

## ON THE GENERALIZED ORDER- $k$ FIBONACCI AND LUCAS NUMBERS

EMRAH KILIÇ and DURSUN TAŞCI

ABSTRACT. In this paper we consider the generalized order- $k$  Fibonacci and Lucas numbers. We give the generalized Binet formula, combinatorial representation and some relations involving the generalized order- $k$  Fibonacci and Lucas numbers.

**1. Introduction.** We consider the generalized *order- $k$*  Fibonacci and Lucas numbers. In [1] Er defined  $k$  sequences of the generalized *order- $k$*  Fibonacci numbers as shown:

$$g_n^i = \sum_{j=1}^k g_{n-j}^i, \quad \text{for } n > 0 \quad \text{and} \quad 1 \leq i \leq k,$$

with boundary conditions for  $1 - k \leq n \leq 0$ ,

$$g_n^i = \begin{cases} 1 & \text{if } i = 1 - n, \\ 0 & \text{otherwise,} \end{cases}$$

where  $g_n^i$  is the  $n$ th term of the  $i$ th sequence. For example, if  $k = 2$ , then  $\{g_n^2\}$  is the usual Fibonacci sequence,  $\{F_n\}$ , and, if  $k = 4$ , then the fourth sequence of the generalized *order-4* Fibonacci numbers is

$$1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, \dots$$

In [6] the authors defined  $k$  sequences of the generalized *order- $k$*  Lucas numbers as shown:

$$l_n^i = \sum_{j=1}^k l_{n-j}^i, \quad \text{for } n > 0 \quad \text{and} \quad 1 \leq i \leq k,$$

---

2000 AMS *Mathematics Subject Classification*. Primary 11B39, 15A24.  
*Key words and phrases*. Generalized order- $k$  Fibonacci and Lucas numbers, Binet formula, combinatorial representation.  
Received by the editors on April 20, 2004, and in revised form on June 7, 2004.

with boundary conditions for  $1 - k \leq n \leq 0$ ,

$$l_n^i = \begin{cases} -1 & \text{if } i = 1 - n, \\ 2 & \text{if } i = 2 - n, \\ 0 & \text{otherwise,} \end{cases}$$

where  $l_n^i$  is the  $n$ th term of the  $i$ th sequence. For example, if  $k = 2$ , then  $\{l_n^2\}$  is the usual Lucas sequence,  $\{L_n\}$ , and, if  $k = 4$ , then the fourth sequence of the generalized *order-4* Lucas numbers is

$$1, 3, 4, 8, 16, 31, 59, 114, 220, 424, 817, 1575, 30636, \dots$$

Also, Er showed that

$$\begin{bmatrix} g_{n+1}^i \\ g_n^i \\ \vdots \\ g_{n-k+2}^i \end{bmatrix} = A \begin{bmatrix} g_n^i \\ g_{n-1}^i \\ \vdots \\ g_{n-k+1}^i \end{bmatrix}$$

where

$$(1.1) \quad A = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

is a  $k \times k$  companion matrix. Then he derived

$$G_{n+1} = AG_n,$$

where

$$(1.2) \quad G_n = \begin{bmatrix} g_n^1 & g_n^2 & \dots & g_n^k \\ g_{n-1}^1 & g_{n-1}^2 & \dots & g_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ g_{n-k+1}^1 & g_{n-k+1}^2 & \dots & g_{n-k+1}^k \end{bmatrix}.$$

Moreover, Er showed that  $G_1 = A$  and  $G_n = A^n$ . The matrix  $A$  is said to be the generalized *order-k* Fibonacci matrix. Furthermore, in [3], recently Karaduman proved that

$$\det G_n = \begin{cases} (-1)^n & \text{if } k \text{ is even,} \\ 1 & \text{if } k \text{ is odd.} \end{cases}$$

In [5], we showed

$$\begin{bmatrix} l_{n+1}^i \\ l_n^i \\ \vdots \\ l_{n-k+2}^i \end{bmatrix} = A \begin{bmatrix} l_n^i \\ l_{n-1}^i \\ \vdots \\ l_{n-k+1}^i \end{bmatrix}$$

and derived

$$H_{n+1} = AH_n$$

where

$$H_n = \begin{bmatrix} l_n^1 & l_n^2 & \cdots & l_n^k \\ l_{n-1}^1 & l_{n-1}^2 & \cdots & l_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ l_{n-k+1}^1 & l_{n-k+1}^2 & \cdots & l_{n-k+1}^k \end{bmatrix}$$

also

$$H_1 = AK$$

where

$$K = \begin{bmatrix} -1 & 2 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \end{bmatrix}.$$

Further, we proved

$$\det H_{n+1} = \begin{cases} -1 & \text{if } k \text{ is odd,} \\ (-1)^{n+1} & \text{if } k \text{ is even,} \end{cases}$$

and showed that

$$(1.3) \quad H_n = G_n K,$$

which is a well-known fact for  $k = 2$ , see [6].

**2. Some relations involving the generalized  $order-k$  Fibonacci and Lucas numbers.** In this section we present and extend some relationships between the generalized  $order-k$  Fibonacci and Lucas numbers by matrix methods. From [1], it is well known that, for all positive integers  $n, m$  and  $1 \leq i \leq k$ ,

$$g_{m+n}^i = \sum_{j=1}^k g_n^j g_{m+1-j}^i,$$

where  $g_n^i$  is the generalized order- $k$  Fibonacci number.

We note that, for example, if  $k = 2$ , then  $\{g_n^2\}$  is the usual Fibonacci sequence. For all  $n, m \in Z^+$ ,

$$\begin{aligned} g_{m+n}^2 &= \sum_{j=1}^2 g_n^j g_{m+1-j}^2 \\ &= g_n^1 g_m^2 + g_n^2 g_{m-1}^2 \end{aligned}$$

and, since  $g_m^1 = g_{m+1}^2$  for  $i = 1, k = 2$  and all  $m \in Z^+$ , we write

$$g_{m+n}^2 = g_{n+1}^2 g_m^2 + g_n^2 g_{m-1}^2.$$

Indeed, we generalize the following relation between Fibonacci numbers

$$F_{n+m} = F_{n+1}F_m + F_nF_{m-1},$$

see [7, p. 176].

**Theorem 1.** *Let  $l_n^i$  be the generalized order- $k$  Lucas number. For all positive integers  $n, m$  and  $1 \leq i \leq k$ ,*

$$l_{n+m}^i = \sum_{j=1}^k g_n^j l_{m+1-j}^i.$$

*Proof.* From [6], we know that  $H_n = G_n K$ , so we can write that

$$H_{n+m} = G_{n+m} K = A^{n+m} K = A^n A^m K = A^n H_m = G_n H_m$$

or

$$H_{n+m} = G_m H_n.$$

Since  $H_{n+m} = G_n H_m$ ,  $l_{n+m}^i = (H_{n+m})_{1,i}$ ,

$$\begin{aligned} l_{n+m}^i &= g_n^1 l_m^i + g_n^2 l_{m-1}^i + \cdots + g_n^k l_{m-k+1}^i \\ &= g_n^1 l_m^i + (g_{n-1}^2 + g_{n-2}^2) l_{m-1}^i + \cdots \\ &\quad + (g_{n-1}^k + g_{n-2}^k + \cdots + g_{n-k}^k) l_{m-k+1}^i. \end{aligned}$$

Thus, we obtain

$$l_{n+m}^i = \sum_{j=1}^k g_n^j l_{m+1-j}^i,$$

so the proof is completed.  $\square$

For example, if  $k = 2$ , then  $g_n^2$  and  $l_n^2$  are the usual Fibonacci and Lucas number, respectively; then

$$\begin{aligned} l_{n+m}^2 &= \sum_{j=1}^2 g_n^j l_{m+1-j}^2 \\ &= g_n^1 l_m^2 + g_n^2 l_{m-1}^2, \end{aligned}$$

and, since  $g_n^1 = g_{n+1}^2$  for  $i = 1$ ,  $k = 2$ , and all  $n \in \mathbb{Z}^+$ , we write

$$l_{n+m}^2 = g_{n+1}^2 l_m^2 + g_n^2 l_{m-1}^2.$$

Indeed, we generalize the following relation involving Lucas and Fibonacci numbers,

$$\begin{aligned} L_{n+m} &= F_{n+1} L_m + F_n L_{m-1} \\ &= (F_n + F_{n-1}) L_m + F_n L_{m-1} \\ &= F_n (L_m + L_{m-1}) + F_{n-1} L_m \\ &= F_n L_{m+1} + F_{n-1} L_m, \end{aligned}$$

see [7, p. 176].

Note that  $G_{n+m+p} = G_n G_{m+p}$ ,  $g_{n+m}^i = (G_{n+m+p})_{p-1,i}$ .

Then we have the following corollary.

**Corollary 1.** *Let  $g_n^i$  be the generalized order- $k$  Fibonacci number. For all  $n, m, p \in \mathbf{Z}^+$  and  $1 \leq i \leq k$ ,*

$$g_{n+m+p}^i = \sum_{j=1}^k g_n^j g_{m+1-p-j}^i.$$

Also we note that  $G_{n-p} G_{m+p} = G_{n+m}$ ,  $g_{n+m}^i = (G_{n+m})_{1,i}$ . Then we have the following corollary.

**Corollary 2.** *Let  $g_n^i$  be the generalized order- $k$  Fibonacci number. Then, for  $n, m, p \in \mathbf{Z}^+$  and  $1 \leq i \leq k$ ,*

$$g_{m+n}^i = \sum_{j=1}^k g_{n-p}^j g_{m+p-j}^i.$$

In [4] Levesque gave a Binet formula for the Fibonacci sequence. In this paper, we derive a generalized Binet formula for the generalized order- $k$  Fibonacci and Lucas sequence by using the determinant.

**3. Generalized Binet formula.** Let  $f(\lambda)$  be the characteristic polynomial of the generalized order- $k$  Fibonacci matrix  $A$ . Then  $f(\lambda) = \lambda^k - \lambda^{k-1} - \dots - \lambda - 1$ , which is a well-known fact. Let  $\lambda_1, \lambda_2, \dots, \lambda_k$  be the eigenvalues of  $A$ . In [5], Miles also showed that  $\lambda_1, \lambda_2, \dots, \lambda_k$  are distinct. Let  $V$  be a  $k \times k$  Vandermonde matrix as follows:

$$V = \begin{bmatrix} \lambda_1^{k-1} & \lambda_2^{k-1} & \lambda_3^{k-1} & \dots & \lambda_k^{k-1} \\ \lambda_1^{k-2} & \lambda_2^{k-2} & \lambda_3^{k-2} & \dots & \lambda_k^{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_k \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}.$$

Let

$$d_k^i = \begin{bmatrix} \lambda_1^{n+k-i} \\ \lambda_2^{n+k-i} \\ \vdots \\ \lambda_k^{n+k-i} \end{bmatrix}$$

and  $V_j^{(i)}$  be a  $k \times k$  matrix obtained from  $V$  by replacing the  $j$ th column of  $V$  by  $d_k^i$ . Then we have the generalized Binet formula for the generalized order- $k$  Fibonacci numbers with the following theorem.

**Theorem 2.** *Let  $g_n^i$  be the generalized order- $k$  Fibonacci number, for  $1 \leq i \leq k$ . Then*

$$t_{ij} = \frac{\det(V_j^{(i)})}{\det(V)}$$

where  $G_n = [t_{ij}]_{k \times k}$ .

*Proof.* Since the eigenvalues of  $A$  are distinct,  $A$  is diagonalizable. It is easy to show that  $AV = VD$ , where  $D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_k)$ . Since  $V$  is invertible,  $V^{-1}AV = D$ . Hence,  $A$  is similar to  $D$ . So we have  $A^n V = VD^n$ . In [1], it is known that  $G_n = A^n$ . So we can write that  $G_n V = VD^n$ . Let  $G_n = [t_{ij}]_{k \times k}$ . Then we have the following linear system of equations:

$$\begin{aligned} t_{i1}\lambda_1^{k-1} + t_{i2}\lambda_1^{k-2} + \dots + t_{ik} &= \lambda_1^{n+k-i} \\ t_{i1}\lambda_2^{k-1} + t_{i2}\lambda_2^{k-2} + \dots + t_{ik} &= \lambda_2^{n+k-i} \\ &\vdots \\ t_{i1}\lambda_k^{k-1} + t_{i2}\lambda_k^{k-2} + \dots + t_{ik} &= \lambda_k^{n+k-i}. \end{aligned}$$

And, for each  $j = 1, 2, \dots, k$ , we get

$$(3.1) \quad t_{ij} = \frac{\det(V_j^{(i)})}{\det(V)}.$$

So the proof is complete.  $\square$

**Corollary 3.** *Let  $g_n^k$  be the generalized order- $k$  Fibonacci number, then*

$$g_n^k = t_{1k} = \frac{\det(V_k^{(1)})}{\det(V)}.$$

*Proof.* If we take  $i = 1$  and  $j = k$ , then  $t_{1,k} = g_n^k$ . Also, employing Theorem 2, the proof is immediately seen.  $\square$

Now we are going to give a generalized Binet formula for the generalized order- $k$  Lucas sequence. Firstly, we give a lemma for a relationship between the generalized order- $k$  Fibonacci and Lucas numbers.

**Lemma 1.** *Let  $l_n^k$  and  $g_n^k$  be the generalized order- $k$  Lucas and Fibonacci numbers, respectively. Then, for  $k \geq 2$ ,*

$$l_n^k = g_n^k + 2g_{n-1}^k.$$

*Proof.* We will use the induction method to prove that  $l_n^k = g_n^k + 2g_{n-1}^k$ . From the definition of the generalized order- $k$  Lucas and Fibonacci numbers, we know that, for all  $k \in \mathbf{Z}^+$  with  $k \geq 2$ ,  $l_1^k = g_1^k = 1$  and  $g_0^k = 0$ . Then, it is true for  $n = 1$ , i.e.,

$$l_1^k = g_1^k + 2g_0^k = 1.$$

Suppose that the equation holds for  $n$ . So we have

$$l_n^k = g_n^k + 2g_{n-1}^k.$$

Now we show that the equation is true for  $n + 1$ . From the definition of  $l_n^k$ , we have

$$(3.2) \quad l_{n+1}^k = l_n^k + l_{n-1}^k + \cdots + l_{n-k+1}^k.$$

Since  $l_n^k = g_n^k + 2g_{n-1}^k$ , we can write equation (3.2) as follows

$$\begin{aligned} l_{n+1}^k &= g_n^k + 2g_{n-1}^k + (g_{n-1}^k + 2g_{n-2}^k) + \cdots + (g_{n-k+1}^k + 2g_{n-k}^k) \\ &= (g_n^k + g_{n-1}^k + \cdots + g_{n-k+1}^k) + 2(g_{n-1}^k + g_{n-2}^k + \cdots + g_{n-k}^k) \\ &= g_{n+1}^k + 2g_n^k. \end{aligned}$$

So the equation holds for  $n + 1$ . Thus the proof is complete.  $\square$



For example, if  $k = 2$ ,  $\{l_n^2\}$  is the usual Lucas sequence,  $\{L_n\}$ , and  $\{g_n^2\}$  is the usual Fibonacci sequence,  $\{F_n\}$ , then it is a well-known fact that

$$L_n = F_n + 2F_{n-1}.$$

If  $k = 4$  and  $n = 9$ , then

$$l_9^4 = g_9^4 + 2g_8^4 = 108 + 2.56 = 220.$$

Then we have the following theorem.

**Theorem 3.** *Let  $l_n^k$  be the generalized order- $k$  Lucas number. Then, for  $k \geq 2$ ,*

$$l_n^k = \frac{\det(V_k^{(1)}) + 2 \det(V_k^{(2)})}{\det(V)}.$$

*Proof.* From Lemma 1, we know that  $l_n^k = g_n^k + 2g_{n-1}^k$ . Also, by Theorem 2, we have  $G_n = [t_{ij}]_{k \times k}$  and  $t_{ij} = \det(V_j^{(i)})/\det(V)$ . If we take  $i = 2, j = k$  in equation (3.1), and since  $t_{2k} = g_{n-1}^k$ , we write

$$\begin{aligned} l_n^k &= g_n^k + 2g_{n-1}^k \\ &= \frac{\det(V_k^{(1)}) + 2 \det(V_k^{(2)})}{\det(V)}. \end{aligned}$$

So the proof is complete.  $\square$

For a further generalization of Theorem 3, we give a lemma which is immediately seen from equation (1.3).

**Lemma 2.** *Let  $l_n^i$  and  $g_n^i$  be the generalized order- $k$  Lucas and Fibonacci numbers for  $1 \leq i \leq k$ . Then*

$$l_n^i = 2g_n^{i-1} - g_n^i$$

for  $1 \leq i \leq k$ .

Then we obtain the following theorem.

**Theorem 4.** *Let  $l_n^i$  be the generalized order- $k$  Lucas number, for  $1 \leq i \leq k$ . Then*

$$l_n^i = \frac{2 \det V_{i-1}^{(1)} + \det V_i^{(1)}}{\det V}.$$

*Proof.* From  $G_n = [t_{ij}] = [g_{n-i+1}^j]$  and Lemma 2, we obtain for  $1 \leq i \leq k$ ,

$$l_n^i = 2t_{1,i-1} - t_{1,i},$$

and by using Theorem 2, the proof is immediately seen. □

**4. Combinatorial representation of the generalized order- $k$  Fibonacci and Lucas numbers.** In this section, we consider a combinatorial representation of  $l_n^k$  and  $g_n^k$  for  $n > 0$ . Recall that  $A$  is the  $k \times k$   $(0, 1)$  matrix given by (1.1):

$$A = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

Also we know from [1] that  $G_n$  is as in (1.2):

$$G_n = [t_{ij}] = \begin{bmatrix} g_n^1 & g_n^2 & \dots & g_n^k \\ g_{n-1}^1 & g_{n-1}^2 & \dots & g_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ g_{n-k+1}^1 & g_{n-k+1}^2 & \dots & g_{n-k+1}^k \end{bmatrix}.$$

**Lemma 3** [4].

$$t_{ij} = \sum_{(m_1, \dots, m_k)} \frac{m_j + m_{j+1} + \dots + m_k}{m_1 + m_2 + \dots + m_k} \times \binom{m_1 + m_2 + \dots + m_k}{m_1, m_2, \dots, m_k}$$

where the summation is over nonnegative integers satisfying  $m_1 + 2m_2 + \dots + km_k = n - i + j$  and defined to be 1 if  $n = i - j$ .

**Corollary 4.** *Let  $g_n^k$  be the generalized order- $k$  Fibonacci number. Then*

$$g_n^k = \sum_{(m_1, \dots, m_k)} \frac{m_k}{m_1 + m_2 + \dots + m_k} \times \binom{m_1 + m_2 + \dots + m_k}{m_1, m_2, \dots, m_k},$$

where the summation is over nonnegative integers satisfying  $m_1 + 2m_2 + \dots + km_k = n - 1 + k$ .

*Proof.* In Lemma 2, if we take  $i = 1$  and  $j = k$ , then the conclusion can be directly from (1.2).  $\square$

**Corollary 5.** *Let  $g_n^k$  be the generalized order- $k$  Fibonacci number. Then*

$$g_{n-1}^k = \sum_{(d_1, \dots, d_k)} \frac{d_k}{d_1 + d_2 + \dots + d_k} \times \binom{d_1 + d_2 + \dots + d_k}{d_1, d_2, \dots, d_k}$$

where the summation is over nonnegative integers satisfying  $d_1 + 2d_2 + \dots + kd_k = n - 2 + k$ .

*Proof.* In Lemma 3, if we take  $i = 2$  and  $j = k$ , then the conclusion follows directly from (1.2).  $\square$

From Corollaries 4 and 5 and Lemma 1, we have the following corollary.

**Corollary 6.** *Let  $l_n^k$  be the generalized order- $k$  Lucas number. Then*

$$l_n^k = \sum_{(m_1, \dots, m_k)} \frac{m_k}{m_1 + m_2 + \dots + m_k} \times \binom{m_1 + m_2 + \dots + m_k}{m_1, m_2, \dots, m_k} + 2 \sum_{(d_1, \dots, d_k)} \frac{d_k}{d_1 + d_2 + \dots + d_k} \times \binom{d_1 + d_2 + \dots + d_k}{d_1, d_2, \dots, d_k}$$

where the summation is over nonnegative integers satisfying  $m_1 + 2m_2 + \dots + km_k = n - 1 + k$  and  $d_1 + 2d_2 + \dots + kd_k = n - 2 + k$ .

*Proof.* From Lemma 1, we know that  $l_n^k = g_n^k + 2g_{n-1}^k$  and if  $i = 1$  and  $j = k$  and,  $i = 2$  and  $j = k$ , in Lemma 3, respectively, then the conclusion can be derived directly from (1.2).  $\square$

## REFERENCES

1. M.C. Er, *Sums of Fibonacci numbers by matrix methods*, Fibonacci Quart. **22** (1984), 204–207.
2. D. Kalman, *Generalized Fibonacci numbers by matrix methods*, Fibonacci Quart. **20** (1982), 73–76.
3. E. Karaduman, *An application of Fibonacci numbers in matrices*, Appl. Math. Comp. **147** (2004), 903–908.
4. C. Levesque, *On  $m^{\text{th}}$ -order linear recurrences*, Fibonacci Quart. **23** (1985), 290–293.
5. E.P. Miles, *Generalized Fibonacci numbers and associated matrices*, Amer. Math. Monthly **67** (1960), 745–752.
6. D. Taşci and E. Kiliç, *On the order- $k$  generalized Lucas numbers*, Appl. Math. Comp. **155** (2004), 637–641.
7. S. Vajda, *Fibonacci & Lucas numbers and the golden section*, Chichester, Brisbane, 1989.

DEPARTMENT OF MATHEMATICS, TOBB UNIVERSITY OF ECONOMICS AND TECHNOLOGY, TR-06560 SÖDÜTÖZÜ, ANKARA, TURKEY  
*E-mail address:* ekilic@etu.edu.tr

GAZI UNIVERSITY, DEPARTMENT OF MATHEMATICS, 06500 TEKNIKULLAR, ANKARA, TURKEY  
*E-mail address:* dtasci@gazi.edu.tr