k+1}\frac{g^i_{2n}}{n!}+\cdots\right)\lambda+\cdots\\
&+\left(\frac{g^i_{n+k-1}}{(k-1)!}+g_1^{1}\frac{g^i_{n+k}}{k!}+g_2^{1}\frac{g^i_{n+k+1}}{(k+1)!}+\cdots\right.\\
&\qquad\qquad\qquad\left.+\,g^{1}_{n-k+1}\frac{g^i_{2n}}{n!}+\cdots\right)\lambda^{k-1}\\
&=\gamma_{1_n}+\gamma_{2_n}\lambda+\gamma_{3_n}\lambda^2+\cdots+\gamma_{k_n}\lambda^{k-1}\\
&=\sum_{j=1}^{k}\gamma_{j_n}\lambda^{j-1},
\end{aligned}$$

where

$$\gamma_{j_n}=\frac{g^i_{n-1+j}}{(j-1)!}+\sum_{t=1}^{\infty}g_t^{k+1-j}\frac{g^i_{n+k-1+t}}{(k-1+t)!};$$

thus, the proof is complete. □

From Theorem 4 by taking $p=m=0$ and Theorem 6, we have

$$\begin{aligned}
F_n(i,k,x)&=\sum_{j=1}^{k}g^j_{n-k+1}F_{k-j}(i,k,x)\\
&=g^1_{n-k+1}F_{k-1}(i,k,x)+g^2_{n-k+1}F_{k-2}(i,k,x)+\cdots\\
&\quad+g^k_{n-k+1}F_0(i,k,x)\\
&=\sum_{j=1}^{k}\gamma_{j_n}\lambda^{j-1}.
\end{aligned}$$

3. Generalized order-k Lucas numbers. In this section we give a more convenient definition for generalization of the Lucas numbers. Then we define generalized Lucas functions and derive some properties of them in the next section. In [**6**], the authors defined the generalized order-k Lucas numbers. However, we see that this definition is not convenient for further steps. So we give the following new definition.

Define k sequences of the generalized order-k Lucas numbers as shown:

$$v_n^i = \sum_{j=1}^{k} v_{n-j}^i, \quad \text{for } n \geq 0 \text{ and } 1 \leq i \leq k,$$

with boundary conditions

$$v_n^i = \begin{cases} 3 & \text{if } n = -i, \\ -1 & \text{if } n = 1-i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } -k \leq n < 0,$$

where v_n^i is the nth term of the ith sequence. When $i = 2$, the generalized order-2 Lucas sequence is reduced to the Lucas sequence $\{L_n\}$. When $i = 4$, the generalized order-4 Lucas sequence is

$$1, 3, 6, 12, 22, 43, 83, 160, 308, 594, \dots .$$

By a property of matrix multiplication, we have

$$(3.1) \quad \left[v_{n+1}^i \quad v_n^i \quad \dots \quad v_{n-k+2}^i \right]^T = A \left[v_n^i \quad v_{n-1}^i \quad \dots \quad v_{n-k+1}^i \right]^T,$$

where A is given by (1.2). To deal with k sequences of the generalized order-k Lucas series simultaneously, we define a $k \times k$ square matrix B_n as follows:

$$B_n = \begin{bmatrix} v_n^1 & v_n^2 & \cdots & v_n^k \\ v_{n-1}^1 & v_{n-1}^2 & \cdots & v_{n-1}^k \\ \vdots & \vdots & & \vdots \\ v_{n-k+1}^1 & v_{n-k+1}^2 & \cdots & v_{n-k+1}^k \end{bmatrix}.$$

Generalizing (3.1), we derive $B_{n+1} = AB_n$. We inductively rewrite it as

$$B_{n+1} = A^n B_1 = A^{n+1} B_0 = A^{n+2} B,$$

where by the definition of sequence $\{v_n^i\}$,

$$B_1 = \begin{bmatrix} 6 & 4 & 4 & \cdots & 4 & 1 \\ 3 & 2 & 2 & \cdots & 2 & 2 \\ 3 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 3 & -1 & \ddots & \vdots & 0 \\ \vdots & \dots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 3 & -1 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 3 & 2 & 2 & \cdots & 2 \\ 3 & -1 & 0 & \cdots & 0 \\ 0 & 3 & -1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 3 & -1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 3 & -1 & & 0 \\ & 3 & \ddots & \\ & & \ddots & -1 \\ 0 & & & 3 \end{bmatrix}.$$

Thus, we infer by $A^n = G_n$,

$$B_{n+1} = A^{n+2}B = G_{n+2}B. \tag{3.2}$$

Theorem 6. *Then*

$$\begin{aligned} v_n^i &= -g_{n+1}^{i-1} + 3g_{n+1}^i \quad \text{for } 2 \le i \le k, \\ v_n^1 &= 3g_{n+1}^1, \end{aligned}$$

where v_n^i *and* g_n^i *are as before.*

Proof. The proof follows from (3.2). □

In particular, when $k = 2$ in (3.2), then

$$\begin{bmatrix} v_{n+1}^1 & v_{n+1}^2 \\ v_n^1 & v_n^2 \end{bmatrix} = \begin{bmatrix} g_{n+2}^1 & g_{n+2}^2 \\ g_{n+1}^1 & g_{n+1}^2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 0 & 3 \end{bmatrix},$$

and since $g_n^1 = g_{n+1}^2$ for $n > 0$, see [**6**], and $v_n^2 = L_n$, $g_n^2 = F_n$, we obtain

$$\begin{aligned} L_n &= -g_{n+1}^1 + 3g_{n+1}^2 = -g_{n+2}^2 + 3g_{n+1}^2 \\ &= -F_{n+2} + 3F_{n+1} \\ &= F_{n+1} + F_{n-1}, \end{aligned}$$

which is a well-known relation between the Fibonacci and Lucas numbers (see [**8**]).

Theorem 7. *Then, for* $n, m > 0$,

$$v_{n+m}^i = \sum_{j=1}^{k} g_n^j v_{m+1-j}^i.$$

Proof. From (3.2), we have $B_n = G_{n+1}B = A^{n+1}B$. Thus, $B_{n+m} = A^{n+m+1}B = A^n A^{m+1}B = G_n B_m$. The theorem is obtained from a property of matrix multiplication. □

Since $B_{n+m} = A^{n+m+1}B = A^{n-p}A^{m+p+1}B = G_{n-p}B_{m+p}$, we have the following result.

Corollary 1. *For* $n, m, p > 0$,

$$v_{n+m}^i = \sum_{j=1}^{k} g_{n-p}^j v_{m+p+1-j}^i .$$

For example, when $i = 2$, then

$$\begin{aligned} v_{n+m}^2 &= \sum_{j=1}^{2} g_{n-p}^j v_{m+p+1-j}^2 \\ &= g_{n-p}^1 v_{m+p}^2 + g_{n-p}^2 v_{m+p-1}^2 , \end{aligned}$$

and since $g_n^1 = g_{n+1}^2 = F_{n+1}$ and $v_n^2 = L_n$,

$$L_{n+m} = F_{n-p+1}L_{m+p} + F_{n-p}L_{m+p-1}$$

and for $p = 0$, we obtain $L_{n+m} = F_{n+1}L_m + F_n L_{m-1}$, see [**8**, page 176].

4. Generalized order-k Lucas functions. We define the generalized Lucas function $L(i, k, x)$ by, for $1 \le i \le k$,

$$L\left(i, k, x\right) = \sum_{i=0}^{\infty} \frac{v_r^i}{r!} x^r .$$

Since

$$\lim_{n\to\infty} \frac{v_n^k \left(n+1\right)}{v_{n+1}^k} \longrightarrow \infty,$$

the function $L(i, k, x)$ is convergent for real number x. The power series $L(i, k, x)$ satisfies the differential equation

$$L^{(k)}(i,k,x) - L^{(k-1)}(i,k,x) - \cdots - L''(i,k,x) - L'(i,k,x) - L(i,k,x) = 0. \tag{4.1}$$

Let the matrices A and V be as in (1.2) and (2.2), respectively.

Theorem 8. *Then the initial-value problem* $\sum_{r=0}^{k-1} L^{(r)}(i,k,x) = L^{(k)}(i,k,x)$, *where* $L^{(r)}(i,k,0) = v_r^i$ *for* $r = 0,1,2,\ldots,k-1$ *has the unique solution* $L(i,k,x) = \sum_{r=1}^{k} s_r e^{\lambda_r x}$ *where*

$$s_r = (-1)^{k+r} \frac{\det V(k \mid r)}{\det V}, \quad r = 1,2,\ldots,k, \tag{4.2}$$

where the λ_i*'s are as before.*

Proof. Since the characteristic equation of A, $s_1 e^{\lambda_1 x} + s_2 e^{\lambda_2 x} + \cdots + s_k e^{\lambda_k x}$ is a solution of (4.1) and since $L(i,k,x) = s_1 e^{\lambda_1 x} + s_2 e^{\lambda_2 x} + \cdots + s_k e^{\lambda_k x}$ and $L^{(r)}(i,k,0) = v_r^i$ for $r = 0,1,2,\ldots,k-1$ and $1 \le i \le k$, we have

$$\begin{aligned}
L(i,k,0) &= s_1 + s_2 + \cdots + s_k = v_0^i \\
L'(i,k,0) &= s_1\lambda_1 + s_2\lambda_2 + \cdots + s_k\lambda_k = v_1^i \\
L''(i,k,0) &= s_1\lambda_1^2 + s_2\lambda_2^2 + \cdots + s_k\lambda_k^2 = v_2^i \\
&\vdots \\
L^{(k-1)}(i,k,0) &= s_1\lambda_1^{k-1} + s_2\lambda_2^{k-1} + \cdots + s_k\lambda_k^{k-1} = v_{k-1}^i.
\end{aligned}$$

Let $s = (s_1, s_2, \ldots, s_k)^T$ and $z = (v_0^i, v_1^i, \ldots, v_{k-1}^i)^T$. Then we have $Vs = z$. Since matrix V is a Vandermonde matrix and $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct, matrix V is nonsingular. Also matrix $V(k \mid r)$ is a Vandermonde matrix and nonsingular for $r = 1,2,\ldots,k$. Thus, we obtain by Cramer's rule

$$s_r = (-1)^{k+r} \frac{\det V(k \mid r)}{\det V}.$$

So the proof is complete. □

We may rewrite (4.1) as

$$\begin{aligned} L^{(k)}(i,k,x) &= L^{(k-1)}(i,k,x) + L^{(k-2)}(i,k,x) + \cdots \\ &\quad + L''(i,k,x) + L'(i,k,x) + L(i,k,x). \end{aligned}$$

Using the notation $L_0(i,k,x) = L(i,k,x)$ and for $t \geq 1$, $L_t(i,k,x) = L^{(t)}(i,k,x)$, we may write

$$L_n(i,k,x) = L^{(n)}(i,k,x) = s_1\lambda_1^n e^{\lambda_1 x} + s_2\lambda_2^n e^{\lambda_2 x} + \cdots + s_k\lambda_k^n e^{\lambda_k x},$$

which gives us the sequence of functions $\{L_n(i,k,x)\}$ with

$$L_n(i,k,x) = L_{n-1}(i,k,x) + L_{n-2}(i,k,x) + \cdots + L_{n-k}(i,k,x), \tag{4.3}$$

where s_i is as in (4.2). We refer to the above functions as generalized order-k Lucas functions.

Theorem 9. *For the generalized order-k Lucas function $L(i,k,x)$,*

$$\begin{aligned} &L_0(i,k,0) = v_0^i, \quad L_1(i,k,0) = v_1^i, \\ &L_2(i,k,0) = v_2^i, \ldots, L_{k-1}(i,k,0) = v_{k-1}^i, \\ &L_k(i,k,0) = L_1(i,k,0) + L_1(i,k,0) + \cdots + L_{k-1}(i,k,0) = v_k^i, \\ &\qquad v_n^i = L_n(i,k,0) = s_1\lambda_1^n + s_2\lambda_2^n + \cdots + s_k\lambda_k^n \\ &\qquad\quad = v_{n-1}^i + v_{n-2}^i + \cdots + v_{n-k}^i, \quad n > k, \end{aligned}$$

where each s_i is given by (4.2).

Let $\mathcal{L}_n(i,k,x) = (L_{n+k-1}(i,k,x), L_{n+k-2}(i,k,x), \ldots, L_n(i,k,x))^T$. By (4.3), we can write that $\mathcal{L}_{n+1}(i,k,x) = A\mathcal{L}_n(i,k,x)$, that is,

$$\begin{bmatrix} L_{n+k}(i,k,x) \\ L_{n+k-1}(i,k,x) \\ \vdots \\ L_{n+1}(i,k,x) \end{bmatrix} = A \begin{bmatrix} L_{n+k-1}(i,k,x) \\ L_{n+k-2}(i,k,x) \\ \vdots \\ L_n(i,k,x) \end{bmatrix}. \tag{4.4}$$

where A is given by (1.2). Generalizing (4.4), we derive

$$H_{n+1} = AH_n,$$

where

$$H_n = \begin{bmatrix} L_{n+k-1}(1,k,x) & L_{n+k-1}(2,k,x) & \cdots & L_{n+k-1}(k,k,x) \\ L_{n+k-2}(1,k,x) & L_{n+k-2}(2,k,x) & \cdots & L_{n+k-2}(k,k,x) \\ \vdots & \vdots & & \vdots \\ L_n(1,k,x) & L_n(2,k,x) & \cdots & L_n(k,k,x) \end{bmatrix}.$$

Inductively, we obtain $H_{n+1} = A^n H_1$ and since $A^n = G_n$, $H_{n+1} = G_n H_1$.

We can generalize the result of Theorem 7 for generalized order-k Lucas functions.

Theorem 10. *For $m, n > 0$ and $1 \leq i \leq k$,*

$$L_{n+m+p}(i,k,x) = \sum_{j=1}^{k} g_{n-k+1}^{j} L_{m+p+k-j}(i,k,x),$$

where $H_n = [h_{ij}] = L_{n+k-i}(j,k,x)$.

Proof. Since $H_{n+1} = G_n H_1$ and so $H_{n+m+p} = G_{n+m+p-1} H_1$, $H_{n+m+p} = G_n H_{m+p}$ and $L_{n+m+p}(i,k,x) = (H_{n+m+p})_{p+1,i}$. Thus, the theorem is proved from a property of matrix multiplication. □

Theorem 11. *For $m, n > 0$ and $1 \leq i \leq k$,*

$$L_{n+m}(i,k,x) = \sum_{j=1}^{k} g_{n-p-k+1}^{j} L_{m+p+k-j}(i,k,x).$$

In particular,

$$L_k(i,k,x) = \sum_{t=0}^{\infty} \frac{v_{k+t}^{i}}{t!} x^t.$$

Proof. Since $H_{n+m} = G_{n+m-1} H_1$ and $H_{n+m} = G_{n-p} G_{m+p-1} H_1 = G_{n-p} H_{m+p}$, we have the conclusion. Since also $\sum_{t=0}^{k-1} L_t(i,k,x) = L_k(i,k,x)$ and

$$\sum_{t=0}^{k-1} L_t(i,k,x) = v_k^i + v_{k+1}^i x + \frac{v_{k+2}^i}{2!} x^2 + \cdots + \frac{v_{n+k}^i}{n!} x^n + \cdots,$$

we obtain

$$L_k(i,k,x)=\sum_{t=0}^{\infty}\frac{v_{k+t}^{i}}{t!}x^t. \qquad \square$$

From Theorem 10, we have

$$\begin{aligned}L_n(i,k,x)&=\sum_{j=1}^{k}g_{n-k+1}^{j}L_{k-j}(i,k,x)\\&=g_{n-k+1}^{1}L_{k-1}(i,k,x)+g_{n-k+1}^{2}L_{k-2}(i,k,x)+\cdots\\&\quad+g_{n-k+1}^{k}L_0(i,k,x).\end{aligned}$$

By Theorem 11, we have the following corollary.

Corollary 2. *Let $L_n(i,k,x)$ and $F_n(i,k,x)$ be the generalized order-k Lucas and Fibonacci functions, respectively. Then*

$$\begin{aligned}&L_n(i,k,x)=3F_{n+1}(i,k,x)-F_{n+1}(i-1,k,x) \quad \textit{for } 2\le i\le k\\&L_n(1,k,x)=3F_{n+2}(k,x).\end{aligned}$$

Proof. From Theorems 6 and 11, we write for $2\le i\le k$,

$$\begin{aligned}L_n(i,k,x)&=\sum_{t=0}^{\infty}\frac{v_{n+t}^{i}}{t!}x^t\\&=\sum_{t=0}^{\infty}\frac{\left(-g_{n+t+1}^{i-1}+3g_{n+t+1}^{i}\right)}{t!}x^t\\&=\sum_{t=0}^{\infty}\frac{-g_{n+t+1}^{i-1}}{t!}x^t+\sum_{t=0}^{\infty}\frac{3g_{n+t+1}^{i}}{t!}x^t\\&=3F_{n+1}(i,k,x)-F_{n+1}(i-1,k,x).\end{aligned}$$

By Theorem 6 and since $g_n^1=g_{n+1}^k$, we write

$$\begin{aligned} L_n(1,k,x) &= \sum_{t=0}^{\infty} \frac{v_{n+t}^1}{t!} x^t \\ &= \sum_{t=0}^{\infty} \frac{3g_{n+1+t}^1}{t!} x^t = \sum_{t=0}^{\infty} \frac{3g_{n+2+t}^k}{t!} x^t \\ &= 3F_{n+2}\left(k,k,x\right) = 3F_{n+2}\left(k,x\right). \end{aligned}$$

So the proof is complete. □

Acknowledgments. The authors would like to thank the referee for a number of helpful suggestions.

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