ON THE USUAL FIBONACCI AND GENERALIZED ORDER-k PELL NUMBERS

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Abstract. In this paper, we give some relations involving the usual Fibonacci and generalized order-k Pell numbers. These relations show that the generalized order-k Pell numbers can be expressed as the summation of the usual Fibonacci numbers. We find families of Hessenberg matrices such that the permanents of these matrices are the usual Fibonacci numbers, F_{2i-1}, and their sums. Also extending these matrix representations, we find families of super-diagonal matrices such that the permanents of these matrices are the generalized order-k Pell numbers and their sums.

1. Introduction

The well-known Fibonacci sequence $\{F_n\}$ is defined by the following recursive relation, for $n > 2$,

$$F_n = F_{n-1} + F_{n-2}.$$

with initial conditions $F_1 = F_2 = 1$.

The Pell sequence $\{P_n\}$ is defined recursively by the equation, for $n > 2$

$$P_n = 2P_{n-1} + P_{n-2} \tag{1.1}$$

where $P_1 = 1$, $P_2 = 2$.

In [5], Ercolano gave the matrix method for generating the Pell sequence as follows:

$$M^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} \tag{1.2}$$

The *permanent* of an n-square matrix $A = (a_{ij})$ is defined by

$$perA = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ of the symmetric group S_n.

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In [20], Minc defined the $n \times n$ super-diagonal $(0,1)$-matrix $F(n,r)$ and showed that the permanent of matrix $F(n,r)$ equals to the generalized order-k Fibonacci number. Also in [18], the author proved the same result of [20] by a different method, the contraction method for permanent of a matrix. In [11], the authors gave the generalized Binet formula and combinatorial representations of the generalized order-k Fibonacci and Lucas numbers. Many studies have been done by several authors about the relationships between the linear recurrence sequences and the permanent or determinant of matrices (for example see [5-12]). Furthermore, in [19], Lehmer gave the relationships between permanent of tridiagonal matrices, recurrence relations, and continued fractions. In [4] and [3], the family of tridiagonal matrices $H(n)$ is defined and the authors show that the determinants of $H(n)$ are the Fibonacci numbers F_n. In a similar family of matrices, the $(1,1)$ element of $H(n)$ is replaced with a 3, then the determinants, [2], now generate the Lucas sequence L_n. Also in [21] and [22], the authors define a family of tridiagonal matrices $M(n)$ and show that the determinants of $M(n)$ are the Fibonacci numbers F_{2n+2}. In [17], the authors showed that the relationships between the tridiagonal determinants and the second order linear recurrences. Then the authors gave the factorizations of these recurrences by considering the determinant of these matrices by product of theirs eigenvalues.

Define k sequences of the generalized order-k Pell numbers as shown [16]:

$$P_n^i = 2P_{n-1}^i + P_{n-2}^i + \ldots + P_{n-k}^i \tag{1.3}$$

for $n > 0$ and $1 \le i \le k$, with initial conditions

$$P_n^i = \begin{cases} 1 & \text{if } i = 1-n, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } 1-k \le n \le 0,$$

where P_n^i is the nth term of the ith sequence. When $k = 2$, the generalized order-k Pell sequence, $\left\{P_n^k\right\}$, is reduced to the usual Pell sequence.

When $i = k$ in (1.3), we call P_n^k the generalized k-Pell number.

For example, if $i = 4$, then $P_{-3}^4 = 1,\ P_{-2}^4 = P_{-1}^4 = P_0^4 = 0$, and then the generalized order-4 Pell sequence is

$$1, 2, 5, 13, 34, 88, 228, \ldots.$$

The fundamental recurrence relation (1.3) can be defined by the vector recurrence relation

$$\begin{bmatrix} P_{n+1}^i \\ P_n^i \\ P_{n-1}^i \\ \vdots \\ P_{n-k+2}^i \end{bmatrix} = \begin{bmatrix} 2 & 1 & \ldots & 1 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix} \begin{bmatrix} P_n^i \\ P_{n-1}^i \\ P_n^i \\ \vdots \\ P_{n-k+1}^i \end{bmatrix} \tag{1.4}$$

for the generalized order-k Pell sequences. Letting

$$R=[r_{ij}]_{k\times k}=\begin{bmatrix} 2 & 1 & \dots & 1 & 1\\ 1 & 0 & \dots & 0 & 0\\ 0 & 1 & \dots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \dots & 1 & 0\end{bmatrix}, \tag{1.5}$$

the matrix R is said to be generalized order-k Pell matrix.

To deal with the k sequences of the generalized order-k Pell sequences simultaneously, we define an $k\times k$ matrix E_n as follows:

$$E_n=[e_{ij}]_{k\times k}=\begin{bmatrix} P_n^1 & P_n^2 & \dots & P_n^k\\ P_{n-1}^1 & P_{n-1}^2 & \dots & P_{n-1}^k\\ \vdots & \vdots & & \vdots\\ P_{n-k+1}^1 & P_{n-k+1}^2 & \dots & P_{n-k+1}^k\end{bmatrix}. \tag{1.6}$$

Generalizing Eq. (1.4), we derive

$$E_{n+1}=R\cdot E_n. \tag{1.7}$$

Since $E_1=R$, the following result is immediate:

$$E_n=R^n$$

Also the following property of the generalized order-k Pell numbers can be found in [16]: Let P_n^i be the generalized order-k Pell number, for $1\le i\le k$. Then the following result is deduced immediately from the fact that $P_{n+m}^i=e_1^TE_mE_ne_i$ for all positive integers n and m

$$P_{n+m}^i=\sum_{j=1}^{k}P_m^jP_{n-j+1}^i. \tag{1.8}$$

For example, if we take $k=i=2$ in the Eq. (1.8), we have

$$P_{n+m}^2=P_m^1P_n^2+P_m^2P_{n-1}^2$$

and, since $P_n^1=P_{n+1}^2$ for all $n\in\mathbb{Z}^+$ and $k=2$, we obtain

$$P_{n+m}^2=P_{m+1}^2P_n^2+P_m^2P_{n-1}^2$$

where P_n^2 is the usual Pell number. Indeed, we generalize the following relation involving the usual Pell numbers (see [6]):

$$P_{n+m}=P_{m+1}P_n+P_mP_{n-1}.$$

The purpose of this paper is to derive relationships between the generalized order-k Pell numbers, the usual Fibonacci numbers, and their sums, and, the permanents of $(0,1,2)$-Hessenberg and super-diagonal matrices. The paper also presents unexpected relations involving the generalized order$-k$ Pell and usual Fibonacci numbers.

2. On the Relations of the Generalized Order−k Pell and Usual Fibonacci numbers

In this section, we show that the generalized order-k Pell numbers can be written in terms of the usual Fibonacci numbers. From the definition of order-k Pell numbers, we write that

$$\begin{aligned}
P_1^k &= 2P_0^k + P_{-1}^k + \ldots + P_{1-k}^k = 1, \\
P_2^k &= 2P_1^k + P_0^k + \ldots + P_{2-k}^k = 2\,(1) = 2, \\
P_3^k &= 2P_2^k + P_1^k + \ldots + P_{3-k}^k = 2\,(2) + 1 = 5, \\
P_4^k &= 2P_3^k + P_2^k + \ldots + P_{4-k}^k = 2\,(5) + 2 + 1 = 13, \\
P_5^k &= 2P_4^k + P_3^k + \ldots + P_{5-k}^k = 2\,(13) + 5 + 2 + 1 = 34, \ldots.
\end{aligned}$$

By the definition of the usual Fibonacci numbers, we know that

$$F_1 = 1, \quad F_3 = 2, \quad F_5 = 5, \quad F_7 = 13, \quad F_9 = 34, \quad \ldots.$$

Thus it is seen that

$$\begin{aligned}
P_1^k &= 1 = F_1,\ P_2^k = 2 = F_3, \\
P_3^k &= 5 = F_5,\ P_4^k = 13 = F_7, \\
P_5^k &= 34 = F_9
\end{aligned}$$

and

$$P_j^k = F_{2j-1} \quad \text{for } 1 \le j \le k+1. \tag{2.1}$$

This process continuous the same as the above with small changes as regularly for $k+2 \le j \le 2k+1$. By the formula (2.1), we can write that

$$\begin{aligned}
P_{k+2}^k &= 2P_{k+1}^k + P_k^k + \ldots + P_2^k \\
&= 2F_{2k+1} + F_{2k-1} + \ldots + F_3. \qquad (2.2)
\end{aligned}$$

From [23], the famous summation formula

$$\sum_{i=1}^{n} F_{2i-1} = F_{2i} \tag{2.3}$$

is well-known. Thus we can write the formula (2.2) by using the formula (2.3)

$$\begin{aligned}
P_{k+2}^k &= F_{2k+1} + F_{2k+1} + F_{2k-1} + \ldots + F_3 + F_1 - F_1 \\
&= F_{2k+1} + \sum_{i=1}^{k+1} F_{2i-1} - F_1 = F_{2k+1} + F_{2k+2} - F_1 = F_{2k+3} - F_1 \qquad (2.4)
\end{aligned}$$

By the Eqs. (2.1), (2.3) and (2.4),

$$\begin{aligned} P^k_{k+3} &= 2P^k_{k+2} + P^k_{k+1} + \ldots + P^k_3 \\ &= 2(F_{2k+3} - F_1) + F_{2k+1} + F_{2k-1} + \ldots + F_5 \end{aligned}$$

or equivalently

$$\begin{aligned} P^k_{k+3} &= F_{2k+3} + F_{2k+3} + F_{2k+1} + F_{2k-1} + \ldots + F_5 + F_3 + F_1 - (F_3 + F_1 + 2F_1) \\ &= F_{2k+3} + \sum_{i=1}^{k+2} F_{2i-1} - (F_3 + 3F_1) = F_{2k+3} + F_{2k+4} - (F_3 + 3F_1) \\ &= F_{2k+5} - (F_3 + 3F_1)\,. \end{aligned} \tag{2.5}$$

Combining the Eqs. (2.1), (2.4), (2.5) and (2.3), we write that

$$\begin{aligned} P^k_{k+4} &= 2P^k_{k+3} + P^k_{k+2} + \ldots + P^k_4 \\ &= 2\left(F_{2k+5} - (F_3 + 3F_1)\right) + (F_{2k+3} - F_1) + F_{2k+1} + \ldots + F_7 \end{aligned}$$

and by some arrangements

$$\begin{aligned} P^k_{k+4} &= 2F_{2k+5} + F_{2k+3} + F_{2k+1} + \ldots + F_7 - (2F_3 + 6F_1 + F_1) \\ &= F_{2k+5} + \sum_{i=1}^{k+3} F_{2i-1} - (F_5 + F_3 + F_1) - (2F_3 + 6F_1 + F_1) \\ &= F_{2k+5} + F_{2k+6} - (F_5 + 3F_3 + 8F_1) \\ &= F_{2k+7} - (F_5 + 3F_3 + 8F_1)\,. \end{aligned}$$

We can shortly write the term

$$P^k_{k+5} = F_{2k+9} - (F_7 + 3F_5 + 8F_3 + 21F_1)\,.$$

Since $F_2 = 1,\ F_4 = 3,\ F_6 = 8,\ F_8 = 21$, we can rewrite the above terms as follows:

$$\begin{aligned} P^k_{k+2} &= F_{2k+3} - F_2F_1, \\ P^k_{k+3} &= F_{2k+5} - (F_2F_3 + F_4F_1)\,, \\ P^k_{k+4} &= F_{2k+7} - (F_2F_5 + F_4F_3 + F_6F_1)\,, \\ P^k_{k+5} &= F_{2k+9} - (F_2F_7 + F_4F_5 + F_6F_3 + F_8F_1) \end{aligned}$$

and in general, for $k + 2 \le j \le 2k + 1$

$$\begin{aligned} P^k_{2k+1} &= F_{4k+1} - (F_2F_{2k-1} + F_4F_{2k-3} + F_6F_{2k-5} + \ldots + F_{2k-2}F_3 + F_{2k}F_1) \\ &= F_{4k+1} - \sum_{j=1}^{k} F_{2j-1}F_{2(k+1-j)} \end{aligned}$$

or more conveniently, we may write that, for $1 \le t \le k$

$$P^k_{k+1+t} = F_{2(k+1+t)-1} - \sum_{i=1}^{t} F_{2i-1} F_{2(t+1-i)}.$$

So we show that the generalized order-k Pell numbers, P^k_j, can be represented by the usual Fibonacci numbers, F_j, for $1 \le j \le 2k+1$. Also we note that these representations can be extended for more j, $j \ge 2k+2$. However the computings are very large and not easy. Moreover, just now we can say that the above rule can be continued by some changes.

A matrix is said to be a $(0,1,2)$-matrix if each of its entries is either $0, 1$ or 2.

3. The Fibonacci Numbers By Hessenberg Matrices Permanents

In this section we define a class of Hessenberg matrices. Then we show that the permanent of Hessenberg matrices equal to the usual Fibonacci numbers, F_{2n+1}.

Let $A = [a_{ij}]$ be an $m \times n$ real matrix row vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$. We say A is *contractible on column* (resp. *row*.) k *if column* (resp. *row*.) k contains exactly two nonzero entries. Suppose A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A by replacing row i with $a_{jk}\alpha_i + a_{ik}\alpha_j$ and deleting row j and column k is called *the contraction of A on column k relative to rows i and j*. If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = \left[A^T_{ij:k}\right]^T$ is called *the contraction of A on row k relative to columns i and j*. Every contraction used in this paper will be on the first column using the first and second rows. We say that A can be contracted to a matrix B if either $B = A$ or exist matrices $A_0, A_1, \ldots A_t$ $(t \ge 1)$ such that $A_0 = A$, $A_t = B$, and A_r is a contraction of A_{r-1} for $r = 1, 2, \ldots, t$.

Now we consider the following Lemma (see [1]).

Lemma 1. *Let A be a nonnegative integral matrix of order $n > 1$ and let B be a contraction of A. Then*

$$per A = per B. \tag{3.1}$$

We define an $n \times n$ upper Hessenberg matrix $H_n = (h_{ij})$ with $h_{ii} = 2$ for $1 \le i \le n$, $h_{i+1,i} = 1$ for $1 \le i \le n-1$ and $h_{ij} = 1$ for $j > i$. Clearly

$$H_n = \begin{bmatrix} 2 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 2 & \cdots & 1 & 1 \\ 0 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 \end{bmatrix}. \tag{3.2}$$

Theorem 1. *Let the Hessenberg matrix H_n be as in (3.2). Then for $n > 1$*

$$perH_n = F_{2n+1}$$

where F_n is the nth Fibonacci number.

Proof. Let $H_n^0 = H_n$, and note that the top row of H_n^0 can be written as $\left[\begin{array}{cccc} F_3 & F_2 & \dots & F_2 \end{array}\right]$. For each $1 \leq i \leq n-2$, form H_n^i from H_n^{i-1} by contracting on its first column. A straightforward proof by induction shows that for each such i, the top row of the $(n-i) \times (n-i)$ matrix H_n^i is $\left[\begin{array}{cccc} F_{2i+3} & F_{2i+2} & \dots & F_{2i+2} \end{array}\right]$, while the remaining rows of H_n^i agree with those of H_{n-i}. It now follows that

$$per\left(H_n\right) = per\left(H_n^{n-2}\right) = 2F_{2n-1} + F_{2n-2} = F_{2n+1}.$$

□

Now we extend the Hessenberg matrix H_n to a super-diagonal matrix. Then we show that permanent of super-diagonal matrix equals to the generalized order-k Pell numbers in the next section.

4. The Generalized Order-k Pell Numbers

Now we show the relationships between the generalized order-k Pell numbers and $(0,1,2)$ super-diagonal matrices.

We define an $n \times n$ $(k+1)^{st}$ super-diagonal $(0,1,2)$-matrix $S(k,n) = (s_{ij})$, $k \leq n$, with $s_{i+1,i} = 1$ for $1 \leq i \leq n-1$, $s_{ii} = 2$ for $1 \leq i \leq n$ and $s_{ij} = 1$ for $i+1 \leq j \leq i+k-1$. Clearly

$$S(k,n) = \left[\begin{array}{ccccccccccc}
2 & 1 & 1 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\
1 & 2 & 1 & 1 & \dots & 1 & 1 & 0 & 0 & \dots & 0 \\
0 & 1 & 2 & 1 & \dots & 1 & 1 & 1 & 0 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & \dots & 0 & 1 & 2 & 1 & \cdots & 1 & 1 & 1 & 0 \\
0 & \dots & \dots & 0 & 1 & 2 & 1 & \cdots & 1 & 1 & 1 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 2 & 1 \\
0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 2
\end{array}\right] \tag{4.1}$$

where $s_{ii} = 2$, $s_{12} = s_{13} = \ldots = s_{1k} = 1$ and $s_{1k+1} = \ldots = s_{1n} = 0$, then $S(k,n)$ is contractible on column 1 relative to the rows 1 and 2.

Theorem 2. *Let the super-diagonal matrix $S(k,n)$ be as in (4.1). Then, for $n > 1$*

$$perS(k,n) = P_{n+1}^k$$

where P_n^k is the nth generalized order-k Pell number.

Proof. We will prove that $perS\left(k,n\right) = P_{n+1}^{k}$ by induction method on t. We consider two cases. Firstly, if $1 \leq t \leq k$ and $k = n$, then the matrix $S\left(k,n\right)$ is reduced to the matrix $S\left(t,t\right)$ which equals to the Hessenberg matrix H_t given by (3.2). From Theorem 1, we know that $perH_t = F_{2t+1}$ and from (2.1), we know that $F_{2t+1} = P_{t+1}^{k}$ for $1 \leq t \leq k+1$. Thus we obtain that

$$perS\left(t,t\right) = perH_t = P_{t+1}^{k}. \tag{4.2}$$

We now consider the second case; let $k < n$ and $k+1 \leq t \leq n$. If $t = k+1$ and we compute the $perS\left(k,k+1\right)$ by the Laplace expansion of the permanent with respect to the first row, then we have

$$perS\left(k,k+1\right) = 2perS\left(k,k\right) + perS\left(k,k-1\right) + \ldots + perS\left(k,1\right)$$

and by (4.2), we can write that

$$perS\left(k,k+1\right) = 2P_{k+1}^{k} + P_{k}^{k} + P_{k-1}^{k} + \ldots + P_{2}^{k}$$

and by (1.3), we obtain

$$perS\left(k,k+1\right) = P_{k+2}^{k}.$$

We suppose that the equation holds for t and $k+1 \leq t \leq n$, then we have

$$perS\left(k,t\right) = P_{t+1}^{k}. \tag{4.3}$$

Now we show that the equation holds for $t+1$. Computing $perS\left(k,t+1\right)$ by the Laplace expansion of the permanent with respect to the first row, we obtain for $k+1 \leq t \leq n$

$$perS\left(k,t+1\right) = 2perS\left(k,t\right) + perS\left(k,t-1\right) + \ldots + perS\left(k,t-k+1\right)$$

and by (4.3) and (1.3), we have

$$perS\left(k,t+2\right) = 2P_{t+1}^{k} + P_{t}^{k} + P_{t-1}^{k} + \ldots + P_{t-k+2}^{k} = P_{t+2}^{k}.$$

So the proof is complete. □

5. Sums of the generalized Pell numbers by matrix methods

In this section, we give the sums of Fibonacci numbers, $\sum\limits_{i=0}^{n-1} F_{2i+1}$, and sums of generalized order-k Pell numbers, $\sum\limits_{i=0}^{n-1} P_{i}^{k}$, by the permanents of two square matrices.

Firstly, we define an $n \times n$ upper $(0,1,2)$-Hessenberg matrix $W_n = (w_{ij})$ with $w_{1j} = 1$ for $1 \leq j \leq n$, $w_{ii} = 2$ for $2 \leq i \leq n$, $w_{i+1,i} = 1$ for

$1 \leq i \leq n-1$ and $w_{ij} = 1$ for $j > i > 1$. Clearly,

$$W_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 2 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 2 \end{bmatrix}. \tag{5.1}$$

By the definition of W_n, it is easily seen that

$$W_{n+1} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & H_n & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}.$$

where H_n is given by (3.2).

Then we have the following Theorem.

Theorem 3. *Let W_n has the form (5.1) and F_n is nth Fibonacci number. Then for $n > 1$*

$$perW_n = F_{2n}.$$

Proof. From Theorem 2, we have $per\,(H_n) = F_{2n+1}$. Expanding the permanent of W_{n+1} along the first column, we have

$$per\,(W_{n+1}) = per\,(W_n) + per\,(H_n) = per\,(W_n) + F_{2n+1}.$$

The conclusion now follows by a simple induction proof. □

We note that by (2.1) and Theorem 4, we have that

$$perW_k = \sum_{j=0}^{k-1} F_{2j+1} = \sum_{j=0}^{k-1} P_{j+1}^k.$$

Second, we define the $n \times n$ matrix $V(k,n)$ as follow

$$V(k,n) = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & 1 & \dots & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 2 & 1 & \dots & 1 & 1 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & 2 & 1 & \cdots & 1 & 1 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 & 2 & 1 & \cdots & 1 & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 2 & 1 \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 2 \end{bmatrix}, \tag{5.2}$$

or by the definition of $S(k,n)$, we write that

$$V(k,n)=\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & S(k,n-1) & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}$$

where $S(k,n)$ is given by (4.1).

Theorem 4. *Let $V(k,n)$ has the form (5.2) and P_n^k is the nth generalized Pell number. Then for $n>1$*

$$perV(k,n)=\sum_{j=1}^{n} P_j^k.$$

Proof. (Induction on n.) If $n=2$, then we have

$$perV(k,2)=per\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}=3.$$

From the definition of the generalized Pell numbers, we know that $P_1^k=1$ and $P_2^k=2$. Thus $perV(k,2)=P_1^k+P_2^k=3$.

If $n=3$, then we have

$$perV(k,n)=per\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}=8.$$

Since $P_1^k=1$, $P_2^k=2$ and $P_3^k=5$, $perV(k,3)=P_1^k+P_2^k+P_3^k$.

We suppose that the equation holds for n. Now we show that the equation holds for $n+1$. Computing $perV(k,n+1)$ by the element of first column, gives us

$$perV(k,n+1) = per\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & 2 & 1 & \dots & 1 & 0 \\ \dots & \dots & \dots & 0 & 1 & 2 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & 0 & 1 & 2 & 1 \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 2 \end{bmatrix}$$

$$+per\begin{bmatrix} 2 & 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ 1 & 2 & 1 & \dots & 1 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & 2 & 1 & \dots & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 & 2 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & 0 & 1 & 2 & 1 \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 2 \end{bmatrix}$$

which, by the definitions of $S(k,n)$ and $V(k,n)$, satisfy

$$perV(k,n+1) = perV(k,n) + perS(k,n).$$

By our assumption and Theorem 3, we obtain that

$$perV(k,n+1) = \sum_{j=1}^{n} P_j^k + P_{n+1}^k = \sum_{j=1}^{n+1} P_j^k.$$

So the proof is complete. □

A matrix A is called *convertible* if there is an $n \times n$ $(1,-1)$ $-$matrix H such that $perA = \det(A \circ H)$, where $A \circ H$ denotes the Hadamard product of A and H. Such a matrix H is called a *converter* of A.

Let T be a $(1,-1)$ $-$matrix of order n, defined by

$$T = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ -1 & 1 & 1 & \dots & 1 & 1 \\ 1 & -1 & 1 & \dots & 1 & 1 \\ 1 & 1 & -1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \dots & -1 & 1 \end{bmatrix}.$$

Then we have the following results.

Let F_n be the nth Fibonacci number. Then, for $n \geq 1$

$$F_{2n+1} = \det(H_n \circ T)$$

and

$$\sum_{j=0}^{n-1} F_{2j+1} = \det\left(W_n \circ T\right).$$

Let P_n^k be the nth generalized order-k Pell number. Then, for $n \geq 2$

$$P_{n+1}^k = \det\left(S_n^k \circ T\right)$$

and

$$\sum_{j=1}^{n} P_j^k = \det\left(V_n^k \circ T\right).$$

References

[1] R. A. Brualdi & P. M. Gibson, "Convex polyhedra of Doubly Stochastic matrices I: Applications of the Permanent Function." *J. Combin. Theory* A 22 (1977): 194-230.

[2] P. F. Byrd, "Problem B-12: A Lucas determinant." *Fib. Quart.* 1 (4) (1963): 78.

[3] N. D Cahill and D. A. Narayan, "Fibonacci and Lucas numbers as tridiagonal matrix determinants." *Fib. Quart.* 42 (3) (2004): 216-221.

[4] N. D. Cahill, J. R. D'Errica, D. A. Narayan and J. Y. Narayan, "Fibonacci determinants." *College Math. J.* 3 (3) (2002): 221-225.

[5] J. Ercolano, "Matrix generators of Pell sequences." *The Fibonacci Quart.*, 17 (1) (1979): 71-77.

[6] A. F. Horadam, "Pell identities." *The Fibonacci Quart.*, 9 (3) (1971): 245-252, 263.

[7] E. Kilic and D. Tasci, "On the second order linear recurrences by tridiagonal matrices." *Ars Combin.*, (to appear).

[8] E. Kilic and D. Tasci, "On the second order linear recurrence satisfied by the permanent of generalized Doubly Stochastic matrices." *Ars Combin.*, (to appear).

[9] E. Kilic and D. Tasci, "On the generalized Fibonacci and Pell sequences by Hessenberg matrices." *Ars Combin.*, (to appear).

[10] E. Kilic, D. Tasci and P. Haukkanen, "On the generalized Lucas sequences by Hessenberg matrices." *Ars Combin.*, (to appear).

[11] E. Kilic and D. Tasci, "On the Generalized Order-k Fibonacci and Lucas Numbers." *Rocky Mountain J. of Math.*, (to appear).

[12] E. Kilic and D. Tasci, "On the permanents of some tridiagonal matrices with applications to the Fibonacci and Lucas numbers." *Rocky Mountain J. of Math.*, (to appear).

[13] E. Kilic and D. Tasci, "On families of Bipartite graphs associated with sums of Fibonacci and Lucas numbers." *Ars Combin.* (to appear).

[14] E. Kilic and D. Tasci, "On families of Bipartite graphs associated with sums of generalized order-k Fibonacci and Lucas numbers." *Ars Combin.* (to appear).

[15] E. Kilic and D. Tasci, "On sums of second order linear recurrences by Hessenberg matrices." *Rocky Mountain J. of Math.*, (to appear).

[16] E. Kilic and D. Tasci, "The generalized Binet formula, representation and sums of the generalized order-k Pell numbers." *Taiwanese J. Math.*, (to appear).

[17] E. Kilic and D. Tasci, "Factorizations and representations of the backward second-order linear recurrences." *J. Comput. Appl. Math.*, (in press).

[18] G. Y. Lee, "A note on Generalized Fibonacci numbers." *The Fibonacci Quart.*, 33 (3) (1995): 273-278.

[19] D. Lehmer, "Fibonacci and related sequences in periodic tridiagonal matrices." *Fib. Quart.* 13 (1975): 150-158.

[20] H. Minc, "Permanents of $(0,1)$- Circulants." *Canad. Math. Bull.*, 7 (2) (1964): 253-263.

[21] G. Strang, *Introduction to Linear Algebra.* 2nd Edition, Wellesley MA: Wellesley-Cambridge, 1998.

[22] G. Strang, K. Borre, *Linear Algebra.* (pp. 555-557) Geodesy and GPS. Wellesley MA: Wellesley-Cambridge, 1997.

[23] S. Vajda, *Fibonacci & Lucas Numbers and The Golden Section*, New York:Chichester-Brisbane-Toronto, 1989.

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