

**ON FAMILIES OF BIPARTITE GRAPHS ASSOCIATED WITH  
SUMS OF FIBONACCI AND LUCAS NUMBERS**

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ABSTRACT. In this paper, we consider the relationships between the sums of the Fibonacci and Lucas numbers and 1-factors of bipartite graphs.

1. INTRODUCTION

The Fibonacci sequence,  $\{F_n\}$ , is defined by the recurrence relation, for  $n > 2$

$$F_n = F_{n-1} + F_{n-2}$$

where  $F_1 = F_2 = 1$ .

The Lucas Sequence,  $\{L_n\}$ , is defined by the recurrence relation, for  $n > 2$

$$L_n = L_{n-1} + L_{n-2}$$

where  $L_1 = 1, L_2 = 3$ .

The *permanent* of an  $n$ -square matrix  $A = (a_{ij})$  is defined by

$$\text{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i\sigma(i)}$$

where the summation extends over all permutations  $\sigma$  of the symmetric group  $S_n$ . A matrix is said to be a  $(0, 1)$ -matrix if each of its entries is either 0 or 1.

In [4], Minc constructed the  $n \times n$   $(0, 1)$ -matrix  $F(n, k)$  where,  $k \leq n + 1$ , with 1 in the  $(i, j)$  position for  $i - 1 \leq j \leq i + k - 1$  and 0 otherwise. Then he showed that  $\text{per} F(n, k) = g_{n+1}^k$  where  $g_n^k$  is the  $n$ th generalized order- $k$  Fibonacci number. When  $k = 2$ ,  $\text{per} F(n, 2) = F_{n+2}$ .

Also Lee defined the matrix  $\mathcal{L}_n$  as follows [3]:

$$\mathcal{L}_n = \begin{bmatrix} 1 & 0 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & & \vdots \\ 0 & 1 & 1 & 1 & \ddots & \\ 0 & 0 & 1 & 1 & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ & & & & & 0 \\ \vdots & & & & & 1 \\ 0 & \dots & 0 & \dots & 0 & 1 & 1 \end{bmatrix}$$

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2000 *Mathematics Subject Classification.* 11B39, 15A15, 15A36.

*Key words and phrases.* Fibonacci sequence; Lucas sequence; 1-factor; Permanent.

and showed that

$$\text{per}\mathcal{L}_n = L_{n-1}$$

where  $L_n$  is the  $n$ th Lucas number.

In this paper, we find families of square matrices such that (i) each matrix is the adjacency matrix of a bipartite graph; and (ii) the permanent of the matrix is a sum of consecutive Fibonacci or Lucas numbers.

A *bipartite graph*  $G$  is a graph whose vertex set  $V$  can be partitioned into two subsets  $V_1$  and  $V_2$  such that every edge of  $G$  joins a vertex in  $V_1$  and a vertex in  $V_2$ . A 1-factor (or *perfect matching*) of a graph with  $2n$  vertices is a spanning subgraph of  $G$  in which every vertex has degree 1. The enumeration or actual construction of 1-factors of a bipartite graph has many applications. Let  $A(G)$  be the adjacency matrix of the bipartite graph  $G$ , and let  $\mu(G)$  denote the number of 1-factors of  $G$ . Then, one can find the following fact in [5]:  $\mu(G) \leq \sqrt{\text{per}A(G)}$ . Also, one can find more applications of permanents in [5].

Let  $G$  be a bipartite graph whose vertex set  $V$  is partitioned into two subsets  $V_1$  and  $V_2$  such that  $|V_1| = |V_2| = n$ . We construct the *bipartite adjacent matrix*  $B(G) = [b_{ij}]$  of  $G$  as following:  $b_{ij} = 1$  if and only if  $G$  contains an edge from  $v_i \in V_1$  to  $v_j \in V_2$ , and 0 otherwise. Then, in [2] and [5], the number of 1-factors of bipartite graph  $G$  equals the permanent of its bipartite adjacency matrix.

Let  $A = [a_{ij}]$  be an  $m \times n$  real matrix row vectors  $\alpha_1, \alpha_2, \dots, \alpha_m$ . We say  $A$  is *contractible on column* (resp. *row*.)  $k$  if column (resp. *row*.)  $k$  contains exactly two nonzero entries. Suppose  $A$  is contractible on column  $k$  with  $a_{ik} \neq 0 \neq a_{jk}$  and  $i \neq j$ . Then the  $(m-1) \times (n-1)$  matrix  $A_{ij:k}$  obtained from  $A$  by replacing row  $i$  with  $a_{jk}\alpha_i + a_{ik}\alpha_j$  and deleting row  $j$  and column  $k$  is called *the contraction of  $A$  on column  $k$  relative to rows  $i$  and  $j$* . If  $A$  is contractible on row  $k$  with  $a_{ki} \neq 0 \neq a_{kj}$  and  $i \neq j$ , then the matrix  $A_{k:ij} = [A_{ij:k}^T]^T$  is called *the contraction of  $A$  on row  $k$  relative to columns  $i$  and  $j$* . Every contraction used in this paper will be on the first column using the first and second rows. We say that  $A$  can be contracted to a matrix  $B$  if either  $B = A$  or if there exist matrices  $A_0, A_1, \dots, A_t$  ( $t \geq 1$ ) such that  $A_0 = A$ ,  $A_t = B$ , and  $A_r$  is a contraction of  $A_{r-1}$  for  $r = 1, 2, \dots, t$ . One can find the following fact in [1]: let  $A$  be a nonnegative integral matrix of order  $n > 1$  and let  $B$  be a contraction of  $A$ . Then

$$\text{per}A = \text{per}B. \tag{1.1}$$

## 2. THE SUMS OF THE FIBONACCI NUMBERS

In this section, we determine a class of bipartite graphs whose number of 1-factors is the summation of the Fibonacci numbers,  $\sum_{i=0}^n F_i$ .

Let  $n$  be positive integer and  $n \geq 3$ . Let  $P_n = [p_{ij}]$  be the  $n \times n$   $(0, 1)$ -tridiagonal matrix with  $p_{ij} = 1$  if and only if  $|i - j| \leq 1$ . Let  $R_n = [r_{ij}]$  be the  $n \times n$   $(0, 1)$ -matrix with  $r_{1j} = 1$  if and only if  $3 \leq j \leq n$ . Now we consider the sum

of these matrices and denote by  $V_n = [s_{ij}] = P_n + R_n$ . Clearly

$$V_n = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & 0 & & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & & & & 0 \\ \vdots & & & & \ddots & \ddots & \ddots & 1 \\ 0 & & 0 & \dots & 0 & 1 & 1 \end{bmatrix}.$$

**Theorem 1.** *Let  $G(V_n)$  be the bipartite graph with bipartite adjacency matrix  $V_n = P_n + R_n$ ,  $n \geq 3$ . Then the number of 1-factors of  $G(V_n)$  is  $\sum_{i=0}^n F_i = F_{n+2} - 1$ .*

*Proof.* If  $n = 3$ , then we have

$$\text{per}V_3 = \text{per} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = F_0 + F_1 + F_2 + F_3 = 4.$$

Let  $V_n^k$  be the  $k$ th contraction of  $V_n$ ,  $1 \leq k \leq n - 2$ . Since the definition of the matrix  $V_n$ , the matrix  $V_n$  can be contracted on column 1 so that

$$V_n^1 = \begin{bmatrix} 2 & 2 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 0 & & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & \ddots & \ddots & \ddots & 1 \\ 0 & & & \dots & 0 & 1 & 1 \end{bmatrix}.$$

Since the matrix  $V_n^1$  can be contracted on column 1 and  $\sum_{i=0}^3 F_i = 4$  and  $F_4 = 3$ ,

$$\begin{aligned} V_n^2 &= \begin{bmatrix} 4 & 3 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 0 & & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & \ddots & \ddots & \ddots & 1 \\ 0 & & & \dots & 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=0}^3 F_i & F_4 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 0 & & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & \ddots & \ddots & \ddots & 1 \\ 0 & & & \dots & 0 & 1 & 1 \end{bmatrix}. \end{aligned}$$

Furthermore, the matrix  $V_n^2$  can be contracted on column 1 so that

$$V_n^3 = \begin{bmatrix} \sum_{i=0}^4 F_i & F_5 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 0 & & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 0 & & & \dots & 0 & 1 & 1 \end{bmatrix}.$$

Continuing this process, we have

$$V_n^k = \begin{bmatrix} \sum_{i=0}^{k+1} F_i & F_{k+2} & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 0 & & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 0 & & & \dots & 0 & 1 & 1 \end{bmatrix}$$

for  $3 \leq k \leq n-4$ . Hence,

$$V_n^{(n-3)} = \begin{bmatrix} \sum_{i=0}^{n-2} F_i & F_{n-1} & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

which, by contraction of  $V_n^{(n-4)}$  on column 1, gives

$$V_n^{(n-2)} = \begin{bmatrix} \sum_{i=0}^{n-1} F_i & F_n \\ 1 & 1 \end{bmatrix}.$$

By applying (1.1), we obtain  $\text{per}V_n = \text{per}V_n^{(n-2)} = \sum_{i=0}^n F_i$ .  $\square$

### 3. ON THE LUCAS NUMBERS AND THEIR SUMS

In this section, we determine two classes of bipartite graphs whose number of 1-factors are the the Lucas numbers and their sums,  $\sum_{i=0}^{n-2} L_i$ . Now we note that our result on the Lucas numbers is different from the result of Lee.

Firstly, let be positive integer such that  $n \geq 4$  and let  $C_n = [c_{ij}]$  be the  $n \times n$   $(0, 1)$ -matrix with  $c_{13} = c_{14} = 1$  and  $c_{ij} = 1$  for  $|i-j| \leq 1$  and 0 otherwise. Clearly

$$C_n = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & & & & 0 \\ \vdots & & & & & & 1 \\ 0 & \dots & & \dots & 0 & 1 & 1 \end{bmatrix}.$$

**Theorem 2.** *Let  $G(C_n)$  be the bipartite graph with bipartite adjacency matrix  $C_n$ ,  $n \geq 4$ . Then the number of 1-factors of  $G(C_n)$  is  $L_n$  where  $L_i$  is the  $i$ th Lucas number.*

*Proof.* If  $n = 4$ , then we have

$$\text{per}C_4 = \text{per} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = 7 = L_4.$$

Let  $C_n^k$  be the  $k$ th contraction of  $C_n$ ,  $1 \leq k \leq n - 2$ . Since the definition of the matrix  $C_n$ , the matrix  $C_n$  can be contracted on column 1 so that

$$C_n^1 = \begin{bmatrix} 2 & 2 & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & 0 & & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ 0 & 1 & 1 & 1 & 1 & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & & 1 \\ 0 & & \dots & 0 & 1 & 1 \end{bmatrix}.$$

Since the matrix  $S_n^1$  can be contracted on column 1 and  $L_2 = 3$ ,  $L_3 = 4$ ,

$$\begin{aligned} C_n^2 &= \begin{bmatrix} 4 & 3 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & & 0 \\ 0 & 1 & 1 & 1 & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & 1 \\ 0 & & \dots & 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} L_3 & L_2 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & & 0 \\ 0 & 1 & 1 & 1 & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & 1 \\ 0 & & \dots & 0 & 1 & 1 \end{bmatrix}. \end{aligned}$$

Furthermore, the matrix  $C_n^2$  can be contracted on column 1 so that

$$C_n^3 = \begin{bmatrix} L_4 & L_3 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & & 0 \\ 0 & 1 & 1 & 1 & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & 1 \\ 0 & & \dots & 0 & 1 & 1 \end{bmatrix}.$$

Continuing this process, we reach

$$C_n^k = \begin{bmatrix} L_{k+1} & L_k & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & & 0 \\ 0 & 1 & 1 & 1 & & \vdots \\ & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & 1 \\ 0 & & \dots & 0 & 1 & 1 \end{bmatrix}$$

for  $3 \leq k \leq n-4$ . Hence,

$$C_n^{(n-3)} = \begin{bmatrix} L_{n-2} & L_{n-3} & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

which, by contraction of  $C_n^{(n-4)}$  on column 1, gives

$$C_n^{(n-2)} = \begin{bmatrix} L_{n-3} + L_{n-2} & L_{n-2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} L_{n-1} & L_{n-2} \\ 1 & 1 \end{bmatrix}.$$

By applying (1.1), we obtain  $\text{per}C_n = \text{per}C_n^{(n-2)} = L_n$ .

So the proof is complete.  $\square$

Secondly, let  $n$  be positive integer such that  $n \geq 4$  and let  $K_n = [k_{ij}]$  be the  $n \times n$   $(0, 1)$ -tridiagonal matrix with entries  $k_{ij} = 1$  for  $|i - j| \leq 1$  and  $2 \leq i, j \leq n$ ,  $k_{11} = 1$  and 0 otherwise. Let  $D_n$  be the  $n \times n$   $(0, 1)$ -matrix with  $d_{1j} = 1$  for  $3 \leq j \leq n$ ,  $d_{24} = 1$  and 0 otherwise. Now we consider the sum of these matrices,  $K_n$  and  $D_n$ , and denote by  $W_n = [w_{ij}] = K_n + D_n$ . Clearly

$$W_n = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & & 0 \\ & \ddots & \ddots & \ddots & \ddots & & \ddots & \vdots \\ \vdots & & & & & & & 0 \\ & & & & & & & 1 \\ 0 & \dots & & \dots & 0 & 1 & 1 \end{bmatrix}.$$

**Theorem 3.** *Let  $G(W_n)$  be the bipartite graph with bipartite adjacency matrix  $W_n = [w_{ij}] = K_n + D_n$ ,  $n \geq 4$ . Then the number of 1-factors of  $G(W_n)$  is  $\sum_{i=0}^{n-2} L_i = L_n - 1$ .*

*Proof.* If  $n = 4$ , then we have

$$\text{per}W_4 = \text{per} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = 6 = L_0 + L_1 + L_2.$$

Let  $W_n^k$  be the  $k$ th contraction of  $W_n$ ,  $1 \leq k \leq n-2$ . Since the definition of the matrix  $W_n$ , the matrix  $W_n$  can be contracted on column 1 so that

$$W_n^1 = \begin{bmatrix} 1 & 2 & 2 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & & & 1 \\ 0 & \dots & & \dots & 0 & 1 & 1 \end{bmatrix}.$$

Since the matrix  $W_n^1$  can be contracted on column 1 and  $\sum_{i=0}^1 L_i = 3$  and  $L_2 = 3$ ,

$$\begin{aligned} W_n^2 &= \begin{bmatrix} 3 & 3 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 0 & & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 0 & & & \dots & 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=0}^1 L_i & L_2 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 0 & & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 0 & & & \dots & 0 & 1 & 1 \end{bmatrix}. \end{aligned}$$

Furthermore, the matrix  $W_n^2$  can be contracted on column 1 so that

$$W_n^3 = \begin{bmatrix} \sum_{i=0}^2 L_i & L_3 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 0 & & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 0 & & & \dots & 0 & 1 & 1 \end{bmatrix}.$$

Continuing this process, we reach

$$W_n^k = \begin{bmatrix} \sum_{i=0}^{k-1} L_i & L_k & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 0 & & 0 \\ 0 & 1 & 1 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 0 & & & \dots & 0 & 1 & 1 \end{bmatrix}$$

for  $4 \leq k \leq n - 4$ . Hence,

$$W_n^{(n-3)} = \begin{bmatrix} \sum_{i=0}^{n-4} L_i & L_{n-3} & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

which, by contraction of  $W_n^{(n-4)}$  on column 1, gives

$$W_n^{(n-2)} = \begin{bmatrix} \sum_{i=0}^{n-3} L_i & L_{n-2} \\ 1 & 1 \end{bmatrix}.$$

By applying (1.1), we have  $\text{per}W_n = \text{per}W_n^{(n-2)} = \sum_{i=0}^{n-2} L_i$ .  $\square$

#### ACKNOWLEDGEMENTS

The authors would like to thank the referee for a number of helpful suggestions.

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