

ON THE SECOND ORDER LINEAR RECURRENCES BY TRIDIAGONAL MATRICES

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ABSTRACT. In this paper, we consider the relationships between the second order linear recurrences and the permanents and determinants of tridiagonal matrices.

1. INTRODUCTION

The well-known Fibonacci, Lucas and Pell numbers can be generalized as follows: Let A and B be nonzero, relatively prime integers such that $D = A^2 - 4B \neq 0$. Define sequences $\{u_n\}$ and $\{v_n\}$ by, for all $n \geq 2$ (see [10]),

$$u_n = Au_{n-1} - Bu_{n-2} \quad (1.1)$$

$$v_n = Av_{n-1} - Bv_{n-2} \quad (1.2)$$

where $u_0 = 0$, $u_1 = 1$ and $v_0 = 2$, $v_1 = A$. If $A = 1$ and $B = -1$, then $u_n = F_n$ (the n th Fibonacci number) and $v_n = L_n$ (the n th Lucas number). If $A = 2$ and $B = -1$, then $u_n = P_n$ (the n th Pell number).

An alternative is to let the roots of the equation $t^2 - At + B = 0$ be, for $n \geq 0$

$$u_n = \frac{\sigma^n - \gamma^n}{\sigma - \gamma} \quad \text{and} \quad v_n = \sigma^n + \gamma^n. \quad (1.3)$$

Also it is well-known that

$$\sigma + \gamma = A \quad \text{and} \quad \sigma\gamma = B.$$

The *permanent* of an n -square matrix $A = (a_{ij})$ is defined by

$$\text{per}A = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i\sigma(i)}$$

where the summation extends over all permutations σ of the symmetric group S_n . Also one can find more applications of permanents in [9].

In [2], [3], the authors consider the relationships between tridiagonal determinants and the Fibonacci and Lucas numbers.

In [5], Lehmer proves a very general result on permanents of tridiagonal matrices whose main diagonal and super-diagonal elements are ones and whose subdiagonal entries are somewhat arbitrary.

Minc, in [8], defined the super-diagonal matrix and showed that the permanent of the matrix equals to the order- k Fibonacci number.

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In [6] and [7], the authors gave the relations involving the generalized Fibonacci and Lucas numbers and the permanent and determinants of the matrices. The result of Minc, [8], and the result of Lee, [6], on the generalized Fibonacci numbers are the same. The authors use the same matrix. However, Lee proved the same result by a different method, contraction method for the permanent (for more detail the contraction method see [1]).

In [4], the authors find the families of matrices such that permanents of the matrices, equal to the sums of Fibonacci and Lucas numbers.

In this paper, we develop the relations involving the second order linear recurrences and the permanents and determinants of tridiagonal matrices.

2. ON THE DETERMINANTS OF SOME TRIDIAGONAL MATRICES

In this section, first, we construct a $n \times n$ tridiagonal toeplitz matrix $T_n = [t_{ij}]$ with entries $t_{k,k} = \alpha + \beta$, $t_{k,k+1} = \beta$ and $t_{k+1,k} = \alpha$ for $1 \leq k \leq n-1$, that is,

$$T_n = [t_{ij}] = \begin{pmatrix} \alpha + \beta & \beta & & & 0 \\ \alpha & \alpha + \beta & \beta & & \\ & \alpha & \alpha + \beta & \ddots & \\ & & \ddots & \ddots & \beta \\ 0 & & & \alpha & \alpha + \beta \end{pmatrix} \quad (2.1)$$

where α, β are real or complex numbers such that $\alpha\beta \neq 0$ and $(\alpha + \beta)^2 \neq 4\alpha\beta$.

Theorem 1. *Let T_n be defined as in (2.1). Then, for all $n \geq 1$*

$$|T_n| = \sum_{j=0}^n \alpha^{n-j} \beta^j.$$

Proof. We prove that $|T_n| = \sum_{j=0}^n \alpha^{n-j} \beta^j$ by induction computing all determinants by cofactor expansion of determinant with respect to row 1. If $n = 1$, then

$$|T_1| = |\alpha + \beta| = \alpha + \beta = \sum_{j=0}^1 \alpha^{1-j} \beta^j.$$

If $n = 2$, then

$$|T_2| = \begin{vmatrix} \alpha + \beta & \beta \\ \alpha & \alpha + \beta \end{vmatrix} = \alpha^2 + \alpha\beta + \beta^2 = \sum_{j=0}^2 \alpha^{2-j} \beta^j.$$

For the inductive step, we suppose that

$$|T_k| = \sum_{j=0}^k \alpha^{k-j} \beta^j. \quad (2.2)$$

We show that the equation holds for $k + 1$. Then

$$|T_{k+1}| = (\alpha + \beta) \begin{vmatrix} \alpha + \beta & \beta & & 0 \\ \alpha & \alpha + \beta & \ddots & \\ & & \ddots & \ddots & \beta \\ 0 & & & \alpha & \alpha + \beta \end{vmatrix} - \beta \begin{vmatrix} \alpha & \beta & & 0 \\ 0 & \alpha + \beta & \beta & \\ 0 & \alpha & \alpha + \beta & \ddots \\ & & \ddots & \ddots & \beta \\ 0 & & & \alpha & \alpha + \beta \end{vmatrix}.$$

By the Eq. (2.2), we may write the last equation as

$$\begin{aligned} |T_{k+1}| &= (\alpha + \beta) |T_k| - \beta \alpha |T_{k-1}| = (\alpha + \beta) \sum_{j=0}^k \alpha^{k-j} \beta^j - \alpha \beta \sum_{j=0}^{k-1} \alpha^{k-1-j} \beta^j \\ &= (\alpha + \beta) \left[\alpha^k + \alpha^{k-1} \beta + \dots + \alpha \beta^{k-1} + \beta^k \right] \\ &\quad - \alpha \beta \left[\alpha^{k-1} + \alpha^{k-2} \beta + \dots + \alpha \beta^{k-2} + \beta^{k-1} \right] \\ &= \sum_{j=0}^{k+1} \alpha^{k+1-j} \beta^j. \end{aligned}$$

So the proof is complete. \square

Now we consider the sequence $\{u_n\}$ in (1.1). It is seen that Theorem 1 is an alternative statement of the *Binet* equations for the sequences type of $\{u_n\}$. Indeed, from Theorem 1, we know that

$$|T_n| = \sum_{j=0}^n \alpha^{n-j} \beta^j = \alpha^n + \alpha^n \beta + \alpha^{n-1} \beta^2 + \dots + \alpha \beta^{n-1} + \beta^n. \quad (2.3)$$

If we multiple and divide the formula (2.3) by $(\alpha - \beta)$, then we obtain that

$$|T_n| = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$$

which equals to the *Binet* equation. For example, when $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, the determinant of the matrix T_n is reduced to the *Binet* formula for the $(n + 1)$ th Fibonacci number, F_{n+1} . Also when $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$, the determinant of T_n is reduced to the *Binet* formula for the $(n + 1)$ th Pell number, P_{n+1} .

Secondly, let we define a $n \times n$ tridiagonal matrix $H_n = (h_{ij})$ with entries $h_{ii} = \alpha + \beta$ for $1 \leq i \leq n$, $h_{1,2} = 2\beta$, $h_{i,i+1} = \beta$ for $2 \leq i \leq n - 1$ and $h_{i+1,i} = \alpha$ for $1 \leq i \leq n - 1$ where α, β be defined as before. Clearly

$$H_n = \begin{pmatrix} \alpha + \beta & 2\beta & & & 0 \\ \alpha & \alpha + \beta & \beta & & \\ & & \alpha & \alpha + \beta & \ddots \\ & & & \ddots & \ddots & \beta \\ 0 & & & & \alpha & \alpha + \beta \end{pmatrix}. \quad (2.4)$$

Theorem 2. Let H_n be a matrix as in the form (2.4). Then, for $n \geq 1$

$$|H_n| = \alpha^n + \beta^n.$$

Proof. (Induction on n) If $n = 1$, then

$$|H_1| = |\alpha + \beta| = \alpha^1 + \beta^1.$$

If $n = 2$, then

$$|H_2| = \begin{vmatrix} \alpha + \beta & 2\beta \\ \alpha & \alpha + \beta \end{vmatrix} = \alpha^2 + \beta^2.$$

We suppose that the equation holds for n . Then we obtain

$$|H_n| = \alpha^n + \beta^n. \quad (2.5)$$

Now we show that the equation holds for $n + 1$. Then, by computing determinant by cofactor expansion with respect to last row

$$\begin{aligned} |H_{n+1}| &= (\alpha + \beta) |H_n| - \beta \begin{vmatrix} \alpha + \beta & 2\beta & & & 0 \\ \alpha & \alpha + \beta & \beta & & \\ & \alpha & \alpha + \beta & \ddots & \\ & & \ddots & \ddots & \beta \\ & & & \alpha & \alpha + \beta & \beta \\ 0 & & & & 0 & \alpha \end{vmatrix} \\ &= (\alpha + \beta) |H_n| - \alpha\beta |H_{n-1}|. \end{aligned}$$

By Eq. (2.5), we write the last equation as

$$|H_{n+1}| = (\alpha + \beta)(\alpha^n + \beta^n) - \alpha\beta(\alpha^{n-1} + \beta^{n-1}) = \alpha^{n+1} + \beta^{n+1}.$$

So the proof is complete. \square

The conclusion of Theorem 2 is an alternative statement to the *Binet* formula of the sequence of $\{v_n\}$ taking by $\alpha = \sigma$ and $\beta = \gamma$. Clearly, for $\alpha = \sigma$ and $\beta = \gamma$

$$|H_n| = v_n$$

where v_n is the n th term of the sequence $\{v_n\}$ and σ and γ are the roots of the characteristic equation of the sequence $\{v_n\}$.

A matrix A is called *convertible* if there is an $n \times n$ $(1, -1)$ -matrix H such that $\text{per} A = \det(A \circ H)$, where $A \circ H$ denotes the Hadamard product of A and H . Such a matrix H is called a *converter* of A .

Let S be a $(1, -1)$ -matrix of order n , defined by

$$S = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ -1 & 1 & 1 & \dots & 1 & 1 \\ 1 & -1 & 1 & \dots & 1 & 1 \\ 1 & 1 & -1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \dots & -1 & 1 \end{bmatrix}.$$

Let we denote the matrices $T_n \circ S$ and $H_n \circ S$ by A_n and B_n , respectively. Thus

$$A_n = \begin{pmatrix} \alpha + \beta & \beta & & & 0 \\ -\alpha & \alpha + \beta & \beta & & \\ & -\alpha & \alpha + \beta & \ddots & \\ & & \ddots & \ddots & \beta \\ 0 & & & -\alpha & \alpha + \beta \end{pmatrix} \quad (2.6)$$

and

$$B_n = \begin{pmatrix} \alpha + \beta & 2\beta & & & 0 \\ -\alpha & \alpha + \beta & \beta & & \\ & -\alpha & \alpha + \beta & \ddots & \\ & & \ddots & \ddots & \beta \\ 0 & & & -\alpha & \alpha + \beta \end{pmatrix}. \quad (2.7)$$

Then we have following Theorems without proof.

Theorem 3. *Let the matrix A_n has the form (2.6). Then for $n \geq 1$*

$$\text{per} A_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}.$$

Theorem 4. *Let the matrix B_n has the form (2.7). Then for $n \geq 1$*

$$\text{per} B_n = \alpha^n + \beta^n.$$

Furthemore, from [9], we have that let A be a tridiagonal matrix, and let $\hat{A} = (\hat{a}_{ij})$ be defined by $\hat{a}_{st} = ia_{st}$ if $s \neq t$ and $\hat{a}_{ss} = a_{ss}$, for all s and t ($i = \sqrt{-1}$). Then we have

$$\text{per}(A) = \det(\hat{A}).$$

Also let we define the following matrices;

$$\hat{A}_n = \begin{pmatrix} \alpha + \beta & i\beta & & & 0 \\ -i\alpha & \alpha + \beta & i\beta & & \\ & -i\alpha & \alpha + \beta & \ddots & \\ & & \ddots & \ddots & i\beta \\ 0 & & & -i\alpha & \alpha + \beta \end{pmatrix} \quad (2.8)$$

and

$$\hat{B}_n = \begin{pmatrix} \alpha + \beta & 2i\beta & & & 0 \\ -i\alpha & \alpha + \beta & i\beta & & \\ & -i\alpha & \alpha + \beta & \ddots & \\ & & \ddots & \ddots & i\beta \\ 0 & & & -i\alpha & \alpha + \beta \end{pmatrix} \quad (2.9)$$

Thus we have following Corollaries without proof.

Corollary 1. *Let the $n \times n$ tridiagonal toeplitz matrix \hat{A}_n as in (2.8). Then, for $n \geq 1$*

$$\det \hat{A}_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}.$$

Corollary 2. *Let the $n \times n$ tridiagonal matrix \hat{B}_n be as in (2.9). Then, for $n \geq 1$*

$$\det \hat{B}_n = \alpha^n + \beta^n.$$

Also it is clear that the value of following determinant is independent of x : (see p.105, [11])

$$\begin{vmatrix} a & x & & & 0 \\ \frac{1}{x} & a & x & & \\ & \frac{1}{x} & a & x & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \frac{1}{x} & a & x \\ & & & & \frac{1}{x} & a \end{vmatrix}.$$

If we define the $n \times n$ following matrices:

$$C_n = \begin{pmatrix} \alpha + \beta & 1 & & & 0 \\ \frac{\alpha}{\beta} & \alpha + \beta & 1 & & \\ & \frac{\alpha}{\beta} & \alpha + \beta & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & \frac{\alpha}{\beta} & \alpha + \beta \end{pmatrix} \quad (2.10)$$

and

$$D_n = \begin{pmatrix} \alpha + \beta & 2 & & & 0 \\ \frac{\alpha}{\beta} & \alpha + \beta & 1 & & \\ & \frac{\alpha}{\beta} & \alpha + \beta & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & \frac{\alpha}{\beta} & \alpha + \beta \end{pmatrix}, \quad (2.11)$$

then we have that

$$\det C_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$$

and

$$\det D_n = \alpha^n + \beta^n$$

where α and β be defined as before.

For example, let we take $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then by using the above results we have that

$$\begin{vmatrix} 1 & i & & & 0 \\ i & 1 & i & & \\ & i & 1 & \ddots & \\ & & \ddots & \ddots & i \\ 0 & & & i & 1 \end{vmatrix} = F_{n+1}$$

where F_n is the n th Fibonacci number. This result is given in [3].

Let also we take $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ in (2.10). Then by using the above results we have $\alpha\beta = -1$ and

$$\begin{vmatrix} 2 & 1 & & & 0 \\ -1 & 2 & 1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & -1 & 2 \end{vmatrix} = P_{n+1}$$

where P_n is the n th Pell number.

Finally, let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. By using the above results, we obtain

$$\text{per} \begin{pmatrix} 1 & 2 & & & 0 \\ 1 & 1 & 1 & & \\ & 1 & 1 & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & 1 \end{pmatrix}_{n \times n} = L_n$$

where L_n is the n th Lucas number.

Indeed, we generalize this representations for all the second order linear recurrences by tridiagonal determinants and permanents.

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