FACTORIZATIONS OF THE PASCAL MATRIX VIA GENERALIZED SECOND ORDER RECURRENT MATRIX

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Abstract. In this paper, we consider positively and negatively subscripted terms of generalized binary sequence $\{U_n\}$ with indices in arithmetic progression. We give a factorization of the Pascal matrix with a matrix associated with the sequence $\{U_{\pm kn}\}$ for a fixed positive integer k, generalizing results of [1, 5, 6, 7]. Some new factorizations and combinatorial identities are derived as applications. Therefore we generalize the earlier results on the factorizations of Pascal matrix.

1. INTRODUCTION

For $n > 0$, the $n \times n$ Pascal matrix $P_n = [p_{ij}]$ is defined as follows [4]:

$$p_{ij} = \begin{cases} \binom{i-1}{j-1} & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

In [6], it is shown that the matrix P_n satisfies

$$P_n = \mathcal{F}_n L_n,$$

where the $n \times n$ Fibonacci matrix $\mathcal{F}_n = [f_{ij}]$ and the matrix $L_n = [l_{ij}]$ are defined by

$$[f_{ij}] = \begin{cases} F_{i-j+1} & \text{if } i - j + 1 \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$l_{ij} = \left(\binom{i-1}{j-1} - \binom{i-2}{j-1} - \binom{i-3}{j-1}\right),$$

respectively and F_n stands for the nth Fibonacci number.

In [7], the authors define an $n \times n$ matrix $R_n = [r_{i,j}]$ as follows:

$$r_{ij} = \binom{i-1}{j-1} - \binom{i-1}{j} - \binom{i-1}{j+1},$$

and show that $P_n = R_n \mathcal{F}_n$. As an example, they give the following result:

$$\begin{aligned} \binom{n-1}{r-1} &= F_{n-r+1} + (n-2) F_{n-r} + \frac{1}{2}\left(n^2 - 5n + 2\right) F_{n-r-1} \\ &+ \sum_{k=r}^{n-3} \binom{n-1}{k-1} \left[2 - \frac{n}{k} - \frac{(n-k)(n-k-1)}{k(k+1)}\right] F_{k-r+1}. \end{aligned}$$

Especially for $r = 1$, they have

$$\sum_{k=1}^{n} \left(\binom{n-1}{k-1} - \binom{n-1}{k} - \binom{n-1}{k+1}\right) F_k = 1.$$

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Furthermore they define an $n \times n$ matrix U_n as in the form:

$$U_n = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ -F_3 & 1 & 1 & 0 & \dots & 0 & 0 \\ -F_4 & 0 & 1 & 1 & \dots & 0 & 0 \\ -F_5 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ -F_n & 0 & 0 & 0 & \dots & 1 & 1 \end{bmatrix},$$

and the matrices $\overline{U}_k$ and $\overline{R}_n$ by $\overline{U}_k = I_{n-k} \oplus U_k$ and $\overline{R}_n = [1] \oplus R_{n-1}$. Then the authors give the following factorization:

$$\begin{aligned} R_n &= \overline{R}_n U_n, \\ R_n &= \overline{U}_1 \overline{U}_2 ... \overline{U}_{n-1} \overline{U}_n. \end{aligned}$$

Let

$$S_0 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad S_{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

$S_k = S_0 \oplus I_k$, for $k \in \mathbb{N}$, $G_1 = I_n$, $G_2 = I_{n-3} \oplus S_{-1}$, and $G_k = I_{n-k} \oplus S_{k-3}$ for $k \geq 3$.

In [5], the authors give the following factorization :

$$\mathcal{F}_n = G_1 G_2 ... G_n, \tag{1.1}$$

where $\mathcal{F}_n$ is defined as before.

In [3], the authors show that the Stirling matrix $S_n = (S(i,j))_{ij}$ of the second kind can be written in terms of the Pascal matrix P_n :

$$S_n = P_n ([1] \oplus S_{n-1}),$$

where the $S(i,j)'$s are the Stirling numbers of the second kind defined by the following recurrence:

$$S(n,k) = S(n-1,k-1) + S(n-1,k).$$

In [1], the authors define the $n \times n$ matrix $W_n = [w_{ij}]$ and Pell matrix $E_n = [e_{ij}]$ as shown

$$w_{ij} = \begin{cases} P_i & \text{if } j = 1, \\ 1 & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases}$$

and $e_{ij} = P_{i-j+1}$ if $i - j + 1 \geq 0$ and 0 otherwise where P_i is the ith Pell number. Then they show that

$$E_n = W_n (I_1 \oplus W_{n-1}) (I_2 \oplus W_{n-2}) ... (I_{n-2} \oplus W_2).$$

The Fibonacci and Lucas sequences have been discussed in so many studies. Further various generalizations and matrix representations of these sequences have been also introduced and investigated by many authors.

For $n > 0$ and nonnegative integers A and B such that $A^2 + 4B \neq 0$, the generalized Fibonacci and Lucas type sequences $\{U_n\}$ and $\{V_n\}$ are defined by

$$\begin{aligned} U_{n+1} &= AU_n + BU_{n-1}, \\ V_{n+1} &= AV_n + BV_{n-1}, \end{aligned}$$

where $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = A$, respectively. When $A = B = 1$, $U_n = F_n$ (nth Fibonacci number) and $V_n = L_n$ (nth Lucas number).

The authors [2] consider positively and negatively subscripted terms of the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$ for a fixed positive integer k. They obtain relationships between these sequences and the determinants of certain tridiagonal matrices. Further, the authors give more general trigonometric factorizations and representations for the terms of $\{U_{\pm kn}\}$ and $\{V_{\pm kn}\}$. Generating functions and combinatorial representations of them are derived. Finally they obtain the following recurrence relations for $k > 0$ and $n > 1$,

$$\begin{aligned} U_{kn} &= V_k U_{k(n-1)} + (-1)^{k+1} B^k U_{k(n-2)}, \\ V_{kn} &= V_k V_{k(n-1)} + (-1)^{k+1} B^k V_{k(n-2)}. \end{aligned}$$

In this paper, we consider positively and negatively subscripted terms of generalized binary sequence $\{U_n\}$. We give a factorization of the Pascal matrix with a matrix associated with the sequence$\{U_{\pm kn}\}$. Also some new factorizations and combinatorial identities are derived as applications of our results. Therefore we generalize the results of some earlier studies on these factorizations.

2. Factorizations of the Pascal matrix via recurrent matrices associated with the $\{U_{\pm kn}\}$

In this section, we define a matrix associated with the sequence $\{U_{\pm kn}\}$. Then we obtain some factorizations of the Pascal matrix by this new matrix and derive new identities as an applications of these factorizations.

Let the $n \times n$ lower triangular matrix $H_n = [h_{ij}]$ be defined as follows:

$$h_{ij} = \begin{cases} U_{\pm(i-j+1)k} & \text{if } i - j + 1 \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly the matrix H_n is in the form

$$H_n = \begin{bmatrix} U_{\pm k} & & & & 0 \\ U_{\pm 2k} & U_{\pm k} & & & \\ U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & & \\ \vdots & \vdots & \vdots & \ddots & \\ U_{\pm kn} & U_{\pm k(n-1)} & U_{\pm k(n-2)} & \cdots & U_{\pm k} \end{bmatrix}.$$

Now we define an $n \times n$ matrix $C_n = [c_{ij}]$ with $c_{ij} = \frac{1}{U_{\pm k}} \left(\binom{i-1}{j-1} - \frac{U_{\pm 2k}}{U_{\pm k}} \binom{i-1}{j} + \binom{i-1}{j+1} (-B)^{\pm k} \right)$ if $i \geq j$ and 0 otherwise.

Then we can give the following theorem.

Theorem 1. *For $n > 0$,*

$$P_n = C_n H_n.$$

Proof. To prove the theorem, it is sufficient to show $P_nH_n^{-1} = C_n$. The inverse of H_n is given by

$$H_n^{-1} = \left(h'_{ij}\right) = \begin{bmatrix} \frac{1}{U_{\pm k}} & & & & & \\ -\frac{V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & & & & \\ \frac{(-B)^{\pm k}}{U_{\pm k}} & -\frac{V_{\pm k}}{U_{\pm k}} & \ddots & & & \\ 0 & \frac{(-B)^{\pm k}}{U_{\pm k}} & \ddots & \frac{1}{U_{\pm k}} & & \\ \vdots & \ddots & \ddots & -\frac{V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & \\ 0 & \dots & 0 & \frac{(-B)^{\pm k}}{U_{\pm k}} & -\frac{V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} \end{bmatrix}.$$

In order to prove that $P_nH_n^{-1} = C_n$, now consider the case for $i \geq 1$ and $j = 1$. By the definitions of P_n and H_n^{-1}, we write

$$\begin{aligned} \sum_{k=1}^{i} p_{i,k}h'_{k,1} &= p_{i,1}h'_{1,1} + p_{i,2}h'_{2,1} + p_{i,3}h'_{3,1} \\ &= \tbinom{i-1}{0}\tfrac{1}{U_{\pm k}} + \tbinom{i-1}{1}\left(-\tfrac{V_{\pm k}}{U_{\pm k}}\right) + \tbinom{i-1}{2}\left(\tfrac{(-B)^{\pm k}}{U_{\pm k}}\right) \\ &= \tfrac{1}{U_{\pm k}} - (i-1)\tfrac{V_{\pm k}}{U_{\pm k}} + \tfrac{(i-2)(i-1)}{2U_{\pm k}}(-B)^{\pm k}. \end{aligned}$$

Since the definition of C_n,

$$c_{i,1} = \tfrac{1}{U_{\pm k}}\left(\tbinom{i-1}{0} - \tbinom{i-1}{1}\tfrac{U_{\pm 2k}}{U_{\pm k}} + \tbinom{i-1}{2}(-B)^{\pm k}\right),$$

and so we get the required conclusion

$$\sum_{k=1}^{i} p_{i,k}h'_{k,1} = c_{i,1}.$$

For $i \geq 1$ and $j \geq 2$, we obtain

$$\begin{aligned} \sum_{k=1}^{n} p_{i,k}h'_{k,j} &= p_{i,j}h'_{j,j} + p_{i,j+1}h'_{j+1,j} + p_{i,j+2}h'_{j+2,j} \\ &= \tbinom{i-1}{j-1}\tfrac{1}{U_{\pm k}} + \tbinom{i-1}{j}\left(-\tfrac{V_{\pm k}}{U_{\pm k}}\right) + \tbinom{i-1}{j+1}\tfrac{(-B)^{\pm k}}{U_{\pm k}} \\ &= \tbinom{i-1}{j-1}\tfrac{1}{U_{\pm k}} - \tbinom{i-1}{j}\tfrac{U_{\pm 2k}}{U_{\pm k}^2} + \tbinom{i-1}{j+1}\tfrac{(-B)^{\pm k}}{U_{\pm k}}. \end{aligned}$$

From the definition of C_n, we write

$$\sum_{k=1}^{n} p_{i,k}h'_{k,j} = c_{i,j}.$$

Then we obtain that $P_nH_n^{-1} = C_n$. Thus the proof is complete. □

As a result of Theorem 1, we may give the following identity without proof.

Corollary 1. *For $n \geq r > 0$,*

$$\tbinom{n-1}{r-1} = \sum_{j=r}^{n}\left(\tbinom{n-1}{j-1} - \tbinom{n-1}{j}V_{\pm k} + \tbinom{n-1}{j+1}(-B)^{\pm k}\right)\tfrac{U_{\pm(j-r+1)k}}{U_{\pm k}}.$$

In particular, if we take $r = 1$ in Corollary 1, we have

$$\sum_{j=1}^{n}\left(\tbinom{n-1}{j-1} - \tbinom{n-1}{j}V_{\pm k} + \tbinom{n-1}{j+1}(-B)^{\pm k}\right)\tfrac{U_{\pm jk}}{U_{\pm k}} = 1.$$

Lemma 1. *For* $1 \le i, j \le n$,

$$\sum_{j=3}^{i} \left(\binom{i-2}{j-2} - \binom{i-2}{j-1} V_{\pm k} + \binom{i-2}{j} (-B)^{\pm k} \right) \frac{U_{\pm jk}}{U_{\pm k}^2}$$
$$= (i-2) \frac{V_{\pm k}^2}{U_{\pm k}} - \left(\binom{i-2}{2} V_{\pm k} + \binom{i-2}{1} \right) \frac{(-B)^{\pm k}}{U_{\pm k}}.$$

Proof. (By induction on i). Clearly the equation holds for $i = 3$. Assume that the equation holds for $i \ge 4$. Thus

$$\begin{aligned}
&\sum_{j=3}^{i+1} \left(\binom{i-1}{j-2} - \binom{i-1}{j-1} V_{\pm k} + \binom{i-1}{j} (-B)^{\pm k} \right) \frac{U_{\pm jk}}{U_{\pm k}^2} \\
= &\sum_{j=3}^{i+1} \left(\binom{i-2}{j-2} - \binom{i-2}{j-1} V_{\pm k} + \binom{i-2}{j} (-B)^{\pm k} \right) \frac{U_{\pm jk}}{U_{\pm k}^2} \\
&+ \sum_{j=3}^{i+1} \left(\binom{i-2}{j-3} - \binom{i-2}{j-2} V_{\pm k} + \binom{i-2}{j-1} (-B)^{\pm k} \right) \frac{U_{\pm jk}}{U_{\pm k}^2} \\
= &\sum_{j=3}^{i} \left(\binom{i-2}{j-2} - \binom{i-2}{j-1} V_{\pm k} + \binom{i-2}{j} (-B)^{\pm k} \right) \frac{U_{\pm jk}}{U_{\pm k}^2} \\
&+ \sum_{j=2}^{i+1} \left(\binom{i-2}{j-2} - \binom{i-2}{j-1} V_{\pm k} + \binom{i-2}{j} (-B)^{\pm k} \right) \frac{U_{\pm (j+1)k}}{U_{\pm k}^2} \\
= &(i-2) \frac{V_{\pm k}^2}{U_{\pm k}} - \left(\binom{i-2}{2} V_{\pm k} + \binom{i-2}{1} \right) \frac{(-B)^{\pm k}}{U_{\pm k}} \\
&+ \sum_{j=2}^{i+1} \left(\binom{i-2}{j-2} - \binom{i-2}{j-1} V_{\pm k} + \binom{i-2}{j} (-B)^{\pm k} \right) \\
&\times \frac{\left(V_{\pm k} U_{\pm jk} - (-B)^{\pm k} U_{\pm (j-1)k} \right)}{U_{\pm k}^2}.
\end{aligned}$$

After some calculations and by Corollary 1, we get

$$\sum_{j=3}^{i+1} \left(\binom{i-1}{j-2} - \binom{i-1}{j-1} V_{\pm k} + \binom{i-1}{j} (-B)^{\pm k} \right) \frac{U_{\pm jk}}{U_{\pm k}^2}$$
$$= (i-1) \frac{V_{\pm k}^2}{U_{\pm k}} - \left(\binom{i-1}{2} V_{\pm k} + \binom{i-1}{1} \right) \frac{(-B)^{\pm k}}{U_{\pm k}}.$$

So the proof is complete. □

Now, we define the $n \times n$ matrices T_n, $\overline{C}_n$ and $\overline{T}_k$ by

$$T_n = \begin{bmatrix}
\frac{1}{U_{\pm k}} & & & & & 0 \\
1 - \frac{U_{\pm 2k}}{U_{\pm k}} & 1 & & & & \\
-\frac{U_{\pm 3k}}{U_{\pm k}} & 1 & 1 & & & \\
-\frac{U_{\pm 4k}}{U_{\pm k}} & 0 & 1 & 1 & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \\
-\frac{U_{\pm nk}}{U_{\pm k}} & 0 & \dots & 0 & 1 & 1
\end{bmatrix},$$

$\overline{C}_n = [1] \oplus C_{n-1}$ and $\overline{T}_k = I_{n-k} \oplus T_k$ where C_n is defined as before.

Lemma 2. *For* $n > 0$,

$$C_n = \overline{C}_n T_n.$$

Proof. We denote the (i,j) element of the matrix $\overline{C}_n$ by $\overline{c}_{i,j}$. Then,

$$\overline{c}_{i,j}=\begin{cases} 1 & \text{if } i=1, j=1, \\ 0 & \text{if } i\neq 1, j=1 \text{ or } i=1, j\neq 1, \\ c_{i-1,j-1} & \text{otherwise.} \end{cases}$$

Let $\overline{C}_n T_n=[K_{i,j}]$ and $T_n=[t_{i,j}]$. Obviously $K_{1,1}=\frac{1}{U_{\pm k}}=c_{1,1}$, $K_{2,2}=\frac{1}{U_{\pm k}}=c_{2,2}$, $K_{2,1}=\frac{1-V_{\pm k}}{U_{\pm k}}=c_{2,1}$ and $K_{i,j}=0$ for $i<j$. Since $t_{i,1}=-\frac{U_{\pm ik}}{U_{\pm k}}$ for $i\geq 3, j=1$ and from Lemma 1, we have

$$\begin{aligned} K_{i,1} &= \sum_{j=2}^{i}\overline{c}_{i,j}t_{j,1}=\sum_{j=2}^{i}c_{i-1,j-1}t_{j,1} \\ &= \sum_{j=2}^{i}\left(\binom{i-2}{j-2}-\frac{U_{\pm 2k}}{U_{\pm k}}\binom{i-2}{j-1}+\binom{i-2}{j}(-B)^{\pm k}\right)\frac{1}{U_{\pm k}}t_{j,1} \\ &= \left(\binom{i-2}{0}-\frac{U_{\pm 2k}}{U_{\pm k}}\binom{i-2}{1}+\binom{i-2}{2}(-B)^{\pm k}\right)\frac{1}{U_{\pm k}}t_{2,1} \\ &\quad -\sum_{j=3}^{i}\left(\binom{i-2}{j-2}-\frac{U_{\pm 2k}}{U_{\pm k}}\binom{i-2}{j-1}+\binom{i-2}{j}(-B)^{\pm k}\right)\frac{1}{U_{\pm k}^2}U_{\pm jk} \\ &= \left(1-(i-2)V_{\pm k}+\binom{i-2}{2}(-B)^{\pm k}\right)\left(\frac{1-V_{\pm k}}{U_{\pm k}}\right) \\ &\quad -\left((i-2)\frac{V_{\pm k}^2}{U_{\pm k}}-\left(\binom{i-2}{2}V_{\pm k}+\binom{i-2}{1}\right)\left(\frac{(-B)^{\pm k}}{U_{\pm k}}\right)\right) \\ &= \left(\binom{i-1}{0}-\binom{i-1}{1}V_{\pm k}+\binom{i-1}{2}(-B)^{\pm k}\right)\frac{1}{U_{\pm k}} \\ &= c_{i,1}. \end{aligned}$$

In general, for $i\geq 2$, $j\geq 2$, from the definition of C_n, we get

$$K_{i,j}=\sum_{m=1}^{i}\overline{c}_{i,m}t_{m,j}=c_{i-1,j-1}.1+c_{i-1,j}.1=c_{i,j}.$$

Thus the proof is complete. □

Lemma 3. *For $n>0$,*

$$C_n=\overline{T}_1\overline{T}_2...\overline{T}_{n-1}\overline{T}_n.$$

Proof. From the definitions of C_n and $\overline{T}_n$, the proof directly follows. □

For example, when $n = 4$ in Lemma 3, we obtain

$$\begin{aligned}
C_4 &= \begin{bmatrix} \frac{1}{U_{\pm k}} & 0 & 0 & 0 \\ \frac{1-V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & 0 & 0 \\ \frac{1-2V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & \frac{2-V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & 0 \\ \frac{1-3V_{\pm k}+3(-B)^{\pm k}}{U_{\pm k}} & \frac{3-3V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & \frac{3-V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{U_{\pm k}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{U_{\pm k}} & 0 \\ 0 & 0 & 1-\frac{U_{\pm 2k}}{U_{\pm k}} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{U_{\pm k}} & 0 & 0 \\ 0 & 1-\frac{U_{\pm 2k}}{U_{\pm k}} & 1 & 0 \\ 0 & -\frac{U_{\pm 3k}}{U_{\pm k}} & 1 & 1 \end{bmatrix} \\
&\quad \times \begin{bmatrix} \frac{1}{U_{\pm k}} & 0 & 0 & 0 \\ 1-\frac{U_{\pm 2k}}{U_{\pm k}} & 1 & 0 & 0 \\ -\frac{U_{\pm 3k}}{U_{\pm k}} & 1 & 1 & 0 \\ -\frac{U_{\pm 4k}}{U_{\pm k}} & 0 & 1 & 1 \end{bmatrix} \\
&= \overline{T}_1 \overline{T}_2 \overline{T}_3 \overline{T}_4.
\end{aligned}$$

Now define

$$M_0 = \begin{bmatrix} U_{\pm k} & 0 & 0 \\ V_{\pm k} & 1 & 0 \\ -(-B)^{\pm k} & 0 & 1 \end{bmatrix}, \; M_{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & U_{\pm k} & 0 \\ 0 & U_{\pm 2k} & U_{\pm k} \end{bmatrix},$$

$M_k = M_0 \oplus I_k$, $k \in \mathbb{N}$, and $A_1 = I_n$, $A_2 = I_{n-3} \oplus M_{-1}$, $A_k = I_{n-k} \oplus M_{k-3}$, $k \geq 3$.

Therefore, we easily obtain the following result without proof.

Lemma 4. *For $n > 0$,*

$$H_n = A_1 A_2 ... A_n.$$

In particular, when $n = 4$,

$$\begin{aligned}
H_4 &= \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 \\ U_{\pm 2k} & U_{\pm k} & 0 & 0 \\ U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & 0 \\ U_{\pm 4k} & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{\pm k} & 0 \\ 0 & 0 & U_{\pm 2k} & U_{\pm k} \end{bmatrix} \times \\
&\quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & U_{\pm k} & 0 & 0 \\ 0 & V_{\pm k} & 1 & 0 \\ 0 & -(-B)^{\pm k} & 0 & 1 \end{bmatrix} \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 \\ V_{\pm k} & 1 & 0 & 0 \\ -(-B)^{\pm k} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= A_1 A_2 A_3 A_4.
\end{aligned}$$

Corollary 2. *For $n > 0$,*

$$P_n = C_n H_n = \overline{T}_1 \overline{T}_2 ... \overline{T}_{n-1} \overline{T}_n A_1 A_2 ... A_n.$$

Proof. From Theorem 1 and Lemma 4, we have the conclusion. □

Now we define an $n \times n$ matrix $C_n' = [c_{i,j}']$ with $c_{i,j}' = \frac{1}{U_{\pm k}}\left(\binom{i-1}{j-1} - \frac{U_{\pm 2k}}{U_{\pm k}}\binom{i-2}{j-1} + \binom{i-3}{j-1}(-B)^{\pm k}\right)$ if $i \geq j$ and 0 otherwise. Therefore, we can give the following theorem.

Theorem 2. *Let P_n, H_n, C_n' be the $n \times n$ matrices defined as above. Thus, we have*

$$P_n = H_n C_n'.$$

Proof. It is sufficient to show $H_n^{-1} P_n = C_n'$. Let $H_n^{-1} P_n = [z_{i,j}]$. Here note that the matrix H_n^{-1} is in the form

$$H_n^{-1} = \begin{bmatrix} \frac{1}{U_{\pm k}} & & & & & 0 \\ -\frac{V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & & & & \\ \frac{(-B)^{\pm k}}{U_{\pm k}} & -\frac{V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & & & \\ 0 & \frac{(-B)^{\pm k}}{U_{\pm k}} & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & -\frac{V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & \\ 0 & \cdots & 0 & \frac{(-B)^{\pm k}}{U_{\pm k}} & -\frac{V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} \end{bmatrix}.$$

Clearly $z_{1,1} = c_{1,1}'$, $z_{2,1} = c_{2,1}'$, $z_{2,2} = c_{2,2}'$ and for $i < j$, $z_{i,j} = c_{ij} = 0$. Since the all elements of the first column of P_n are 1, we have $z_{i,j} = \frac{1 - V_{\pm k} + (-B)^{\pm k}}{U_{\pm k}}$ for $i \geq 3$ and $j = 1$. For $i, j \geq 2$, from the definition of C_n', we obtain

$$\begin{aligned} z_{i,j} &= \sum_{j=1}^{n} h_{i,k}' p_{k,j} = h_{i,i}' p_{i,j} + h_{i,i-1}' p_{i-1,j} + h_{i,i-2}' p_{i-2,j} \\ &= \frac{1}{U_{\pm k}}\binom{i-1}{j-1} + \left(-\frac{V_{\pm k}}{U_{\pm k}}\right)\binom{i-2}{j-1} + \frac{(-B)^{\pm k}}{U_{\pm k}}\binom{i-3}{j-1} \\ &= c_{i,j}'. \end{aligned}$$

Thus the proof is complete. □

From Theorem 2, we get the following result:

Corollary 3. *For $n \geq r > 0$,*

$$\binom{n-1}{r-1} = \sum_{j=r}^{n}\left(\frac{U_{\pm(n-j+1)k}}{U_{\pm k}}\right)\left(\binom{j-1}{r-1} - \binom{j-2}{r-1}V_{\pm k} + \binom{j-3}{r-1}(-B)^{\pm k}\right).$$

In particular, when $r = 1$, we obtain

$$\sum_{j=1}^{n}\left(\frac{U_{\pm(n-j+1)k}}{U_{\pm k}}\right)\left(1 - \binom{j-2}{0}V_{\pm k} + \binom{j-3}{0}(-B)^{\pm k}\right) = 1.$$

We define an $n \times n$ matrix Q_n by

$$Q_n = \begin{bmatrix} \frac{1}{U_{\pm k}} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1-V_{\pm k}}{U_{\pm k}} & 1 & 0 & 0 & \cdots & 0 \\ \frac{1-V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & 1 & 1 & 0 & \cdots & 0 \\ \frac{1-V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{1-V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

If we take $\overline{C}_n' = [1] \oplus C_{n-1}'$, the following result is easily seen.

Lemma 5. *For $n > 0$,*

$$C'_n = Q_n \overline{C}'_n.$$

When $n = 4$, we get

$$\begin{aligned}
C'_4 &= \begin{bmatrix} \frac{1}{U_{\pm k}} & 0 & 0 & 0 \\ \frac{1-V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & 0 & 0 \\ \frac{1-V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & \frac{2-V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & 0 \\ \frac{1-V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & \frac{3-2V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & \frac{3-V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{U_{\pm k}} & 0 & 0 & 0 \\ \frac{1-V_{\pm k}}{U_{\pm k}} & 1 & 0 & 0 \\ \frac{1-V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & 1 & 1 & 0 \\ \frac{1-V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{U_{\pm k}} & 0 & 0 \\ 0 & \frac{1-V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & 0 \\ 0 & \frac{1-V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & \frac{2-V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} \end{bmatrix} \\
&= Q_4 \overline{C}'_4.
\end{aligned}$$

Lemma 6. *Let the matrix Q_k be defined as before and $\overline{Q}_k = I_{n-k} \oplus Q_k$. Then*

$$C'_n = \overline{Q}_n \overline{Q}_{n-1} \ldots \overline{Q}_2 \overline{Q}_1.$$

We can give the following example:

$$\begin{aligned}
C'_3 &= \begin{bmatrix} \frac{1}{U_{\pm k}} & 0 & 0 \\ \frac{1-V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} & 0 \\ \frac{1-V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & \frac{2-V_{\pm k}}{U_{\pm k}} & \frac{1}{U_{\pm k}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{U_{\pm k}} & 0 & 0 \\ \frac{1-V_{\pm k}}{U_{\pm k}} & 1 & 0 \\ \frac{1-V_{\pm k}+(-B)^{\pm k}}{U_{\pm k}} & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{U_{\pm k}} & 0 \\ 0 & \frac{1-V_{\pm k}}{U_{\pm k}} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{U_{\pm k}} \end{bmatrix} \\
&= \overline{Q}_3 \overline{Q}_2 \overline{Q}_1.
\end{aligned}$$

Now, we consider an $n \times n$ matrix $T'_n = \left[t'_{i,j}\right]$ with

$$t'_{i,j} = \begin{cases} U_{\pm ik} & \text{if } i \geq 1, j = 1, \\ 1 & \text{if } i = j,\ i,j \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Now we can give the following results:

Lemma 7. *For $n > 1$,*

$$H_n = T'_n \left([1] \oplus H_{n-1}\right).$$

Proof. Let $T'_n \left([1] \oplus H_{n-1}\right) = (y_{i,j})$. Since $(1,1)-$element of the matrix $[1] \oplus H_{n-1}$ is 1 and other elements are zero in the first column of this matrix, we get $y_{i,1} = U_{\pm ik}$. For $i \geq 1$, $j \geq 2$ and $i \geq j$, using definitions of T'_n and $[1] \oplus H_{n-1}$, we obtain $y_{i,j} = U_{\pm(i-j+1)k}$. For $i < j$, we obtain $y_{i,j} = 0$. Finally we get $y_{i,j} = h_{ij}$ for $1 \leq i,j \leq n$ which completes the proof. □

When $n=6$ in Lemma 7,

$$
\begin{aligned}
H_6 &= \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 & 0 & 0 \\ U_{\pm 2k} & U_{\pm k} & 0 & 0 & 0 & 0 \\ U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & 0 & 0 & 0 \\ U_{\pm 4k} & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & 0 & 0 \\ U_{\pm 5k} & U_{\pm 4k} & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & 0 \\ U_{\pm 6k} & U_{\pm 5k} & U_{\pm 4k} & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} \end{bmatrix} \\
&= \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 & 0 & 0 \\ U_{\pm 2k} & 1 & 0 & 0 & 0 & 0 \\ U_{\pm 3k} & 0 & 1 & 0 & 0 & 0 \\ U_{\pm 4k} & 0 & 0 & 1 & 0 & 0 \\ U_{\pm 5k} & 0 & 0 & 0 & 1 & 0 \\ U_{\pm 6k} & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & U_{\pm k} & 0 & 0 & 0 & 0 \\ 0 & U_{\pm 2k} & U_{\pm k} & 0 & 0 & 0 \\ 0 & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & 0 & 0 \\ 0 & U_{\pm 4k} & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & 0 \\ 0 & U_{\pm 5k} & U_{\pm 4k} & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} \end{bmatrix} \\
&= T_6' \left([1] \oplus H_5 \right).
\end{aligned}
$$

Lemma 8. *If we define* $\overline{T}_k' = I_{n-k} \oplus T_k'$, *then*

$$H_n = \overline{T}_n' \overline{T}_{n-1}' ... \overline{T}_2' \overline{T}_1'.$$

For example, when $n=5$ in Lemma 8, we have

$$
\begin{aligned}
H_5 &= \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 & 0 \\ U_{\pm 2k} & U_{\pm k} & 0 & 0 & 0 \\ U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & 0 & 0 \\ U_{\pm 4k} & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & 0 \\ U_{\pm 5k} & U_{\pm 4k} & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} \end{bmatrix} \\
&= \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 & 0 \\ U_{\pm 2k} & 1 & 0 & 0 & 0 \\ U_{\pm 3k} & 0 & 1 & 0 & 0 \\ U_{\pm 4k} & 0 & 0 & 1 & 0 \\ U_{\pm 5k} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & U_{\pm k} & 0 & 0 & 0 \\ 0 & U_{\pm 2k} & 1 & 0 & 0 \\ 0 & U_{\pm 3k} & 0 & 1 & 0 \\ 0 & U_{\pm 4k} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & U_{\pm k} & 0 & 0 \\ 0 & 0 & U_{\pm 2k} & 1 & 0 \\ 0 & 0 & U_{\pm 3k} & 0 & 1 \end{bmatrix} \\
&\times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & U_{\pm k} & 0 \\ 0 & 0 & 0 & U_{\pm 2k} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & U_{\pm k} \end{bmatrix} \\
&= \overline{T}_5' \overline{T}_4' \overline{T}_3' \overline{T}_2' \overline{T}_1'.
\end{aligned}
$$

Now we define an $n \times n$ matrix D_n as in the form:

$$D_n = \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 & \cdots & 0 \\ V_{\pm k} & 1 & 0 & 0 & \cdots & 0 \\ -(-B)^{\pm k} & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Then we have the following factorization.

Lemma 9. *For $n > 1$,*

$$H_n = ([1] \oplus H_{n-1})\, D_n.$$

Proof. Since the $(i,j)-$element of $[1] \oplus H_{n-1}$ is h_{ij} and the definition of D_n, the result is readily seen. □

For $n = 4$ in Lemma 9, we obtain

$$\begin{aligned} H_4 &= \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 \\ U_{\pm 2k} & U_{\pm k} & 0 & 0 \\ U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & 0 \\ U_{\pm 4k} & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & U_{\pm k} & 0 & 0 \\ 0 & U_{\pm 2k} & U_{\pm k} & 0 \\ 0 & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} \end{bmatrix} \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 \\ V_{\pm k} & 1 & 0 & 0 \\ -(-B)^{\pm k} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= ([1] \oplus H_3)\, D_4. \end{aligned}$$

If we define an $n \times n$ matrix $\overline{D}_k$ with $\overline{D}_k = I_{n-k} \oplus D_k$, then we can give the following result.

Lemma 10. *For $n > 1$,*

$$H_n = \overline{D}_1 \overline{D}_2 ... \overline{D}_{n-1} \overline{D}_n.$$

When $n = 4$ in Lemma 10, we get

$$\begin{aligned} H_4 &= \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 \\ U_{\pm 2k} & U_{\pm k} & 0 & 0 \\ U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} & 0 \\ U_{\pm 4k} & U_{\pm 3k} & U_{\pm 2k} & U_{\pm k} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & U_{\pm k} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{\pm k} & 0 \\ 0 & 0 & V_{\pm k} & 1 \end{bmatrix} \times \\ &\quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & U_{\pm k} & 0 & 0 \\ 0 & V_{\pm k} & 1 & 0 \\ 0 & -(-B)^{\pm k} & 0 & 1 \end{bmatrix} \begin{bmatrix} U_{\pm k} & 0 & 0 & 0 \\ V_{\pm k} & 1 & 0 & 0 \\ -(-B)^{\pm k} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \overline{D}_1 \overline{D}_2 \overline{D}_3 \overline{D}_4. \end{aligned}$$

3. Conclusion

In the present paper we introduce the $n \times n$ matrix H_n whose entries are U_{kn} satisfying the general second order recurrence formula $U_{kn} = V_k U_{k(n-1)} + (-1)^{k+1} B^k U_{k(n-2)}$ with initial conditions $0, U_k$ for $k > 0$ and $n > 1$. We use the matrix H_n instead of the $n \times n$ Fibonacci matrix $\mathcal{F}_n$ in factorizations $P_n = R_n \mathcal{F}_n$ and $P_n = \mathcal{F}_n L_n$ given in [7] and [6], respectively. Here we obtain new matrices correspond to the matrices R_n and L_n. Therefore we give more generalized factorizations of the $n \times n$ Pascal matrix P_n. Further, using these factorizations, the sequence $\{U_{\pm kn}\}$ and the matrix H_n associated with the sequence $\{U_{\pm kn}\}$, we generalize various results in [1, 3, 5, 6, 7].

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