

RECIPROCAL SUMS OF l -TH POWER OF GENERALIZED BINARY SEQUENCES WITH INDICES

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ABSTRACT. Recently in [5], the author considered certain reciprocal sums of general second order recurrence $\{W_n\}$. In this paper, we generalize the results of Xi and we give some new results for the reciprocal sums of l -th power of general second order recurrence $\{W_{kn}\}$ for arbitrary positive integer k .

1. INTRODUCTION

Let a, b, P and Q integers such that $PQ \neq 0$ and $P^2 - 4Q \neq 0$. Define the sequence $\{W_n\}$ as follows: for $n > 1$

$$W_n = PW_{n-1} - QW_{n-2} \quad (1.1)$$

where $W_0 = a, W_1 = b$. The sequence $\{W_n\}$ and its some properties have been studied by several authors. Horadam [3] gave the Binet form of $\{W_n\}$ as shown :

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}$$

where $\alpha, \beta = \left(P \pm \sqrt{P^2 - 4Q}\right) / 2$, $A = b - \beta a, B = b - \alpha a$. We denote W_n by $W_n(a, b; P, Q)$. As important special cases, denote $W_n(0, 1, P, Q)$ and $W_n(2, P, P, Q)$ by U_n and V_n , respectively.

Some authors have studied the both finite and infinite reciprocal sums of terms of certain sequences. In [1], the authors derived

$$\sum_{n=1}^m \frac{Q^n}{W_n W_{n+k}} = U_m \sum_{n=1}^k \frac{Q^n}{W_n W_{n+m}} \text{ and } \sum_{n=1}^{\infty} \frac{Q^n}{W_n W_{n+k}} = \frac{1}{ABU_k} \left(\sum_{n=1}^k \frac{W_{n+1}}{W_n} - k\alpha \right) \quad (1.2)$$

where $P > 0$ and k, m nonnegative integers. For the case $Q = -1$, the identities in (1.2) are obtained by Good [2]. Regarding taking l -th powers of terms in the sums, the author [5] generalized the results of [1, 2]. For

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example, he derived the following infinitive reciprocal sums:

$$\begin{aligned} & \sum_{n=k}^{\infty} \frac{Q^n}{W_n^l W_{n+m}^l} \sum_{i=0}^{l-1} [(W_{k+1} - W_k \beta) Q^{n-k} \alpha^m - (W_{k+1} - W_k \alpha) \\ & \times \beta^{2(n-k)+m}]^{l-1-i} [(W_{k+1} - W_k \beta) Q^{n-k} \beta^m - (W_{k+1} - W_k \alpha) \beta^{2(n-k)+m}]^i \\ = & \frac{(P^2 - 4Q)^{l/2} Q^k}{(\alpha^m - \beta^m)(W_{k+1} - W_k \beta)} \sum_{i=0}^{m-1} \frac{\beta^{li}}{W_{k+i}^l}. \end{aligned}$$

In [5], the authors gave the general results including the earlier results by taking l -th powers of terms in the reciprocal sums. In this paper, we generalize the result of [5] regarding reciprocal sums of l -th powers of the terms with indices.

2. THE MAIN RESULTS

In this section we consider both finite and infinite reciprocal sums of products of r -consecutive terms of the sequence $\{W_n\}$. Clearly we will consider the finite and infinite reciprocal sums of $\{W_{rn}\}_{n=0}^{\infty}$ for arbitrary positive integer r . For later use and the readers convenience, we have the following result from [4]:

Lemma 1. *Let W_n be the n th term of sequence $\{W_n\}$. Then for $n, r > 0$,*

$$W_{rn} = V_r W_{r(n-1)} - Q^r W_{r(n-2)} \quad (2.1)$$

where V_n and Q be as before.

For our purpose, we use the generating function of the sequence $\{W_{rn}\}$. We give our first result.

Theorem 1. *Let $P > 0$,*

$$\begin{aligned} & \sum_{n=k}^t \frac{Q^{rn}}{W_{rn}^l W_{r(n+1)}^l} \sum_{i=0}^{l-1} \{[(W_{r(k+1)} - W_{rk} \beta^r) Q^{r(n-k)} \alpha^r \\ & - (W_{r(k+1)} - W_{rk} \alpha^r) \beta^{2r(n-k)+r}]^{l-1-i} \times \\ & [(W_{r(k+1)} - W_{rk} \beta^r) Q^{r(n-k)} \beta^r - (W_{r(k+1)} - W_{rk} \alpha^r) \beta^{2r(n-k)+r}]^i\} \\ = & \frac{(\alpha^r - \beta^r)^{l-1} Q^{rk}}{(W_{r(k+1)} - W_{rk} \beta^r)} \left[\frac{1}{W_{rk}^l} - \frac{\beta^{lr(t+1-k)}}{W_{r(t+1)}^l} \right]. \end{aligned} \quad (2.2)$$

Proof. For ease write and arbitrary $r > 0$, let $f(x) = \sum_{n=k}^{\infty} W_{rn} x^n$. Thus consider

$$f(x) - W_{rk} x^k - W_{r(k+1)} x^{k+1} = V_r x (f(x) - W_{rk} x^k) - Q^r x^2 f(x).$$

Hence

$$f(x) = x^k \frac{W_{rk} + x(W_{r(k+1)} - (\alpha^r + \beta^r)W_{rk})}{1 - (\alpha^r + \beta^r)x + (\alpha\beta)^r x^2}.$$

Since $1 - (\alpha^r + \beta^r)x + (\alpha\beta)^r x^2 = (1 - \alpha^r x)(1 - \beta^r x)$, $f(x)$ can be decomposed into partial fractions:

$$f(x) = \frac{x^k}{\alpha^r - \beta^r} \left(\frac{W_{r(k+1)} - \beta^r W_{rk}}{1 - \alpha^r x} - \frac{W_{r(k+1)} - \alpha^r W_{rk}}{1 - \beta^r x} \right).$$

From the coefficients of x^n in both sides above equations, we get

$$W_{rn} = \frac{W_{r(k+1)} - \beta^r W_{rk}}{\alpha^r - \beta^r} \alpha^{r(n-k)} - \frac{W_{r(k+1)} - \alpha^r W_{rk}}{\alpha^r - \beta^r} \beta^{r(n-k)}.$$

Let $T_{rn} = \frac{\beta^{r(n-k)}}{W_{rn}}$. If we compute the difference of two consecutive terms of $\{T_{nk}^l\}$, we obtain

$$\begin{aligned} T_{rn}^l - T_{r(n+1)}^l &= \frac{(\beta^{r(n-k)} W_{r(n+1)})^l - (\beta^{r(n+1-k)} W_{rn})^l}{W_{rn}^l W_{r(n+1)}^l} \\ &= \frac{1}{W_{rn}^l W_{r(n+1)}^l (\alpha^r - \beta^r)^l} \{ [(W_{r(k+1)} - W_{rk} \beta^r) Q^{r(n-k)} \alpha^r \\ &\quad - (W_{r(k+1)} - W_{rk} \alpha^r) \beta^{2r(n-k)+r}]^l \\ &\quad - [(W_{r(k+1)} - W_{rk} \beta^r) Q^{r(n-k)} \beta^r - (W_{r(k+1)} - W_{rk} \alpha^r) \beta^{2r(n-k)+r}]^l \} \\ &= \frac{(W_{r(k+1)} - W_{rk} \beta^r) Q^{r(n-k)}}{W_{rn}^l W_{r(n+1)}^l (\alpha^r - \beta^r)^{l-1}} \sum_{i=0}^{l-1} \{ [(W_{r(k+1)} - W_{rk} \beta^r) Q^{r(n-k)} \alpha^r \\ &\quad - (W_{r(k+1)} - W_{rk} \alpha^r) \beta^{2r(n-k)+r}]^{l-1-i} [(W_{r(k+1)} - W_{rk} \beta^r) Q^{r(n-k)} \beta^r - \\ &\quad (W_{r(k+1)} - W_{rk} \alpha^r) \beta^{2r(n-k)+r}]^i \}. \end{aligned}$$

Thus

$$\begin{aligned} &T_{rn}^l - T_{r(n+1)}^l \\ &= \frac{(W_{r(k+1)} - W_{rk} \beta^r) Q^{r(n-k)}}{W_{rn}^l W_{r(n+1)}^l (\alpha^r - \beta^r)^{l-1}} \\ &\quad \times \sum_{i=0}^{l-1} \{ [(W_{r(k+1)} - W_{rk} \beta^r) Q^{r(n-k)} \alpha^r - (W_{r(k+1)} - W_{rk} \alpha^r) \beta^{2r(n-k)+r}]^{l-1-i} \\ &\quad \times [(W_{r(k+1)} - W_{rk} \beta^r) Q^{r(n-k)} \beta^r - (W_{r(k+1)} - W_{rk} \alpha^r) \beta^{2r(n-k)+r}]^i \}. \quad (2.3) \end{aligned}$$

Then we obtain

$$\begin{aligned} &\sum_{n=k}^t \frac{Q^{rn}}{W_{rn}^l W_{r(n+1)}^l} \sum_{i=0}^{l-1} \{ [(W_{r(k+1)} - W_{rk} \beta^r) Q^{r(n-k)} \alpha^r \\ &\quad - (W_{r(k+1)} - W_{rk} \alpha^r) \beta^{2r(n-k)+r}]^{l-1-i} \\ &\quad \times [(W_{r(k+1)} - W_{rk} \beta^r) Q^{r(n-k)} \beta^r - (W_{r(k+1)} - W_{rk} \alpha^r) \beta^{2r(n-k)+r}]^i \} \\ &= \frac{(\alpha^r - \beta^r)^{l-1} Q^{rk}}{(W_{r(k+1)} - W_{rk} \beta^r)} \sum_{n=k}^t (T_{rn}^l - T_{r(n+1)}^l) \\ &= \frac{(\alpha^r - \beta^r)^{l-1} Q^{rk}}{(W_{r(k+1)} - W_{rk} \beta^r)} \left[\frac{1}{W_{rk}^l} - \frac{\beta^{lr(t+1-k)}}{W_{r(t+1)}^l} \right]. \end{aligned}$$

Thus the proof is complete. \square

For example, when $l = 1$ in (2.2), we have the following result for $P > 0$

$$\sum_{n=k}^t \frac{Q^{rn}}{W_{rn}W_{r(n+1)}} = \frac{Q^{rk}}{(W_{r(k+1)} - W_{rk}\beta^r)} \left(\frac{1}{W_{rk}} - \frac{\beta^{r(t+1-k)}}{W_{r(t+1)}} \right).$$

The case $r = 1$ in the above result can be found in [5]. As a numerical example, if we take $W_n(0, 1, 1, -1) = F_n$, then

$$\sum_{n=1}^t \frac{1}{F_{2n}F_{2(n+1)}} = \beta^2 - \frac{\beta^{2t+2}}{F_{2(t+1)}}.$$

Theorem 2. For $P > 0$,

$$\begin{aligned} & \sum_{n=k}^{\infty} \frac{Q^{rn}}{W_{rn}^l W_{r(n+1)}^l} \sum_{i=0}^{l-1} \{[(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n-k)}\alpha^r \\ & - (W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r}]^{l-1-i} \\ & \times [(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n-k)}\beta^r - (W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r}]^i\} \\ & = \frac{(\alpha^r - \beta^r)^{l-1} Q^{rk}}{W_{rk}^l (W_{r(k+1)} - W_{rk}\beta^r)}. \end{aligned}$$

Proof. From Theorem 1,

$$\begin{aligned} & \sum_{n=k}^{\infty} \frac{Q^{rn}}{W_{rn}^l W_{r(n+1)}^l} \sum_{i=0}^{l-1} \{[(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n-k)}\alpha^r \\ & - (W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r}]^{l-1-i} \\ & \times [(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n-k)}\beta^r - (W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r}]^i\} \\ & = \frac{(\alpha^r - \beta^r)^{l-1} Q^{rk}}{(W_{r(k+1)} - W_{rk}\beta^r)} \left(\frac{1}{W_{rk}^l} - \lim_{t \rightarrow \infty} \frac{\beta^{lr(t+1-k)}}{W_{r(t+1)}^l} \right) \\ & = \frac{(\alpha^r - \beta^r)^{l-1} Q^{rk}}{W_{rk}^l (W_{r(k+1)} - W_{rk}\beta^r)}. \end{aligned}$$

Since

$$\left| \frac{\alpha}{\beta} \right| > 1 \quad \text{for } P > 0,$$

and

$$\begin{aligned} & \lim_{t \rightarrow \infty} T_{rt}^l = \lim_{t \rightarrow \infty} \left(\frac{\beta^{r(t-k)}}{W_{rt}} \right)^l \\ & = \lim_{t \rightarrow \infty} \left[\frac{W_{r(k+1)} - \beta^r W_{rk}}{\alpha^r - \beta^r} \left(\frac{\alpha}{\beta} \right)^{r(t-k)} - \frac{W_{r(k+1)} - \alpha^r W_{rk}}{\alpha^r - \beta^r} \right]^{-l} = 0. \quad (2.4) \end{aligned}$$

The proof is complete. \square

If we take $l = 1$ in Theorem 2, we get

$$\sum_{n=k}^{\infty} \frac{Q^{rn}}{W_{rn}W_{r(n+1)}} = \frac{Q^{rk}}{W_{rk}(W_{r(k+1)} - W_{rk}\beta^r)}.$$

Theorem 3. For $P > 0$ and $m > p$,

$$\begin{aligned} & \sum_{n=k}^t \frac{Q^{r(n+p)}}{W_{r(n+p)}^l W_{r(n+m)}^l} \sum_{i=0}^{l-1} \{ [(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}\alpha^{r(m-p)} \\ & - (W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^{l-1-i} \\ & \times [(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}\beta^{r(m-p)} - (W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^i \} \\ = & \frac{U_r^l Q^{rk}}{(W_{r(k+1)} - W_{rk}\beta^r)U_{r(m-p)}} \sum_{i=p}^{m-1} \left[\frac{\beta^{ri}}{W_{r(k+i)}^l} - \frac{\beta^{r(t+i+1-k)}}{W_{r(t+i+1)}^l} \right]. \end{aligned} \quad (2.5)$$

Proof. By (2.3), we write

$$\begin{aligned} T_{r(n+p)}^l - T_{r(n+m)}^l &= \left(\frac{\beta^{r(n+p-k)}}{W_{r(n+p)}} \right)^l - \left(\frac{\beta^{r(n+m-k)}}{W_{r(n+m)}} \right)^l \\ &= \frac{(\beta^{r(n+p-k)}W_{r(n+m)})^l - (\beta^{r(n+m-k)}W_{r(n+p)})^l}{W_{r(n+p)}^l W_{r(n+m)}^l} \\ &= \frac{(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}U_{r(m-p)}}{W_{r(n+p)}^l W_{r(n+m)}^l U_r^l} \\ &\quad \times \sum_{i=0}^{l-1} \{ [(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}\alpha^{r(m-p)} \\ &\quad - (W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^{l-1-i} \\ &\quad \times [(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n-k)}\beta^{r(m-p)} \\ &\quad - (W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^i \}. \end{aligned}$$

Thus

$$\begin{aligned} T_{r(n+p)}^l - T_{r(n+m)}^l &= \frac{(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}U_{r(m-p)}}{W_{r(n+p)}^l W_{r(n+m)}^l U_r^l} \\ &\quad \times \sum_{i=0}^{l-1} \{ [(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}\alpha^{r(m-p)} \\ &\quad - (W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^{l-1-i} \\ &\quad [(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n-k)}\beta^{r(m-p)} \\ &\quad - (W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^i \} \end{aligned}$$

Hence we have

$$\begin{aligned}
& \sum_{n=k}^t \frac{Q^{r(n+p)}}{W_{r(n+p)}^l W_{r(n+m)}^l} \sum_{i=0}^{l-1} \{ [(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}\alpha^{r(m-p)} \\
& - (W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^{l-1-i} \\
& \times [(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}\beta^{r(m-p)} \\
& - (W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^i \} \\
& = \frac{U_r^l Q^{rk}}{(W_{r(k+1)} - W_{rk}\beta^r)U_{r(m-p)}} \sum_{n=k}^t (T_{r(n+p)}^l - T_{r(n+m)}^l) \\
& = \frac{U_r^l Q^{rk}}{(W_{r(k+1)} - W_{rk}\beta^r)U_{r(m-p)}} \sum_{i=p}^{m-1} \left(\frac{\beta^{rli}}{W_{r(k+i)}^l} - \frac{\beta^{r(t+i+1-k)}}{W_{r(t+i+1)}^l} \right).
\end{aligned}$$

So we have the conclusion. \square

As an example, when $l = 1$ in Theorem 3, one can obtain

$$\sum_{n=k}^t \frac{Q^{r(n+p)}}{W_{r(n+p)} W_{r(n+m)}} = \frac{U_r Q^{rk}}{(W_{r(k+1)} - W_{rk}\beta^r)U_{r(m-p)}} \sum_{i=p}^{m-1} \left(\frac{\beta^{ri}}{W_{r(k+i)}} - \frac{\beta^{r(t+i+1-k)}}{W_{r(t+i+1)}} \right).$$

Theorem 4. For $P > 0$ and $m > p$,

$$\begin{aligned}
& \sum_{n=k}^{\infty} \frac{Q^{r(n+p)}}{W_{r(n+p)}^l W_{r(n+m)}^l} \sum_{i=0}^{l-1} \{ [(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}\alpha^{r(m-p)} \\
& - (W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^{l-1-i} \\
& \times [(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}\beta^{r(m-p)} \\
& - (W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^i \} \\
& = \frac{U_r^l Q^{rk}}{(W_{r(k+1)} - W_{rk}\beta^r)U_{r(m-p)}} \sum_{i=p}^{m-1} \frac{\beta^{rli}}{W_{r(k+i)}^l}.
\end{aligned}$$

Proof. Considering (2.4) and (2.5), we have

$$\begin{aligned}
& \sum_{n=k}^t \frac{Q^{r(n+p)}}{W_{r(n+p)}^l W_{r(n+m)}^l} \sum_{i=0}^{l-1} \{ [(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}\alpha^{r(m-p)} \\
& - (W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^{l-1-i} \\
& \times [(W_{r(k+1)} - W_{rk}\beta^r)Q^{r(n+p-k)}\beta^{r(m-p)} \\
& - (W_{r(k+1)} - W_{rk}\alpha^r)\beta^{2r(n-k)+r(m+p)}]^i \} \\
& = \frac{U_r^l Q^{rk}}{(W_{r(k+1)} - W_{rk}\beta^r)U_{r(m-p)}} \sum_{i=p}^{m-1} \left(\frac{\beta^{rli}}{W_{r(k+i)}^l} - \lim_{t \rightarrow \infty} \frac{\beta^{r(t+i+1-k)}}{W_{r(t+i+1)}^l} \right) \\
& = \frac{U_r^l Q^{rk}}{(W_{r(k+1)} - W_{rk}\beta^r)U_{r(m-p)}} \sum_{i=p}^{m-1} \frac{\beta^{rli}}{W_{r(k+i)}^l}.
\end{aligned}$$

Thus the proof is complete. \square

For example, by $W_n(0, 1, 1, -1) = F_n$, from Theorem 4, we get

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}F_{2n+4}} = \frac{8-3\sqrt{5}}{9}.$$

Also for $W_n(2, 1, 1, -1) = L_n$, we have

$$\sum_{n=1}^{\infty} \left(\frac{1}{L_{2n+2}^2 L_{2n+6}^2} \left(\frac{(35+21\sqrt{5})}{2} - (40 - 18\sqrt{5}) \beta^{4n} \right) \right) = \frac{3953}{31752} - \frac{5287}{95256} \sqrt{5}$$

and

$$\begin{aligned} \sum_{n=1}^t \frac{1}{L_{4(n+3)}L_{4(n+6)}} &= \left(\frac{1}{480} \sqrt{5} - \frac{16125092335}{3461453534098} \right) \\ &\quad - \left(\frac{7}{1440} \sqrt{5} - \frac{1}{96} \right) \left(\frac{\beta^{4(t+3)}}{L_{4(t+4)}} + \frac{\beta^{4(t+4)}}{L_{4(t+5)}} + \frac{\beta^{4(t+5)}}{L_{4(t+6)}} \right) \end{aligned}$$

Also choosing by appropriate parameter, many special cases can be obtained.

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