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The generalized Pell (p, i)-numbers and their Binet formulas, combinatorial representations, sums

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Abstract

The theory of generalized Pell p-numbers was introduced by Stakhov and then have been studied by several authors. In this paper, we consider the usual Pell numbers and as similar to the Fibonacci p-numbers, we give fair generalization of the Pell numbers, which we call *the generalized Pell* (p, i)*-numbers* for $0 \leqslant i \leqslant p$. First we give relationships between the generalized Pell (p, i)-numbers and give the generating matrices for these numbers. Also we derive the generalized Binet formulas, sums, combinatorial representations and generating function of the generalized Pell p-numbers. Also using matrix methods, we derive an explicit formula for the sums of the generalized Fibonacci p-numbers. Finally, we derive relationships between generalized Pell (p, i)-numbers and their sums and permanents of certain matrices.

1. Introduction

The well-known Fibonacci sequence is defined by the following recurrence relations: for $n > 0$

$$F_{n+1} = F_n + F_{n-1}$$

where $F_0 = 0$, $F_1 = 1$.

In literature, one can find many interesting generalizations of the Fibonacci sequence. However, one of the most interesting generalization of the recurrence is given by Stakhov. The generalization is called Fibonacci p-numbers, and defined by the following equation for any given $p(p = 1, 2, 3, \ldots)$ and $n > p + 1$

$$F_p(n) = F_p(n-1) + F_p(n-p-1)$$

with initial conditions $F_p(1) = \cdots = F_p(p) = F_p(p+1) = 1$.

When $p = 1$, then the sequence of Fibonacci p-numbers, $\{F_p(n)\}$, is reduced to the well-known Fibonacci sequence $\{F_n\}$.

The Fibonacci p-numbers, their miscellaneous properties and many applications have been studied by some authors (for more details see [1,3–10,19–50,53]). For example, the generating matrix, the Binet like formulas, applications to the coding theory and the generalized Cassini formula, i.e., of the Fibonacci p-numbers are given by Stakhov.

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Also in [10], recently the author consider the Fibonacci p-numbers and then give the Binet formula, combinatorial representation, sums of terms of the sequence of Fibonacci p-numbers by matrix methods.

Matrix methods are very useful tools to solve many problems for stemming from number theory.

Furthermore, similar to the Fibonacci sequence, the well known Pell sequence is defined by the following equations: for $n > 0$

$$P_{n+1} = 2P_n + P_{n-1}$$

where $P_0 = 0,\ P_1 = 1$.

In 1843, Binet gave a formula which is called "Binet formula" for the usual Fibonacci numbers by using the roots of the characteristic equation $x^2 - x - 1 = 0$: $\alpha, \beta = (1 \mp \sqrt{5})/2$

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

where α is called Golden Proportion, $\alpha = \frac{1+\sqrt{5}}{2}$ (for details see [5,53,49]).

Similarly, the Binet formula for the Pell sequence is as follows:

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}$$

where $\gamma, \delta = 1 \mp \sqrt{2}$.

In [17], Levesque gave a Binet formula for the Fibonacci sequence by using a generating function. In [2], the authors considered $n \times n$ companion matrix and its nth power of, then gave the combinatorial representation of the sequence generated by the nth power of the matrix. Further in [44], the authors derived analytical formulas for the Fibonacci p-numbers and then showed these formulas are similar to the Binet formulas for the classical Fibonacci numbers. Also, in [9], the authors gave the generalized Binet formulas and the combinatorial representations for the generalized order-k Fibonacci [3] and Lucas [48] numbers. In [8], the authors defined the generalized order-k Pell numbers and gave the Binet formula for the generalized Pell sequence. For the common generalization of the generalized order-k Fibonacci and Pell numbers, and its generating matrix, sums and combinatorial representation, we refer to [7].

The generating matrix of the Fibonacci numbers as shown:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}. \tag{1}$$

Also the generating matrix of the Pell sequence is as follows:

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}. \tag{2}$$

The generating matrix of the Fibonacci p-numbers was given by Stakhov [42] as follows: Let Q_p be the $(p+1) \times (p+1)$ companion matrix as follows:

$$Q_p = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \dots & \ddots & \dots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}$$

and the nth power of the matrix Q_p is

$$Q_p^n = \begin{bmatrix} F_p(n+1) & F_p(n-p+1) & \dots & F_p(n-1) & F_p(n) \\ F_p(n) & F_p(n-p) & \dots & F_p(n-2) & F_p(n-1) \\ \vdots & \vdots & & \vdots & \vdots \\ F_p(n-p+2) & F_p(n-2p+2) & \dots & F_p(n-p) & F_p(n-p+1) \\ F_p(n-p+1) & F_p(n-2p+1) & \dots & F_p(n-p-1) & F_p(n-p) \end{bmatrix}.$$

Indeed, the matrix Q_1 and its nth power is reduced to the well known form given by (1) for the usual Fibonacci numbers.

While we derive our results throughout this paper, we obey the straightforward generalization of the well known earlier results to obtain some new results on the generalized Pell numbers.

The generating functions of the Fibonacci and Pell numbers are given by, resp.:

$$G(x) = \frac{1}{1-x-x^2}$$

and

$$H(x) = \frac{1}{1-2x-x^2}.$$

The *permanent* of an n-square matrix $A = (a_{ij})$ is defined by

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ of the symmetric group S_n.

The permanent of a matrix is analogous to the determinant, where all of the signs used in the Laplace expansion of minors are positive.

The relationships between recurrence relations and determinant of certain matrices have been studied by some authors (see [11–16,18,51,52]).

The Fibonacci and Pell numbers can be expressed by the successive determinants of the following matrices: resp.,

$$\begin{vmatrix} 1 & 1 & & & \\ 1 & 1 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 1 & 1 \\ & & & 1 & 1 \end{vmatrix}_{n\times n} = F_{n+1} \quad \text{and} \quad \begin{vmatrix} 2 & 1 & & & \\ 1 & 2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 2 & 1 \\ & & & 1 & 2 \end{vmatrix}_{n\times n} = P_{n+1}.$$

In this paper, we define generalized Pell p-numbers and give the generalized Binet formula, generating functions, combinatorial representations and sums of the generalized Pell p-numbers.

2. Generalized Pell p-numbers

In this section we give definition of generalized Pell (p,i)-numbers and then we present their generating matrix. Also by matrix methods, we derive the generalized Binet formula of the generalized Pell (p,i)-numbers.

Define the generalized Pell (p,i)-numbers as shown, for any given $p(p = 1, 2, 3, \ldots)$ $n > p+1$ and $0 \leqslant i \leqslant p$

$$P_p^{(i)}(n) = 2P_p^{(i)}(n-1) + P_p^{(i)}(n-p-1) \tag{3}$$

with initial conditions $P_p^{(i)}(1) = \cdots = P_p^{(i)}(i) = 0$ and $P_p^{(i)}(i+1) = P_p^{(i)}(i+2) = \cdots = P_p^{(i)}(p+1) = 1$. Note that if $i = 0$, then the initial conditions are $P_p^{(0)}(1) = P_p^{(0)}(2) = \cdots = P_p^{(0)}(p+1) = 1$.

For example, it is obvious that when $i = p = 1$, then the generalized Pell (1,1)-number is the $(n+1)$th usual Pell number. If we take $p = 2$ and $i = 0$, then the sequence of the generalized Pell (2,0)-numbers, $\{P_2^{(0)}(n)\}$ is

$$1, 1, 1, 3, 7, 15, 33, 73, 161, \ldots.$$

Also when $p = 2$ and $i = 1$, then the initial conditions of the generalized Pell (2,1)-numbers are $P_2^{(1)}(1) = 0, P_2^{(1)}(2) = P_2^{(1)}(3) = 1$. Then the sequence of the generalized Pell (2,1)-numbers, $\{P_2^{(1)}(n)\}$, is

$$0, 1, 1, 2, 5, 11, 24, 53, 117, 258, \ldots.$$

Now we give indirect generating matrix of the generalized Pell (p,i)-sequences by an auxiliary matrix.

Define the $(p+1)\times(p+1)$ companion matrix A in the following form:

$$A = \begin{bmatrix} 2 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & 0 \\ \vdots & \ldots & \ddots & 0 & \vdots \\ 0 & \ldots & 0 & 1 & 0 \end{bmatrix}. \tag{4}$$

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Also define the $(p+1)\times(p+1)$ auxiliary matrix E shown:

$$E = \begin{bmatrix} 3 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & 1 & \dots & 1 \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 1 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}.$$

In order to give one of the our main results, we also define the $(p+1)\times(p+1)$ matrix H_n as follows:

$$H_n = \begin{bmatrix} P_p^{(0)}(n+p+2) & P_p^{(p-1)}(n+p+1) & P_p^{(p-2)}(n+p+1) & \dots & P_p^{(1)}(n+p+1) & P_p^{(0)}(n+p+1) \\ P_p^{(0)}(n+p+1) & P_p^{(p-1)}(n+p) & P_p^{(p-2)}(n+p) & \dots & P_p^{(1)}(n+p) & P_p^{(0)}(n+p) \\ \vdots & \vdots & & & \vdots & \vdots \\ P_p^{(0)}(n+3) & P_p^{(p-1)}(n+2) & P_p^{(p-2)}(n+2) & \dots & P_p^{(1)}(n+2) & P_p^{(0)}(n+2) \\ P_p^{(0)}(n+2) & P_p^{(p-1)}(n+1) & P_p^{(p-2)}(n+1) & \dots & P_p^{(1)}(n+1) & P_p^{(0)}(n+1) \end{bmatrix}$$

Then we have the following Theorem.

Theorem 1. *For n, $p > 0$,*

$$H_n = A^n E.$$

Proof. We will use the induction method on n. If $n = 1$, then

$$H_1 = \begin{bmatrix} P_p^{(0)}(p+3) & P_p^{(p-1)}(p+2) & \dots & P_p^{(1)}(p+2) & P_p^{(0)}(p+2) \\ P_p^{(0)}(p+2) & P_p^{(p-1)}(p+1) & \dots & P_p^{(1)}(p+1) & P_p^{(0)}(p+1) \\ \vdots & \vdots & & \vdots & \vdots \\ P_p^{(0)}(4) & P_p^{(p-1)}(3) & \dots & P_p^{(1)}(3) & P_p^{(0)}(3) \\ P_p^{(0)}(3) & P_p^{(p-1)}(2) & \dots & P_p^{(1)}(2) & P_p^{(0)}(2) \end{bmatrix}.$$

By the definition of the generalized Pell (p,i)-numbers, we have the matrix H_1:

$$H_1 = \begin{bmatrix} 7 & 2 & 2 & 2 & \dots & 2 & 3 \\ 3 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 0 & 1 & 1 \end{bmatrix}.$$

Also by a simple calculation, we obtain

$$H_1 = A \cdot E.$$

Thus the proof is complete for $n = 1$. Now we suppose that the equation holds for $n-1$. Then we show that the equation holds for n. Thus, by our assumption and since the matrix A is a companion matrix, we write

$$H_n = A^n E = AA^{n-1}E = A \cdot H_{n-1}.$$

From the matrix multiplication and the definition of the generalized Pell (p,i)-numbers, we have the conclusion. □

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Now we define the $(p+1)\times(p+1)$ matrix G_n as follows:

$$G_n = \begin{bmatrix} P_p^{(p)}(n+p+1) & P_p^{(p)}(n+1) & P_p^{(p)}(n+2) & \dots & P_p^{(p)}(n+p) \\ P_p^{(p)}(n+p) & P_p^{(p)}(n) & P_p^{(p)}(n+1) & \dots & P_p^{(p)}(n+p-1) \\ \vdots & \vdots & & & \vdots \\ P_p^{(p)}(n+2) & P_p^{(p)}(n-p+2) & P_p^{(p)}(n-p+3) & \dots & P_p^{(p)}(n+1) \\ P_p^{(p)}(n+1) & P_p^{(p)}(n-p+1) & P_p^{(p)}(n-p+2) & \dots & P_p^{(p)}(n) \end{bmatrix}. \quad (5)$$

Theorem 2. *For $n > 0$,*

$$A^n = G_n.$$

Proof. From the recurrence relation of the generalized Pell (p,p)-numbers, we can write following vector relation:

$$\begin{bmatrix} P_p^{(p)}(n+p+1) \\ P_p^{(p)}(n+p) \\ \vdots \\ P_p^{(p)}(n+2) \\ P_p^{(p)}(n+1) \end{bmatrix} = \begin{bmatrix} 2 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & 0 \\ \vdots & \dots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_p^{(p)}(n+p) \\ P_p^{(p)}(n+p-1) \\ \vdots \\ P_p^{(p)}(n+1) \\ P_p^{(p)}(n) \end{bmatrix}.$$

We can generalize the above vector recurrence relation to the $(p+1)$ columns, then we obtain

$$G_n = AG_{n-1}.$$

By the inductive argument, we may write

$$G_n = A^{n-1}G_1.$$

By the definition of the generalized Pell (p,p)-numbers, we obtain $G_1 = A$. Thus we have the conclusion □

Thus the matrix A is said to be a generalized Pell p-matrix.

Combining the results of Theorems 1 and 2, we have the following useful result.

Corollary 3. *For n, $p > 0$,*

$$\begin{aligned} P_p^{(0)}(n+p+1) &= 3P_p^{(p)}(n+p) + \sum_{j=0}^{p-1} P_p^{(p)}(n+j) \\ P_p^{(p-j)}(n+p+1) &= P_p^{(p)}(n+p+1) + \sum_{k=1}^{j} P_p^{(p)}(n+k) \quad \textit{for } 1 \leqslant j \leqslant p \end{aligned} \quad (6)$$

Proof. The proof follows from the matrix multiplication by considering $H_n = A^n E$ and $A^n = G_n$. □

When $j = p$ in the Eq. (6), we obtain

$$P_p^{(0)}(n+p+1) = P_p^{(p)}(n+p+1) + P_p^{(p)}(n+1) + P_p^{(p)}(n+2) + \dots + P_p^{(p)}(n+p).$$

If we substract the above equation from (6), then we reach at the well known recurrence relation of the generalized Pell p-numbers:

$$P_p^{(p)}(n+p+1) = 2P_p^{(p)}(n+p) + P_p^{(p)}(n).$$

According to the above corollary, it is seen that the sequence of the generalized Pell (p,p)-numbers can be thought for the sequence of the generalized Pell (p,i)-numbers as basis for $0 \leqslant i \leqslant p-1$. Clearly one can obtain the terms $P_p^{(p-j)}(n)$ from the linear combination of the terms $P_p^{(p)}(n)$ for $1 \leqslant j \leqslant p$.

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The Simpson formula of the Fibonacci numbers can be obtained from determinant of the following matrix:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

and thus

$$(-1)^n = \det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n\right) = \begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = F_{n+1}F_{n-1} - F_n^2,$$

that is,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

Similarly, from this point of view, the generalized Simpson formula for the generalized Pell (p,i)-numbers can be obtained from the determinant of their generating matrix, that is, $\det G_n$. From companion matrices, we can easily derive

$$\det G_n = (-1)^{n+1}.$$

For example, when $p = 2$, then we obtain

$$G_n = \begin{bmatrix} P_p^{(2)}(n+3) & P_p^{(2)}(n+1) & P_p^{(2)}(n+2) \\ P_p^{(2)}(n+2) & P_p^{(2)}(n) & P_p^{(2)}(n+1) \\ P_p^{(2)}(n+1) & P_p^{(2)}(n-1) & P_p^{(2)}(n) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n.$$

According to the our result, by the determinantal representation, we find the Simpson formula of the generalized Pell (2,2)-numbers as shown:

$$\begin{aligned} &P_p^{(2)}(n-1)\left(P_p^{(2)}(n+2)\right)^2 - 2P_p^{(2)}(n+2)P_p^{(2)}(n+1)P_p^{(2)}(n) + \left(P_p^{(2)}(n+1)\right)^3 - P_p^{(2)}(n+3)P_p^{(2)}(n-1)P_p^{(2)}(n+1) \\ &\quad + P_p^{(2)}(n+3)\left(P_p^{(2)}(n)\right)^2 \\ &\quad = (-1)^{n+1}. \end{aligned}$$

3. The generalized Binet formula of the generalized Pell (p,i)-numbers

In this section, we give the generalized Binet formula for the generalized Pell p-numbers. Before this, we give some results.

Lemma 4. *Let* $a_p = \frac{2}{p+1}\left(\frac{2p}{p+1}\right)^p$. *Then* $a_p < a_{p+1}$ *for* $p > 1$.

Proof. Note that for all $k > 1$,

$$\frac{1}{2}\left(\frac{k+1}{k}\right)^2 < \frac{k^2}{k^2-1}.$$

Since also $k^2 > k^2 - 1$ for all $k > 1$. Thus we can write for $k > 2$

$$\frac{1}{2}\left(\frac{k+1}{k}\right)^2 < \left(\frac{k^2}{k^2-1}\right)^{k-1}.$$

Then we may write

$$\frac{1}{2}\left(\frac{k+1}{k}\right)^2 < \left(\frac{k}{2(k-1)} \times \frac{2k}{(k+1)}\right)^{k-1} = \left(\frac{k}{2(k-1)}\right)^{k-1}\left(\frac{2k}{k+1}\right)^{k-1}.$$

Therefore, we obtain

$$\frac{1}{k}\left(\frac{2(k-1)}{k}\right)^{k-1} < \frac{1}{k+1}\left(\frac{2k}{k+1}\right)\left(\frac{2k}{k+1}\right)^{k-1}$$

and so

$$\frac{2}{k}\left(\frac{2(k-1)}{k}\right)^{k-1} < \frac{2}{k+1}\left(\frac{2k}{k+1}\right)^k$$

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which gives us for $k > 2$

$$a_{k-1} < a_k.$$

Thus the proof is complete. □

Lemma 5. *The characteristic equation of all the Pell (p,i)-numbers $x^{p+1} - 2x^p - 1 = 0$ does not have multiple roots for $p > 1$.*

Proof. Let $f(z) = z^{p+1} - 2z^p - 1$. Suppose that α is a multiple root of $f(z) = 0$. Note that $\alpha \neq 0$ and $\alpha \neq 1$. Since α is a multiple root, $f(\alpha) = \alpha^{p+1} - 2\alpha^p - 1 = 0$ and $f'(\alpha) = (p+1)\ \alpha^p - 2p\alpha^{p-1} = 0$. Then

$$f'(\alpha) = \alpha^{p-1}((p+1)\alpha - 2p) = 0.$$

Thus $\alpha = \frac{2p}{p+1}$, and hence

$$0 = -f(\alpha) = -\alpha^{p+1} + 2\alpha^p + 1 = \alpha^p(2-\alpha) + 1 = \left(\frac{2p}{p+1}\right)^p \left(2 - \frac{2p}{p+1}\right) + 1 = \frac{2}{p+1}\left(\frac{2p}{p+1}\right)^p + 1 = a_p + 1.$$

Since, by Lemma 4, $a_2 = \frac{32}{27} > 1$ and $a_p < a_{p+1}$ for $p > 1$, $a_p \neq -1$, which is a contradiction. Therefore, the equation $f(z) = 0$ does not have multiple roots. □

Suppose that $f(\lambda)$ be the characteristic polynomial of the generalized Pell p-matrix A. Then $f(\lambda) = \lambda^{p+1} - 2\lambda^p - 1$, which is a well-known fact from the companion matrices. Let $\lambda_1, \lambda_2, \cdots, \lambda_{p+1}$ be the eigenvalues of the matrix A. Then, by Lemma 5, it is known that $\lambda_1, \lambda_2, \ldots, \lambda_{p+1}$ are distinct. Let Λ be an $(p+1) \times (p+1)$ Vandermonde matrix as follows:

$$\Lambda = \begin{bmatrix} \lambda_1^p & \lambda_1^{p-1} & \ldots & \lambda_1 & 1 \\ \lambda_2^p & \lambda_2^{p-1} & \ldots & \lambda_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_{p+1}^p & \lambda_{p+1}^{p-1} & \ldots & \lambda_{p+1} & 1 \end{bmatrix}. \tag{7}$$

We denote Λ^T by V. Let

$$d_k^i = \begin{bmatrix} \lambda_1^{n+p+1-i} \\ \lambda_2^{n+p+1-i} \\ \vdots \\ \lambda_{p+1}^{n+p+1-i} \end{bmatrix}$$

and $V_j^{(i)}$ be a $(p+1) \times (p+1)$ matrix obtained from V by replacing the jth column of V by d_k^i.

Then we can give the generalized Binet formula for the generalized Pell (p,p)-numbers with the following Theorem.

Theorem 6. *Let $P_p^{(p)}(n)$ be the nth generalized Pell (p,p)-number, then*

$$g_{ij} = \frac{\det(V_j^{(i)})}{\det(V)}$$

where $G_n = [g_{ij}]$.

Proof. Since the eigenvalues of the matrix A are distinct, the matrix A is diagonalizable. It is easy to show that $AV = VD$, where $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{p+1})$. Since the Vandermonde matrix V is invertible, $V^{-1}AV = D$. Hence, the matrix A is similar to the diagonal matrix D. So we have the matrix equation $A^nV = VD^n$. Since $A^n = G_n = [g_{ij}]$, we have the following linear system of equations:

$$\begin{aligned} g_{i1}\lambda_1^p + g_{i2}\lambda_1^{p-1} + \cdots + g_{i,p+1} &= \lambda_1^{p+n+1-i} \\ g_{i1}\lambda_2^p + g_{i2}\lambda_2^{p-1} + \cdots + g_{i,p+1} &= \lambda_2^{p+n+1-i} \\ &\vdots \\ g_{i1}\lambda_{p+1}^p + g_{i2}\lambda_{p+1}^{p-1} + \cdots + g_{i,p+1} &= \lambda_{p+1}^{p+n+1-i}. \end{aligned}$$

Thus, for each $i, j = 1, 2, \ldots, p+1$, we obtain

$$g_{ij} = \frac{\det(V_j^{(i)})}{\det(V)}.$$

So the proof is complete. □

Thus we give the Binet formula for the nth Pell (p,p)-number $P_p^{(p)}(n)$ by the following corollary.

Corollary 7. *Let $P_p^{(p)}(n)$ be the nth Pell (p,p)-number. Then*

$$P_p^{(p)}(n) = \frac{\det(V_2^{(2)})}{\det(V)} = \frac{\det(V_{p+1}^{(p+1)})}{\det(V)}.$$

Proof. The conclusion is immediate result of Theorem 6 by taking $i = j = p+1$ or $i = j = 2$. □

Also we have the Binet formulas for all the generalized Pell (p,i)-numbers for $1 \leqslant i \leqslant p$.

Corollary 8. *For $n > p > 0$ and $1 \leqslant i \leqslant p$,*

$$P_p^{(p-j)}(n+p+1) = \sum_{k=0}^{j} \frac{\det(V_{j+1}^{(1)})}{\det(V)}$$

Let C be a $k \times k$ companion matrix as follows:

$$C(c_1, c_2, \ldots, c_k) = \begin{bmatrix} c_1 & c_2 & c_3 & \ldots & c_k \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then one can find the following result in [2]:

Theorem 9. *The (i,j) entry $c_{ij}^{(n)}(c_1, c_2, \ldots, c_k)$ in the matrix $C^n(c_1, c_2, \ldots, c_k)$ is given by the following formula:*

$$c_{ij}^{(n)}(c_1, c_2, \ldots, c_k) = \sum_{(t_1, t_2, \ldots m t_k)} \frac{t_j + t_{j+1} + \cdots + t_k}{t_1 + t_2 + \cdots + t_k} \times \binom{t_1 + t_2 + \cdots + t_k}{t_1, t_2, \ldots, t_k} c_1^{t_1} \ldots c_k^{t_k} \tag{8}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + kt_k = n - i + j$, and the coefficients in (8) is defined to be 1 if $n = i - j$.

Then we have the following Corollaries.

Corollary 10. *Let $P_p^{(p)}(n)$ be the generalized Pell (p,p)-number. Then*

$$P_p^{(p)}(n) = \sum_{(m_1, \ldots, m_{p+1})} \frac{m_{p+1}}{m_1 + m_2 + \cdots + m_{p+1}} \times \binom{m_1 + m_2 + \cdots + m_{p+1}}{m_1, m_2, \ldots, m_{p+1}} 2^{m_1}$$

where the summation is over nonnegative integers satisfying $m_1 + 2m_2 + \cdots + (p+1)m_{p+1} = n$.

Proof. In Theorem 9, when $i = p+1$ and $j = p+1$, then the conclusion can be directly from Theorem 2. □

Corollary 11. *Let $P_p^{(p)}(n)$ be the generalized Pell (p,p)-number. Then*

$$P_p^{(p)}(n) = \sum_{(m_1, \ldots, m_{p+1})} \frac{m_2 + \cdots + m_{p+1}}{m_1 + m_2 + \cdots + m_{p+1}} \times \binom{m_1 + m_2 + \cdots + m_{p+1}}{m_1, m_2, \ldots, m_{p+1}} 2^{m_1}$$

where the summation is over nonnegative integers satisfying $m_1 + 2m_2 + \cdots + (p+1)m_{p+1} = n$.

Proof. In Theorem 9, if we take $i = 2$ and $j = 2$, then we have the corollary from Theorem 2. □

In the above results, we give the combinatorial representation of the generalized Pell (p,p)-numbers. Now considering the result of Corollary 3, we can give the combinatorial representations of all generalized Pell (p,i)-numbers for $1 \leqslant i \leqslant p$ by the following corollary.

Corollary 12. *For $n > p > 0$ and $1 \leqslant i \leqslant p$,*

$$P_p^{(p-j)}(n+p+1) = \sum_{(m_1,\ldots,m_{p+1})} \frac{m_2 + \cdots + m_{p+1}}{m_1 + m_2 + \cdots + m_{p+1}} \times \binom{m_1 + m_2 + \cdots + m_{p+1}}{m_1, m_2, \ldots, m_{p+1}} 2^{m_1} + \sum_{k=1}^{j} \left(\sum_{(t_1,\ldots,t_{p+1})} \frac{t_2 + \cdots + t_{p+1}}{t_1 + t_2 + \cdots + t_{p+1}} \times \binom{t_1 + t_2 + \cdots + t_{p+1}}{t_1, t_2, \ldots, t_{p+1}} 2^{t_1} \right)$$

where the first summation is over nonnegative integers satisfying $m_1 + 2m_2 + \cdots + (p+1)m_{p+1} = n + p + 1$ and the second summation is over nonnegative integers satisfying $m_1 + 2m_2 + \cdots + (p+1)\, m_{p+1} = n + k$.

We consider the generating function of the generalized Pell (p,p)-numbers. We give the following Lemma.

Lemma 13. *Let $P_p^{(p)}(n)$ be the nth generalized Pell number, then for $n > p + 1$ and $p > 1$*

$$x^n = P_p^{(p)}(n+1)x^p + \sum_{j=0}^{p-1} P_p^{(p)}(n-j)x^j.$$

Proof. For the first case, we suppose that $p = 2$ and so $n = 4$, then by the definition of the Pell (p,p)-numbers, our claim is as follows:

$$x^4 = P_p^{(2)}(5)x^2 + P_p^{(2)}(3)x + P_p^{(2)}(4).$$

Since $P_p^{(2)}(5) = 4, P_p^{(2)}(3) = 1, P_p^{(2)}(4) = 2$ and the characteristic equation of the Pell p-numbers is $x^3 - 2x^2 - 1$, we obtain

$$x^4 = x \cdot x^3 = x(2x^2+1) = 2x^3 + x = 2(2x^2+1) + x = 4x^2 + x + 2 = P_p^{(2)}(5)x^2 + P_p^{(2)}(3)x + P_p^{(2)}(4).$$

Thus the proof is complete for the first case. Suppose that the equation holds for n. Now we show that the equation holds for $n+1$. Thus by our assumption and the characteristic equation of the generalized Pell p-numbers, $x^{p+1} = 2x^p + 1$,

$$\begin{aligned} x^{n+1} &= x^n x = \left(P_p^{(p)}(n+1)x^p + \sum_{j=0}^{p-1} P_p^{(p)}(n-j)x^j \right) x = P_p^{(p)}(n+1)x^{p+1} + \sum_{j=0}^{p-1} P_p^{(p)}(n-j)x^{j+1} \\ &= P_p^{(p)}(n+1)(2x^p+1) + \sum_{j=0}^{p-1} P_p^{(p)}(n-j)x^{j+1} \\ &= 2P_p^{(p)}(n+1)x^p + P_p^{(p)}(n-p+1)x^p + P_p^{(p)}(n-p+2)x^{p-1} + \cdots + P_p^{(p)}(n)x + P_p^{(p)}(n+1) \\ &= \left(2P_p^{(p)}(n+1) + P_p^{(p)}(n-p+1)\right)x^p + P_p^{(p)}(n-p+2)x^{p-1} + \cdots + P_p^{(p)}(n)x + P_p^{(p)}(n+1) \\ &= P_p^{(p)}(n+2)x^p + P_p^{(p)}(n-p+2)x^{p-1} + \cdots + P_p^{(p)}(n)x + P_p^{(p)}(n+1) = P_p^{(p)}(n+2)x^p + \sum_{j=0}^{p-1} P_p^{(p)}(n+1-j)x^j \end{aligned}$$

which is as desired. □

Now we give the generating function of the generalized Pell p-numbers: Let

$$g_p(x) = P_p^{(p)}(p+1) + P_p^{(p)}(p+2)x + P_p^{(p)}(p+3)x^2 + \cdots + F_p(n+p+1)x^n + \cdots.$$

Then

$$g_p(x) - 2xg_p(x) - x^{p+1}g_p(x) = (1 - 2x - x^{p+1})g_p(x).$$

By the definition of Pell (p,p)-numbers, we have $(1 - 2x - x^{p+1})g_p(x) = P_p^{(p)}(p+1) = 1$. Thus

$$g_p(x) = (1 - 2x - x^{p+1})^{-1}$$

for $0 \leqslant 2x + x^{p+1} < 1$.

Let $f_p(x) = 2x + x^{p+1}$. Then, for $0 \leqslant f_p(x) < 1$, we have the following Lemma.

Lemma 14. *For positive integers t and n, the coefficient of* x^n *in* $(f_p(x))^t$ *is*

$$\sum_{j}^{t} \binom{t}{j} 2^{t-j}, \frac{n}{p+1} \leqslant t \leqslant n$$

where the integers j satisfying pj + t = n.

Proof. From the above results, we write

$$(f_p(x))^t = (2x + x^{p+1})^t = x^t(2 + x^p)^t = x^t \sum_{j=0}^{t} \binom{t}{j} 2^{t-j} x^{pj}.$$

In the above equation, we consider the coefficient of x^n. For positive integers t and j such that $pj + t = n$ and $j \leqslant t$, the coefficients of x^n is

$$\sum_{j}^{t} \binom{t}{j} 2^{t-j}, \frac{n}{p+1} \leqslant t \leqslant n.$$

So we have the conclusion. □

Then we can give a combinatorial representation for the generalized Pell p-numbers by the following Theorem.

Theorem 15. *For positive integers t and n*

$$P_p^{(p)}(n + p + 1) = \sum_{\frac{n}{p+1} \leqslant t \leqslant n} \sum_{j}^{t} \binom{t}{j} 2^{t-j}$$

where the integers j satisfying pj + t = n.

Proof. Since

$$g_p(x) = P_p^{(p)}(p + 1) + P_p^{(p)}(p + 2)x + \cdots + P_p^{(p)}(n + p + 1)x^n + \cdots = \frac{1}{1 - 2x - x^{p+1}}$$

and $f_p(x) = 2x + x^{p+1}$, the coefficient of x^n is the $(n + p + 1)$ th generalized Pell p-number, $P_p(n + p + 1)$ in $g_p(x)$. Thus

$$g_p(x) = \frac{1}{1 - 2x - x^{p+1}} = \frac{1}{1 - f_p(x)} = 1 + f_p(x) + (f_p(x))^2 + \cdots + (f_p(x))^n + \cdots = 1 + x(2 + x^p)$$
$$+ x^2 \sum_{j=0}^{2} \binom{2}{j} 2^{2-j} x^{pj} + \cdots + x^n \sum_{j=0}^{n} \binom{n}{j} 2^{n-j} x^{pj} + \cdots.$$

As we need the coefficient of x^n, we only consider the first $n + 1$ terms on the right-side. Thus by Lemma 14, the proof is complete. □

The above combinatorial representation of the generalized Pell (p, p)-numbers can be obtained for all Pell (p, i)-numbers for $0 \leqslant i \leqslant p - 1$ by considering Corollary 3. For this purpose we give the following Corollary without proof.

Corollary 16. *For* $n > p > 0$ *and* $1 \leqslant r \leqslant p$,

$$P_p^{(p-r)}(n + p + 1) = P_p^{(p)}(n + p + 1) + \sum_{k=1}^{r} P_p^{(p)}(n + k) = \sum_{\frac{n}{p+1} \leqslant v \leqslant n} \sum_{s}^{v} \binom{v}{s} 2^{v-s} + \sum_{r=1}^{p} \sum_{\frac{n-r}{p+1} \leqslant t \leqslant n-r} \sum_{j}^{t} \binom{t}{j} 2^{t-j}$$

where the integers v satisfying ps + v = n and the integers j satisfying pj + t = n − r.

Now we give exponential representation for the generalized Pell p-numbers.

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$$\begin{aligned}\ln g_p(x) &= \ln[1-(2x+x^{p+1})]^{-1} = -\ln[1-(2x+x^{p+1})] \\ &= -\left[-(2x+x^{p+1}) - \frac{1}{2}(2x+x^{p+1})^2 - \cdots - \frac{1}{n}(2x+x^{p+1})^n - \cdots\right] \\ &= x\left[(2+x^p) + \frac{1}{2}(2+x^p)^2 + \cdots + \frac{1}{n}(2+x^p)^n + \cdots\right] = x\sum_{n=0}^{\infty}\frac{1}{n}(2+x^p)^n.\end{aligned}$$

Thus,

$$g_p(x) = \exp\left(x\sum_{n=0}^{\infty}\frac{1}{n}(2+x^p)^n\right).$$

4. Sums of the generalized Pell p-numbers by matrix methods

In this section we define a $(p+2)\times(p+2)$ matrix T, then we give the sums of the generalized Pell p-numbers subscripted from $p+1$ to n by the nth power of the matrix T.

Definition 17. For $p \geqslant 1$, let $T = (t_{ij})$ denote the $(p+2)\times(p+2)$ matrix with $t_{22} = 2$, $t_{11} = t_{21} = t_{2,p+2} = 1$, $t_{i+1,i} = 1$ for $2 \leqslant i \leqslant p+1$ and 0 otherwise.

Clearly, by the definition of the matrix A given by (4),

$$T = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2 & 0 & \dots & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & & 0 & 1 & 0 \end{bmatrix} \quad \text{or} \quad T = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & & & \\ 0 & & A & \\ \vdots & & & \\ 0 & & & \end{bmatrix}. \tag{9}$$

Let S_n denote the sums of the generalized Pell p-numbers from $p+1$ to n, that is,

$$S_n = \sum_{i=1}^{n} P_p^{(p)}(p+i). \tag{10}$$

Now we define a $(p+2)\times(p+2)$ matrix C_n as in the following form:

$$C_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ S_n & & & \\ S_{n-1} & & A^n & \\ \vdots & & & \\ S_{n-p} & & & \end{bmatrix} \tag{11}$$

where $A^n = G_n$ given by (5).

Then we have the following Theorem.

Theorem 18. *Let the $(p+2)\times(p+2)$ matrices T and C_n be as in (9) and (11), respectively. Then for $n \geqslant 1$*

$$C_n = T^n.$$

Proof. Since the definition of the sums, we can write

$$S_n = P_p^{(p)}(n+p) + S_{n-1}.$$

Also we have that $A^n = G_n$. Thus, combining these results gives us the following matrix recurrence relation

$$C_n = C_{n-1}T$$

by the inductive argument, we write

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$$C_n = C_1 T^{n-1}.$$

By the definition of the generalized Pell numbers and the matrix C_n, we obtain $C_1 = T$ and so

$$C_n = T^n.$$

So the proof is complete. □

It is seen that the matrix C is commutative under the matrix multiplication.

We define two $(p+2)\times(p+2)$ matrices. First, we define the matrix W as follows:

$$W = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -1/2 & \lambda_1^p & \lambda_2^p & \dots & \lambda_{p+1}^p \\ -1/2 & \lambda_1^{p-1} & \lambda_2^{p-1} & \dots & \lambda_{p+1}^{p-1} \\ \vdots & \vdots & \vdots & & \vdots \\ -1/2 & \lambda_1 & \lambda_2 & \dots & \lambda_{p+1} \\ -1/2 & 1 & 1 & \dots & 1 \end{bmatrix} \tag{12}$$

and the diagonal matrix D_1 as follows:

$$D_1 = \begin{bmatrix} 1 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{p+1} \end{bmatrix} \tag{13}$$

where λ_i's are the eigenvalues of the matrix A for $1 \leqslant i \leqslant p+1$.

We give the following Theorem for the computing the sums of the generalized Pell p-numbers $p+1$ from to n by using a matrix method.

Theorem 19. *Let the sums of the generalized Pell numbers S_n be as in (10). Then*

$$S_n = \left(\sum_{i=0}^{p} P_p^{(p)}(n+p+1-i) - 1\right) \Big/ 2.$$

Proof. Expanding the $\det W$ by the Laplace expansion of determinant with respect to the first row gives us $\det W = \det V$ where V is the transpose of the matrix Λ given by (7). By considering Lemma 2, the eigenvalues of the matrix W are distinct because the characteristic equation of the matrix W is $(x^p - 2x^{p-1} - 1)(x-1)$. Also the eigenvalues of W are 1, $\lambda_1, \dots, \lambda_{p+1}$. So the matrix W is diagonalizable. We can write that $TW = WD_1$. Then

$$T^n W = WD_1^n. \tag{14}$$

Since $T^n = C_n$, we write that $C_n W = WD_1^n$. Since $S_n = (C_n)_{2,1}$ and from matrix multiplication,

$$S_n - \frac{1}{2}\left(\sum_{i=0}^{p} P_p^{(p)}(n+p+1-i)\right) = -\frac{1}{2}. \tag{15}$$

Thus

$$S_n = \left(\sum_{i=0}^{p} P_p^{(p)}(n+p+1-i) - 1\right) \Big/ 2$$

So the proof is complete. □

In [53], the author presents an enumeration problem for the paths from A to c_n then show that the number of paths from A to c_n equal to the nth usual Fibonacci number. Now we interest in a problem of paths related with the generalized Pell numbers. The figure about the problem is as follows (see Fig. 1).

It is seen that the number of path from A to $c_1, c_2, \dots, c_{p+1}$ and c_i to c_{i+p+1} are all 1 for all i. Also the number of path from any c_i to c_{i+1} is 1 for $i \geqslant p+1$. Thus we know that the initial conditions of the generalized Pell $(p,0)$-numbers, that is, $P_p^{(0)}(1), P_p^{(0)}(2), \dots, P_p^{(0)}(p+1)$ are 1. Now we consider the case $n > p+1$. The number of the path

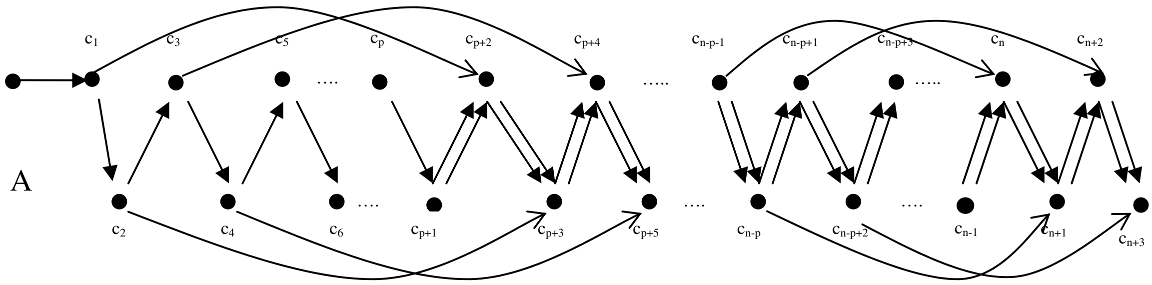

Fig. 1.

from A to c_{p+2} is 3. By the induction method one can see that the number of the path from A to c_n is the nth generalized Pell $(p, 0)$-number.

5. Generalized Pell numbers and permanents of certain matrices

In this section, we give relationships between the determinants of certain matrices and the generalized Pell (p, i)-numbers. First, we start with the case $i = p$, then we give general case.

Definition 20. Let $M(n,p) = [m_{ij}]$ be the $n \times n$ super-diagonal matrix with $m_{i,i} = 2$, $m_{i+1,i} = m_{i,i+p} = 1$ for a fixed integer p and all i, j, and 0 otherwise.

Clearly,

$$M(n,p) = \begin{pmatrix} 2 & 0 & \dots & 0 & \overset{(p+1)\text{th}}{\overset{\downarrow}{1}} & 0 & \dots & 0 \\ 1 & 2 & 0 & \dots & 0 & 1 & 0 & \\ 0 & 1 & 2 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \ddots & \ddots & \ddots & \dots \\ 0 & 0 & \dots & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 2 \end{pmatrix} \tag{16}$$

Then we have the following Theorem.

Theorem 21. *For* $n > 1$

$$\operatorname{per} M(n,p) = P_p^{(p)}(n+p+1)$$

where per $M(1,p) = 2$.

Proof. Firs we start with considering the case $n < p + 1$. It is easy to see that the matrix reduced to the bidiagonal matrix

$$M(p,p) = \begin{bmatrix} 2 & 0 & & & 0 \\ 1 & 2 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2 & 0 \\ 0 & & & 1 & 2 \end{bmatrix}.$$

Then the per $M(p,p) = 2^p$. Also we know that $P_p^{(p)}(n) = 0$ for $1 < n < p+1$ and $P_p^{(p)}(p+k) = 2^{k-1}$ for $2 \leqslant k \leqslant p+1$. Thus it is verified that for $1 < k < p+1$

$$\operatorname{per} M(k,p) = P_p^{(p)}(p+k+1) = 2^k.$$

So we have the conclusion for $n < p+1$. Suppose that the equation holds for $n > p$. Then we show that the equation holds for $n+1$. If we expand the per $M(n,p)$ by the Laplace expansion of permanent according to the first row, then we easily obtain that

$$\operatorname{per} M(n+1,p) = 2\operatorname{per} M(n,p) + \operatorname{per} M(n-p).$$

By our assumption and the recurrence relation of generalized Pell (p,p)-numbers,

$$\operatorname{per} M(n+1,p) = 2P_p^{(p)}(n+p+1) + P_p^{(p)}(n+1) = P_p^{(p)}(n+p+2).$$

Thus the proof is complete. □

Now we consider more general case, that is, relationship between the determinant of certain matrix and the generalized Pell (p,i)-numbers.

For this purpose, for $n > p$, $0 \leqslant i \leqslant p$, the matrix $H(n,p,i) = \left[h_{kj}^{(i)}\right]$ is defined by

$$\left[h_{kj}^{(i)}\right] = \begin{cases} h_{kk}^{(i)} = 2 & \text{for } 1 \leqslant k \leqslant n-p, \\ h_{kk}^{(i)} = 1 & \text{for } n-p+1 \leqslant k \leqslant n, \\ h_{k,k+p}^{(i)} = 1 & \text{for } 1 \leqslant k \leqslant n-p-i, \\ h_{k+1,k}^{(i)} = 1 & \text{for } 1 \leqslant k \leqslant n-1, \\ 0 & \text{otherwise.} \end{cases}$$

For example, when $n = 8$, $p = 4$ and $i = 2$, the matrix $H(8,4,2)$ is as follows:

$$H(8,4,2) = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then we have the following interesting result.

Theorem 22. *For $n > p > 0$ and $0 \leqslant i \leqslant p$,*

$$\operatorname{per} H(n,p,i) = P_p^{(i)}(n+1)$$

Proof. (Induction on n) First we consider the case $n = p+1$, then the values of i is changing between 0 and 1, and matrix $H(p+1,p,i)$ is reduced the following form

$$H(p,p,0) = \begin{bmatrix} 2 & & & 1 \\ 1 & 1 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 1 \end{bmatrix} \quad \text{and} \quad H(p,p,1) = \begin{bmatrix} 2 & & & 0 \\ 1 & 1 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 1 \end{bmatrix}$$

It is easy to see that per $H(p+1,p,0) = 3$ and per $H(p+1,p,1) = 2$. Since definition of generalized Pell sequence, we have

$$P_p^{(i)}(p+2) = 2P_p^{(i)}(p+1) + P_p^{(i)}(1)$$

and so

$$\operatorname{per} H(p+1,p,0) = P_p^{(0)}(p+2) = 3 \quad \text{and} \quad \operatorname{per} H(p+1,p,1) = P_p^{(1)}(p+2) = 3.$$

Thus the proof is complete for the first case. Suppose that the equation holds for $n-1>p$. Then expanding the per$H(n,p,i)$ by the Laplace expansion of permanent with respect to the first row, by the definition of matrix $H(n,p,i)$, gives us

$$\operatorname{per} H(n,p,i) = 2\operatorname{per} H(n-1,p,i) + \operatorname{per} H(n-p-1,p,i) = 2P_p^{(i)}(n) + P_p^{(i)}(n-p) = P_p^{(i)}(n+1).$$

Thus we have the conclusion. □

For giving the similar direction to the sums of the generalized Pell (p,i)-numbers, we define a new matrix.

Considering the definition of matrix $H(n,p,i)$, for $0 \leqslant i \leqslant p$, define the $n \times n$ matrix $L(n,p,i) = [l_{kj}]$ with the first row $l_{1,j} = 1$ for $1 \leqslant j \leqslant n-i$ and $l_{1j} = 0$ otherwise;

$$L(n,p,i) = \begin{bmatrix} 1 & & \dots & \overset{\text{i.th}}{\overset{\downarrow}{1}} & 0 & \dots & 0 \\ 1 & & & & & & \\ 0 & H(n-1,p,i) & & & & & \\ & & & & & & \\ 0 & & & & & & \end{bmatrix}. \tag{17}$$

Also $i = 0$, then the first row of matrix $L(n,p,0)$ consist of all 1.

Then we have the following Corollary.

Corollary 23. *For $n > p > 0$ and $0 \leqslant i \leqslant p$,*

$$\operatorname{per} L(n,p,i) = \sum_{k=0}^{n} P_p^{(i)}(k,p,i).$$

Proof. If we extend the per $L(n,p,i)$ with respect to the first row, then we obtain

$$perL(n,p,i) = \operatorname{per} H(n-1,p,i) + \operatorname{per} L(n-1,p,i).$$

By the result of Theorem 21 and the inductive argument, the proof is easily seen. □

For example, if we take $p = 4$ and $i = 2$, then the few terms of generalized Pell sequence $\{P_4^{(2)}(n)\}$ with initial conditions is

$$0, 0, 1, 1, 1, 2, 4, 9, 19, 39, 80, 164, \dots.$$

For $n = 8$, by Corollary 23, we have

$$\operatorname{per} L(10,4,2) = \operatorname{per} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \sum_{k=0}^{10} P_4^{(2)}(k) = 76.$$

A matrix A is called *convertible* if there is an $n \times n$ $(1,-1)$-matrix H such that per $A = \det(A \circ H)$, where $A \circ H$ denotes the Hadamard product of A and H. Such a matrix H is called a *converter* of A.

Let S be a $(1,-1)$-matrix of order n, defined by

$$S = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ -1 & 1 & \dots & 1 & 1 \\ 1 & -1 & \dots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \dots & -1 & 1 \end{bmatrix}.$$

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Denote $M(n,p)\circ S$, $H(n,p,i)\circ S$ and $L(n,p,i)\circ S$ by $\widehat{M}(n,p)$, $\widehat{H}(n,p,i)$ and $\widehat{L}(n,p,i)$ respectively. Then the following facts can be easily seen.

For $n>1$,

$$\det \widehat{M}(n,p) = P_p^{(p)}(n+p+1).$$

For $n>p>0$ and $0\leqslant i\leqslant p$,

$$\det \widehat{H}(n,p,i) = P_p^{(i)}(n+1),$$

$$\det \widehat{L}(n,p,i) = \sum_{k=0}^{n} P_p^{(i)}(k,p,i).$$

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