ON THE GENERALIZED LUCAS SEQUENCES BY HESSENBERG MATRICES

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ABSTRACT. We show that there are relationships between a generalized Lucas sequence and the permanent and determinant of some Hessenberg matrices.

1. INTRODUCTION

The Fibonacci sequence, $\{F_n\}$, is defined by the recurrence relation

$$F_{n+1} = F_n + F_{n-1}, \qquad n \geq 1,$$

where $F_0 = 0$, $F_1 = 1$. The Lucas sequence, $\{L_n\}$, is defined by the recurrence relation

$$L_{n+1} = L_n + L_{n-1}, \qquad n \geq 1,$$

where $L_0 = 2$, $L_1 = 1$.

The well-known Fibonacci and Lucas numbers can be generalized as follows: Let A be nonzero real number. Define the generalized Fibonacci sequence, $\{u_n\}$, and the generalized Lucas sequence, $\{v_n\}$, by

$$u_{n+1} = Au_n + u_{n-1}, \qquad n \geq 1, \tag{1.1}$$

$$v_{n+1} = Av_n + v_{n-1}, \qquad n \geq 1, \tag{1.2}$$

where $u_0 = 0$, $u_1 = 1$ and $v_0 = 2$, $v_1 = A$ (see [1, 6, 25]). If $A = 1$, then $u_n = F_n$ (the nth Fibonacci number). If $A = 2$, then $u_n = P_n$ (the nth Pell number). If $A = 1$, then $v_n = L_n$ (the nth Lucas number). For later use we note that $u_2 = A$, $u_3 = A^2 + 1$, $u_4 = A^3 + 2A$, $v_2 = A^2 + 2$, $v_3 = A^3 + 3A$ and $v_4 = A^4 + 4A^2 + 2$. The sequences $\{u_n\}$ and $\{v_n\}$ may also be referred to as the Fibonacci and Lucas polynomial sequences.

Let the roots of the equation $t^2 - At - 1 = 0$ be σ and γ. Then for $n \geq 0$

$$u_n = \frac{\sigma^n - \gamma^n}{\sigma - \gamma} \quad \text{and} \quad v_n = \sigma^n + \gamma^n.$$

The sequences $\{u_n\}$ and $\{v_n\}$ have been studied by several authors (see [1, 6]). The following identities can be found in [1] and [6]:

$$u_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} A^{n-2k}, \qquad n \geq 0, \tag{1.3}$$

$$v_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} A^{n-2k}, \qquad n \geq 1. \tag{1.4}$$

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The *permanent* of an n-square matrix $A = (a_{ij})$ is defined by

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation extends over all permutations σ of the symmetric group S_n. Also one can find applications of permanents in [24].

The permanent of a matrix is analogous to the determinant, where all of the signs used in the Laplace expansion of minors are positive.

There are many connections between permanents or determinants of tridiagonal matrices and the Fibonacci and Lucas numbers. For example, Minc [23] defines an $n \times n$ super-diagonal $(0,1)$-matrix $F(n,k)$ for $n+1 \geq k$, and shows that the permanent of $F(n,k)$ equals a generalized kth order Fibonacci number. Note that when $k = 2$, the matrix $F(n,2)$ reduces to a tridiagonal matrix and its permanent equals a usual Fibonacci number. Also in [26] and [27], the authors define a family of tridiagonal matrices $M(n)$ and show that the determinants of $M(n)$ are the Fibonacci numbers F_{2n+2}. In [5] and [4], a family of tridiagonal matrices $H(n)$ is defined and the authors show that the determinants of $H(n)$ are the Fibonacci numbers F_n. In a similar family of matrices, the $(1,1)$ element of $H(n)$ is replaced with a 3. The determinants, [3], now generate the Lucas numbers L_n. Recently, in [13], the authors defined two tridiagonal matrices and then gave the relationships of the permanents and determinants of these matrices and the second order linear recurrences (1.1) and (1.2). In [20], Lehmer discussed the relationships between permanents of tridiagonal matrices, recurrence relations, and continued fractions.

In [15], the authors present a result involving the permanent of a $(-1,0,1)$-matrix and the Fibonacci number F_{n+1}. The authors then explore similar directions involving the positive subscripted Fibonacci and Lucas numbers as well as their negatively subscripted counterparts. Finally the authors explore the generalized kth order Lucas numbers, (see [28] and [14] for further details on the generalized Fibonacci and Lucas numbers), and their permanents.

In [18] and [19], the authors defined two $(0,1)$-matrices and then showed that the permanents of these matrices are the generalized Fibonacci and Lucas numbers. In particular, one of these $(0,1)$-matrices is the $n \times n$ super-diagonal $(0,1)$-matrix $F(n,k)$. Therefore, the result of Minc, [23], and the result of Lee, [18], on the generalized Fibonacci numbers are the same because they use the same matrix. However, Lee proved the same result by a different method, contraction method for the permanent (for further details of the contraction method see [2]).

In [11], the authors defined two $(0,1)$-matrices and then showed the relations involving the sums of the Fibonacci and Lucas numbers, and the permanents of these matrices. In [12], the authors defined two $(0,1)$-matrices and then showed the relations involving the sums of the generalized kth order Fibonacci and Lucas numbers, and the permanents of these matrices.

In [16] the authors show that the permanents of certain generalized doubly stochastic matrices satisfy a second order linear recurrence.

For $n \geq 2$, a lower *Hessenberg* matrix, $A_n = (a_{ij})$, is an $n \times n$ matrix, where $a_{ij} = 0$, whenever $j > i + 1$, and $a_{j,j+1} \neq 0$ for some j. Clearly,

$$A_n = \begin{bmatrix} a_{11} & a_{12} & 0 & \dots & 0 \\ a_{21} & a_{22} & a_{23} & \ddots & 0 \\ a_{31} & a_{32} & a_{33} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & a_{n-1,n} \\ a_{n1} & a_{n2} & \dots & a_{n,n-1} & a_{n,n} \end{bmatrix}.$$

In [5] the authors give the following determinant formula for A_n: for $n \geq 2$,

$$\det A_n = a_{n,n} \det A_{n-1} + \sum_{r=1}^{n-1} \left((-1)^{n-r} a_{n,r} \prod_{j=r}^{n-1} a_{j,j+1} \det A_{r-1} \right),$$

where $A_0 = 1$ and $A_1 = a_{11}$.

Furthermore, the authors consider the Fibonacci sequence, $\{F_n\}$, and then define an $n \times n$ lower Hessenberg matrix D_n and then state that the determinants of the first few matrices are $\det D_1 = 2$, $\det D_2 = 3$ and $\det D_3 = 5$, and, it turns out that this sequence is precisely $\{F_n\}$ starting at $n = 3$.

In [17], we define some Hessenberg matrices. Then we show that the determinants or permanents of these matrices are equal to the terms u_n, u_{2n+1} and u_{2n}. In [17], the following results can be found: Let the $n \times n$ lower Hessenberg matrices H_n and T_n be defined as

$$H_n = \begin{bmatrix} A^2+1 & 1 & 0 & \dots & 0 & 0 \\ 1 & A^2+1 & 1 & \dots & \vdots & 0 \\ 1 & 1 & A^2+1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ 1 & 1 & \dots & 1 & A^2+1 & 1 \\ 1 & 1 & 1 & \dots & 1 & A^2+1 \end{bmatrix} \tag{1.5}$$

and

$$T_n = \begin{bmatrix} A^2+1 & 1 & 0 & \dots & 0 & 0 \\ 1 & A^2+1 & 1 & \dots & \vdots & 0 \\ 1 & 1 & A^2+1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ 1 & 1 & \dots & 1 & A^2+1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}. \tag{1.6}$$

Then $\det H_n = A^{n-1} u_{n+2}$ for $n \geq 1$ and $\det T_n = A^2 \det H_{n-2}$ for $n \geq 3$. Let the $n \times n$ lower Hessenberg matrices W_n and R_n be defined as

$$W_n = \begin{bmatrix} A^2+1 & -1 & 0 & \dots & 0 & 0 \\ A^2 & A^2+1 & -1 & \dots & \vdots & 0 \\ A^2 & A^2 & A^2+1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 & 0 \\ A^2 & A^2 & \dots & A^2 & A^2+1 & -1 \\ A^2 & A^2 & A^2 & \dots & A^2 & A^2+1 \end{bmatrix}, \tag{1.7}$$

where $n \geq 1$, and

$$R_n = \begin{bmatrix} A^2 & -1 & 0 & \dots & 0 & 0 \\ A^2 & A^2+1 & -1 & \dots & \vdots & 0 \\ A^2 & A^2 & A^2+1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 & 0 \\ A^2 & A^2 & \dots & A^2 & A^2+1 & -1 \\ A^2 & A^2 & A^2 & \dots & A^2 & A^2+1 \end{bmatrix}, \tag{1.8}$$

where $n \geq 2$, and $R_1 = [A^2]$. Then $\det W_n = u_{2n+1}$ and $\det R_n = A u_{2n}$ for $n \geq 1$.

We note that using the definitions of the sequences $\{u_n\}$ and $\{v_n\}$, we have the following result without proof:

$$u_{n+2} + u_n = v_{n+1}, \qquad n \geq 0. \tag{1.9}$$

In this paper, we consider relationships between certain Hessenberg determinants or permanents, and the generalized Lucas sequence $\{v_n\}$.

2. On The Generalized Lucas Sequence By Hessenberg Matrices

Let $n \geq 1$. We define the $n \times n$ lower Hessenberg matrix $Q_n = (q_{ij})$ with $q_{11} = A^3 + 3$, $q_{ii} = A^2 + 1$ for $2 \leq i \leq n$, $q_{i,i+1} = 1$ for $1 \leq i \leq n-1$, $q_{ij} = 1$ for $i > j$ and $q_{ij} = 0$ otherwise. Clearly, for $n \geq 2$,

$$Q_n = \begin{bmatrix} A^2+3 & 1 & 0 & \dots & 0 & 0 \\ 1 & A^2+1 & 1 & \dots & \vdots & 0 \\ 1 & 1 & A^2+1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ 1 & 1 & \dots & 1 & A^2+1 & 1 \\ 1 & 1 & 1 & \dots & 1 & A^2+1 \end{bmatrix}. \tag{2.1}$$

In addition, $Q_1 = [A^2 + 3]$.

Theorem 1. *Suppose that the $n \times n$ lower Hessenberg matrix Q_n has the form* (2.1). *Then for $n \geq 1$*

$$\det Q_n = A^{n-2} v_{n+2}.$$

Proof. The case $n=1$ is trivial. Let $n \geq 2$. Writing the first row $[A^2+3, 1, 0, \dots, 0]$ as $[A^2+1, 1, 0, \dots, 0] + [2, 0, 0, \dots, 0]$ we obtain

$$\begin{aligned}\det Q_n &= \det H_n + 2\det H_{n-1}\\ &= A^{n-1}u_{n+2} + 2A^{n-2}u_{n+1}\\ &= A^{n-2}(Au_{n+2} + u_{n+1} + u_{n+1})\\ &= A^{n-2}(u_{n+3} + u_{n+1}).\end{aligned}$$

Thus from (1.9), we obtain Theorem 1. □

For example, if $A = 1$, then by Theorem 1 we have that

$$\begin{vmatrix} 4 & 1 & 0 & 0 & \dots & 0 & 0\\ 1 & 2 & 1 & 0 & \dots & 0 & 0\\ 1 & 1 & 2 & 1 & \ddots & \vdots & 0\\ \vdots & & \ddots & \ddots & \ddots & 0 & \vdots\\ 1 & 1 & \dots & 1 & 2 & 1 & 0\\ 1 & 1 & 1 & \dots & 1 & 2 & 1\\ 1 & 1 & 1 & 1 & \dots & 1 & 2 \end{vmatrix}_{n\times n} = L_{n+2},$$

where L_n is the nth Lucas number.

A matrix A is called *convertible* if there is an $n \times n$ $(1, -1)$-matrix H such that $\operatorname{per} A = \det(A \circ H)$, where $A \circ H$ denotes the Hadamard product of A and H. Such a matrix H is called a *converter* of A.

Let S be the $(1, -1)$-matrix of order n defined by

$$S = \begin{bmatrix} 1 & -1 & 1 & \dots & 1 & 1\\ 1 & 1 & -1 & \dots & 1 & 1\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 1 & 1 & 1 & \dots & -1 & 1\\ 1 & 1 & 1 & \dots & 1 & -1\\ 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}.$$

We denote the matrix $Q_n \circ S$ by D_n. That is,

$$D_n = \begin{bmatrix} A^2+3 & -1 & 0 & \dots & 0 & 0\\ 1 & A^2+1 & -1 & \dots & \vdots & 0\\ 1 & 1 & A^2+1 & \ddots & 0 & \vdots\\ \vdots & \vdots & \ddots & \ddots & -1 & 0\\ 1 & 1 & \dots & 1 & A^2+1 & -1\\ 1 & 1 & 1 & \dots & 1 & A^2+1 \end{bmatrix}. \tag{2.2}$$

Then we have the following theorem without proof.

Theorem 2. *Suppose that the $n\times n$ lower Hessenberg matrix D_n has the form* (2.2). *Then for $n \geq 1$*

$$\operatorname{per} D_n = A^{n-2}v_{n+2}.$$

Let $\{x_n\}$ be any second order linear recurrence sequence. Denote

$$x_{n+1} = Ax_n + Bx_{n-1}, \qquad n \geq 1,$$

with $x_0 = C$, $x_1 = D$. Then, for $n \geq 0$,

$$x_n = \begin{vmatrix} C & D & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & B & 0 & \dots & 0 & 0 \\ 0 & -1 & A & B & \dots & 0 & 0 \\ 0 & 0 & -1 & A & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & A & B \\ 0 & 0 & 0 & 0 & \dots & -1 & A \end{vmatrix}_{(n+1)\times(n+1)} . \tag{2.3}$$

In particular,

$$v_n = \begin{vmatrix} 2 & A & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & A & 1 & \dots & 0 & 0 \\ 0 & 0 & -1 & A & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & A & 1 \\ 0 & 0 & 0 & 0 & \dots & -1 & A \end{vmatrix}_{(n+1)\times(n+1)} .$$

These two matrices are tridiagonal and thus also Hessenberg matrices. Note that Q_n's are *nonnegative* Hessenberg matrices whose determinants generate the generalized Lucas sequence $\{v_n\}$.

3. On the terms v_{2n+1} and v_{2n}

In this section, we define two lower Hessenberg matrices and show that their determinants are equal to the terms v_{2n+1} and v_{2n}.

We define the $n \times n$ lower Hessenberg matrix $E_n = (e_{ij})$ with $e_{11} = A^2 + 3$, $e_{ii} = A^2 + 1$ for $2 \leq i \leq n$, $e_{i,i+1} = -1$ for $1 \leq i \leq n-1$, $e_{ij} = A^2$ for $3 \leq i \leq n$, $2 \leq j \leq i-1$, $e_{i1} = A^2 + 2$ for $2 \leq i \leq n$ and $e_{ij} = 0$ otherwise. That is, for $n \geq 2$,

$$E_n = \begin{bmatrix} A^2+3 & -1 & 0 & \dots & 0 & 0 \\ A^2+2 & A^2+1 & -1 & \dots & \vdots & 0 \\ A^2+2 & A^2 & A^2+1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 & 0 \\ A^2+2 & A^2 & \dots & A^2 & A^2+1 & -1 \\ A^2+2 & A^2 & A^2 & \dots & A^2 & A^2+1 \end{bmatrix} . \tag{3.1}$$

In addition, $E_1 = [A^2 + 3]$.

Theorem 3. *Suppose that the $n \times n$ lower Hessenberg matrix E_n has the form* (3.1). *Then for $n \geq 1$*

$$\det E_n = \frac{v_{2n+1}}{A}.$$

Proof. (Induction on n.) If $n = 1$ or $n = 2$, then we have

$$\det E_1 = A^2 + 3 = \frac{v_3}{A},$$

$$\det E_2 = \begin{vmatrix} A^2+3 & -1 \\ A^2+2 & A^2+1 \end{vmatrix} = A^4 + 5A^2 + 5 = \frac{v_5}{A}.$$

Assume that Theorem 3 holds for $n < k$ $(k \geq 3)$. Consider the case $n = k$. We expand $\det E_k$ with respect to the last column to obtain

$$\det E_k = (A^2+1)\det E_{k-1} + \det E'_{k-1},$$

where

$$\det E'_{k-1} = \begin{vmatrix} A^2+3 & -1 & 0 & \dots & 0 & 0 \\ A^2+2 & A^2+1 & -1 & \ddots & 0 & 0 \\ A^2+2 & A^2 & A^2+1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ A^2+2 & A^2 & A^2 & \ddots & A^2+1 & -1 \\ A^2 & A^2 & A^2 & \dots & A^2 & A^2 \end{vmatrix}_{(k-1)\times(k-1)}.$$

Writing the last column $[0, 0, \dots, 0, -1, A^2]^\mathsf{T}$ of $\det E'_{k-1}$ as $[0, 0, \dots, 0, -1, A^2 + 1]^\mathsf{T} + [0, 0, \dots, 0, 0, -1]^\mathsf{T}$ we obtain

$$\det E'_{k-1} = \det E_{k-1} - \det E_{k-2}.$$

Thus

$$\det E_k = (A^2+2)\det E_{k-1} - \det E_{k-2}. \tag{3.2}$$

We apply the induction assumption to obtain

$$\det E_k = (A^2+2)\frac{v_{2k-1}}{A} - \frac{v_{2k-3}}{A}.$$

Thus

$$\begin{aligned} \det E_k &= \frac{1}{A}(A^2 v_{2k-1} + v_{2k-1} + v_{2k-1} - v_{2k-3}) \\ &= \frac{1}{A}(A^2 v_{2k-1} + v_{2k-1} + A v_{2k-2}) \\ &= \frac{1}{A}(A v_{2k} + v_{2k-1}) = \frac{v_{2k+1}}{A}. \end{aligned}$$

Thus completes the induction. □

Equation (3.2) shows that the sequence $\{\alpha_n\}_{n=0}^{\infty} \equiv \{v_{2n+1}\}_{n=0}^{\infty}$ is a second order linear recurrence sequence. In fact,

$$\alpha_n = (A^2+2)\alpha_{n-1} - \alpha_{n-2}, \qquad n \geq 2,$$

with $\alpha_0 = A$, $\alpha_1 = A^3 + 3A$. Thus $\{\alpha_n\}$ or $\{v_{2n+1}\}$ is also generated by determinants of tridiagonal matrices of type (2.3) and respective permanents. For the sake of brevity we do not present these determinants and permanents here. Note that the matrix M_n in (3.5) is a nonnegative Hessenberg matrix whose permanents generate the sequence $\{v_{2n+1}\}$.

It can be shown that the matrix W_n given in (1.7) also satisfies (3.2). Thus the sequence $\{u_{2n+1}\}_{n=0}^{\infty}$ is a second order linear recurrence sequence.

For $n \geq 3$, let

$$X_n = \begin{bmatrix} A^2 & -1 & 0 & \dots & 0 & 0 \\ A^2 & A^2+1 & -1 & \dots & \vdots & 0 \\ A^2 & A^2 & A^2+1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 & 0 \\ A^2 & A^2 & \dots & A^2 & A^2+1 & -1 \\ A^2 & A^2 & A^2 & \dots & A^2 & A^2+2 \end{bmatrix}_{n\times n}. \tag{3.3}$$

In addition,

$$X_1 = [A^2], \qquad X_2 = \begin{bmatrix} A^2 & -1 \\ A^2 & A^2+2 \end{bmatrix}.$$

Theorem 4. *Suppose that the $n\times n$ lower Hessenberg matrix X_n has the form* (3.3)*. Then for $n \geq 1$*

$$\det X_n = Av_{2n-1}.$$

Proof. If $n = 1$, then Theorem 4 holds. Let $n \geq 2$. We write the last row $[A^2, A^2, \dots, A^2, A^2+2]$ of X_n as $[A^2, A^2, \dots, A^2, A^2+1] + [0, 0, \dots, 0, 1]$ to obtain

$$\det X_n = \det R_n + \det R_{n-1} = Au_{2n} + Au_{2n-2}.$$

From (1.9) we obtain Theorem 4. □

Now, we give a relationship between the term v_{2n} and a Hessenberg matrix. For this purpose, we define the $n \times n$ lower Hessenberg matrix $Z_n = (z_{ij})$ with $z_{11} = A^2 + 2$, $z_{ii} = A^2 + 1$ for $2 \leq i \leq n$, $z_{i,i+1} = -1$ for $1 \leq i \leq n-1$, $z_{ij} = A^2$ for $i > j$ and $z_{ij} = 0$ otherwise. That is, for $n \geq 2$,

$$Z_n = \begin{bmatrix} A^2+2 & -1 & 0 & \dots & 0 & 0 \\ A^2 & A^2+1 & -1 & \dots & \vdots & 0 \\ A^2 & A^2 & A^2+1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 & 0 \\ A^2 & A^2 & \dots & A^2 & A^2+1 & -1 \\ A^2 & A^2 & A^2 & \dots & A^2 & A^2+1 \end{bmatrix}. \tag{3.4}$$

In addition, $Z_1 = [A^2 + 2]$.

Theorem 5. *Suppose that the $n\times n$ lower Hessenberg matrix Z_n has the form* (3.4)*. Then for $n \geq 1$*

$$\det Z_n = v_{2n}.$$

Proof. If $n = 1$, then Theorem 5 holds. Let $n \geq 2$. We write the first row $[A^2 + 2, -1, 0, \dots, 0]$ of Z_n as $[A^2 + 1, -1, 0, \dots, 0] + [1, 0, 0, \dots, 0]$ to obtain

$$\det Z_n = \det W_n + \det W_{n-1} = u_{2n+1} + u_{2n-1}.$$

From (1.9) we obtain Theorem 5. □

Theorem 5 can also be proven in a way similar to the proof of Theorem 3 to obtain

$$\det Z_n = (A^2 + 2) \det Z_{n-1} - \det Z_{n-2}.$$

Thus the sequence $\{\beta_n\}_{n=0}^{\infty} \equiv \{v_{2n}\}_{n=0}^{\infty}$ is a second order linear recurrence sequence. In fact,

$$\beta_n = (A^2+2)\beta_{n-1} - \beta_{n-2}, \qquad n \geq 2,$$

with $\beta_0 = 2$, $\beta_1 = A^2+2$. Thus $\{\beta_n\}$ or $\{v_{2n}\}$ is also generated by determinants of tridiagonal matrices of type (2.3) and respective permanents. For the sake of brevity we do not present these determinants and permanents here. Note that the matrix Y_n in (3.6) is a nonnegative Hessenberg matrix whose permanents generate the sequence $\{v_{2n}\}$.

It can be shown that the matrix R_n given in (1.8) also satisfies (3.2). Thus the sequence $\{u_{2n}\}_{n=0}^{\infty}$ is a second order linear recurrence sequence.

Let the $n \times n$ $(1,-1)$-matrix S be as before and denote the matrices $E_n \circ S$ and $Z_n \circ S$ by M_n and Y_n, respectively. Clearly

$$M_n = \begin{bmatrix} A^2+3 & 1 & 0 & \cdots & 0 & 0 \\ A^2+2 & A^2+1 & 1 & \cdots & \vdots & 0 \\ A^2+2 & A^2 & A^2+1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ A^2+2 & A^2 & \cdots & A^2 & A^2+1 & 1 \\ A^2+2 & A^2 & A^2 & \cdots & A^2 & A^2+1 \end{bmatrix} \tag{3.5}$$

and

$$Y_n = \begin{bmatrix} A^2+2 & 1 & 0 & \cdots & 0 & 0 \\ A^2 & A^2+1 & 1 & \cdots & \vdots & 0 \\ A^2 & A^2 & A^2+1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ A^2 & A^2 & \cdots & A^2 & A^2+1 & 1 \\ A^2 & A^2 & A^2 & \cdots & A^2 & A^2 \end{bmatrix}. \tag{3.6}$$

Then we have the following theorems without proof.

Theorem 6. *Suppose that the $n \times n$ lower Hessenberg matrix M_n has the form* (3.5). *Then for $n \geq 1$*

$$\operatorname{per} M_n = \frac{v_{2n+1}}{A}.$$

Theorem 7. *Suppose that the $n \times n$ lower Hessenberg matrix Y_n has the form* (3.6). *Then for $n \geq 1$*

$$\operatorname{per} Y_n = v_{2n}.$$

Considering the identity (1.4), we can give the following results:

$$\det Q_{n-2} = \operatorname{per} D_{n-2} = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} A^{2n-2k-4} \quad \text{for } n \geq 3, \tag{3.7}$$

$$\det E_n = \operatorname{per} M_n = \sum_{k=0}^{\lfloor \frac{2n+1}{2} \rfloor} \frac{2n+1}{2n-k+1} \binom{2n-k+1}{k} A^{2n-2k} \quad \text{for } n \geq 1 \tag{3.8}$$

and

$$\det Z_n = \operatorname{per} Y_n = \sum_{k=0}^{n} \frac{2n}{2n-k} \binom{2n-k}{k} A^{2n-2k} \quad \text{for } n \geq 1. \tag{3.9}$$

Similar results hold also for $\det X_n$ and $\text{per}(X_n \circ S)$. Further matrices possessing properties similar to Theorems 1–7 and formulas (3.7)–(3.9) can be derived applying various determinant rules. For example,

$$v_{2n+1} = A^{-1} \begin{vmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & A^2 & -1 & \ddots & 0 & 0 \\ A^2 & A^2 & A^2+1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ A^2 & A^2 & A^2 & \dots & A^2+1 & -1 \\ A^2 & A^2 & A^2 & \dots & A^2 & A^2+1 \end{vmatrix}_{(n+2)\times(n+2)}.$$

This formula follows from expanding the determinant with respect to the first row. Further binomial formulas can be found from [9, 25].

In this paper we have provided connections between recurrence sequences and determinants and permanents of Hessenberg matrices and also certain binomial sums. Connections to other objects in mathematics can also be found, e.g. to powers of matrices, continued fractions, generating functions and rational arithmetical functions, see [7, 8, 9, 10, 20, 21, 22, 25].

References

[1] M. Bicknell. "A Primer On the Pell Sequence and Related Sequences." *Fib. Quart.* **13** (4) (1975), 345–349.

[2] R. A. Brualdi & P. M. Gibson. "Convex Polyhedra of Doubly Stochastic Matrices I: Applications of the Permanent Function." *J. Combin. Theory* **A 22** (1977), 194–230.

[3] P. F. Byrd. "Problem B-12: A Lucas Determinant." *Fib. Quart.* **1** (4) (1963), 78.

[4] N. D. Cahill and D. A. Narayan. "Fibonacci and Lucas Numbers as Tridiagonal Matrix Determinants." *Fib. Quart.* **42** (3) (2004), 216–221.

[5] N. D. Cahill, J. R. D'Errica, D. A. Narayan, and J. Y. Narayan. "Fibonacci Determinants." *College Math. J.* **3** (3) (2002), 221–225.

[6] P. Filipponi and A. F. Horadam. "Second Derivative Sequences of Fibonacci and Lucas Polynomials." *Fib. Quart.* **31** (3) (1993), 194–204.

[7] P. Haukkanen. "A note on rational arithmetic functions of order $(2, 1)$." *Fib. Quart.* **31** (4) (1993), 302–306.

[8] P. Haukkanen. "A note on a curious polynomial identity." *Nieuw Arch. Wisk.* (4) **13** (1995), no. 2, 181–186.

[9] P. Haukkanen. "Binomial formulas for specially multiplicative functions." *Math. Stud.* **64** (1995), no. 1-4, 155–161.

[10] D. Kalman. "Generalized Fibonacci numbers by matrix methods." *Fib. Quart.* **20** (1) (1982), 73–76.

[11] E. Kilic and D. Tasci. "On Families of Bipartite Graphs Associated with Sums of Fibonacci and Lucas Numbers." *Ars Combin.* (to appear).

[12] E. Kilic and D. Tasci. "On Families of Bipartite Graphs Associated with Sums of Generalized Order-k Fibonacci and Lucas Numbers." *Ars Combin.* (to appear).

[13] E. Kilic and D. Tasci. "On the Second Order Linear Recurrences by Tridiagonal Matrices." *Ars Combin.* (to appear).

[14] E. Kilic and D. Tasci. "On the Generalized Order-k Fibonacci and Lucas Numbers." *Rocky Mountain J. of Math.* (to appear).

[15] E. Kilic and D. Tasci. "On the Permanents of Some Tridiagonal Matrices with Applications to the Fibonacci and Lucas Numbers." *Rocky Mountain J. of Math.* (to appear).

[16] E. Kilic and D. Tasci. "On the Second Order Linear Recurrence Satisfied by the Permanent of Generalized Doubly Stochastic Matrices." *Ars Combin.* (to appear).

[17] E. Kilic and D. Tasci. "On The Generalized Fibonacci and Pell Sequences By Hessenberg Matrices." *Ars Combin.* (to appear).

[18] G.-Y. Lee and S.-G. Lee. "A Note on Generalized Fibonacci Numbers." *Fib. Quart.* **33** (3) (1995), 273–278.
[19] G.-Y. Lee. "k-Lucas Numbers and Associated Bipartite Graphs." *Linear Algebra Appl.* **320** (2000), 51–61.
[20] D. Lehmer. "Fibonacci and Related Sequences in Periodic Tridiagonal Matrices." *Fib. Quart.* **13** (1975), 150–158.
[21] P. J. McCarthy and R. Sivaramakrishnan. "Generalized Fibonacci sequences via arithmetical functions." *Fib. Quart.* **28** (4) (1990), 363–370.
[22] J. Mc Laughlin. "Combinatorial identities deriving from the nth power of a 2×2 matrix." *Integers* **4** (2004), A19, 15 pp.
[23] H. Minc. "Permanents of $(0, 1)$-Circulants." *Canad. Math. Bull.* **7** (2) (1964), 253–263.
[24] H. Minc. *Permanents, Encyclopedia of Mathematics and its Applications.* Addison-Wesley, New York, 1978.
[25] N. Robbins. *Beginning Number Theory.* Dubuque, Iowa: Wm. C. Brown Publishers 1993.
[26] G. Strang. *Introduction to linear Algebra.* 2nd Edition, Wellesley MA, Wellesley-Cambridge, 1998.
[27] G. Strang and K. Borre. *Linear Algebra.* Geodesy and GPS. Wellesley MA, Wellesley-Cambridge, 1997, pp. 555–557.
[28] D. Tasci and E. Kilic. "On the Order-k Generalized Lucas Numbers." *Appl. Math. Comput.* **155** (3) (2004), 637–641.

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