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On the order-k generalized Lucas numbers

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Abstract

In this paper we give a new generalization of the Lucas numbers in matrix representation. Also we present a relation between the generalized order-k Lucas sequences and Fibonacci sequences.

Keywords: Fibonacci numbers; Lucas numbers; The generalized order-k Lucas numbers

1. Introduction

It is well known that $\{F_n\}$, the Fibonacci sequence is defined by a recurrence relation, that is, $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 3$ such that $F_1 = 1$ and $F_2 = 1$. Also, one can obtain the Fibonacci sequence by matrix methods. Indeed, it is clear that

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Kalman [1] mentioned that this is a special case of a sequence which is defined recursively as a linear combination of the preceding k terms:

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$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [1], Kalman obtained a number of closed-form formulas for the generalized sequence by matrix method.

In [2] Er defined k sequences of the generalized *order-k Fibonacci* numbers as shown: for $n > 0$ and $1 \leqslant i \leqslant k$

$$g_n^i = \sum_{j=1}^{k} c_j g_{n-j}^i \tag{1}$$

with boundary conditions for $1 - k \leqslant n \leqslant 0$,

$$g_n^i = \begin{cases} 1 & \text{if } i = 1 - n, \\ 0 & \text{otherwise}, \end{cases} \tag{2}$$

where c_j, $1 \leqslant j \leqslant k$, are constant coefficients, g_n^i is the nth term of ith sequence.

Er showed that

$$\begin{bmatrix} g_{n+1}^i \\ g_n^i \\ \vdots \\ g_{n-k+2}^i \end{bmatrix} = A \begin{bmatrix} g_n^i \\ g_{n-1}^i \\ \vdots \\ g_{n-k+1}^i \end{bmatrix}, \tag{3}$$

where

$$A = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_{k-1} & c_k \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{4}$$

being a $k \times k$ companion matrix. Then he derived

$$G_{n+1} = AG_n, \tag{5}$$

where

$$G_n = \begin{bmatrix} g_n^1 & g_n^2 & \cdots & g_n^k \\ g_{n-1}^1 & g_{n-1}^2 & \cdots & g_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ g_{n-k+1}^1 & g_{n-k+1}^2 & \cdots & g_{n-k+1}^k \end{bmatrix}, \tag{6}$$

generalizing the matrix equation in (3).

Recently, Karaduman [3] showed that $G_1 = A$, and therefore, $G_n = A^n$. Also he proved that

$$\det G_n = \begin{cases} (-1)^n & \text{if } k \text{ is even}, \\ 1 & \text{if } k \text{ is odd}. \end{cases}$$

In this paper we give an order-k generalization of the usual Lucas sequence $\{L_n\}$, which is defined recursively as $L_n = L_{n-1} + L_{n-2}$ for $n \geqslant 3$ with initial condition $L_1 = 1$ and $L_2 = 3$. We present a matrix representation of the order-k generalization of the Lucas sequence. Then we calculate the determinant of the matrix obtained by our sequence. Furthermore, we find a relation between G_n obtained by the sequence of the generalized order-k Fibonacci numbers and the matrix obtained by the sequence of the generalized order-k Lucas numbers.

2. The main results

Define k sequences of the generalized order-k Lucas numbers as shown:

$$l_n^i = \sum_{j=1}^{k} l_{n-j}^i, \tag{7}$$

for $n > 0$ and $1 \leqslant i \leqslant k$, with boundary (initial) conditions

$$l_n^i = \begin{cases} 2 & \text{if } i = 2 - n, \\ -1 & \text{if } i = 1 - n, \\ 0 & \text{otherwise}, \end{cases} \tag{8}$$

for $1 - k \leqslant n \leqslant 0$, where l_n^i is the nth term of the ith sequence.

When $i = 1$ and $k = 2$, the generalized order-k Lucas sequence reduces to negative usual Fibonacci sequence, i.e., $l_n^1 = -F_{n+1}$ for all $n \in \mathbb{Z}^+$.

When we choose $k = 4$ and $i = 3$, the generalized order-k Lucas sequence is as follows:

$$\ldots, l_{-2}^3 = -1, l_{-1}^3 = 2, l_0^3 = 0, l_1^3 = 1, l_2^3 = 2, l_3^3 = 5, l_4^3 = 8,$$
$$l_5^3 = 16, l_6^3 = 31, \ldots$$

By (7), we can write

$$\begin{bmatrix} l_{n+1}^i \\ l_n^i \\ l_{n-1}^i \\ \vdots \\ l_{n-k+2}^i \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} l_n^i \\ l_{n-1}^i \\ l_{n-2}^i \\ \vdots \\ l_{n-k+1}^i \end{bmatrix}, \tag{9}$$

for the generalized order-k Lucas sequences. Letting

$$A = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}. \tag{10}$$

To deal with the k sequences of the generalized order-k Lucas sequences simultaneously, we define a $k \times k$ matrix H_n as follows:

$$H_n = \begin{bmatrix} l_n^1 & l_n^2 & \cdots & l_n^k \\ l_{n-1}^1 & l_{n-1}^2 & \cdots & l_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ l_{n-k+1}^1 & l_{n-k+1}^2 & \cdots & l_{n-k+1}^k \end{bmatrix}. \tag{11}$$

Generalizing Eq. (9), we have

$$H_{n+1} = A \cdot H_n. \tag{12}$$

Lemma 1. *Let A and H_n be as in* (10) *and* (11), *respectively. Then $H_{n+1} = A^n \cdot H_1$, where*

$$H_1 = \begin{bmatrix} -1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 2 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}.$$

Proof. By (12), we have $H_{n+1} = AH_n$. Then by induction and a property of matrix multiplication, we have

$$H_{n+1} = A^n \cdot H_1.$$

Also $H_1 = A \cdot K$, where

$$K = \begin{bmatrix} -1 & 2 & 0 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{bmatrix}. \tag{13}$$

Thus, $H_{n+1} = A^{n+1} \cdot K$. $\square$

Theorem 2. *Let* H_n *be as in* (11). *Then*

$$\det H_{n+1} = \begin{cases} -1 & \textit{if } k \textit{ is odd}, \\ (-1)^{n+1} & \textit{if } k \textit{ is even}. \end{cases}$$

Proof. From Lemma 1, we have $H_{n+1} = A^{n+1} \cdot K$. Then

$$\det H_{n+1} = (\det A)^{n+1} \cdot \det K,$$

where

$$\det A = (-1)^{k+1} \quad \text{and} \quad \det K = (-1)^k.$$

Thus

$$\det H_{n+1} = \begin{cases} -1 & \text{if } k \text{ is odd}, \\ (-1)^{n+1} & \text{if } k \text{ even}. \end{cases}$$

The following theorem presents a relation between the generalized order-k Lucas sequence and Fibonacci sequence, which was given by Er in [2]. □

Theorem 3. *Let* G_n *and* H_n *be as in* (6) *and* (11), *respectively. Then* $H_n = G_n \cdot K$, *where* K *is the* $k \times k$ *matrix in* (13).

Proof. In [2] Er, showed that $G_n = A^n$. Also we obtain $H_n = A^n \cdot K$. Thus we have

$$H_n = G_n \cdot K. \qquad \square$$

In Theorem 3, letting $k = 2$. Then we have

$$\begin{bmatrix} l_n^1 & l_n^2 \\ l_{n-1}^1 & l_{n-2}^2 \end{bmatrix} = \begin{bmatrix} g_n^1 & g_n^2 \\ g_{n-1}^1 & g_{n-2}^2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}.$$

Therefore, $l_n^2 = 2g_n^1 - g_n^2$. Since $g_n^1 = g_{n+1}^2$ for all $n \in \mathbb{Z}$, we have $l_n^2 = 2g_{n+1}^2 - g_n^2$, where l_n^2 and g_n^2 is the usual Lucas and Fibonacci numbers, respectively.

Indeed, we generalize a relation Lucas and Fibonacci numbers, i.e., $L_n = 2F_{n+1} - F_n$ (see p. 176, [4]).

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