CERTAIN BINOMIAL SUMS WITH RECURSIVE COEFFICIENTS

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Abstract. In this short note, we establish some identities containing sums of binomials with coefficients satisfying third order linear recursive relations. As a result and in particular, we obtain general forms of earlier identities involving binomial coefficients and Fibonacci type sequences.

1. Introduction

There are many types of identities containing sums of certain functions of binomial coefficients and Fibonacci, Lucas or Pell numbers. Let us give a few examples of such identities (see [2, 5]):

$$\sum_{k=0}^{n}\binom{n}{k}F_k = F_{2n},\quad \sum_{k=0}^{n}\binom{n}{k}2^kF_k = F_{3n}, \sum_{k=0}^{2n}\binom{2n}{k}F_{2k} = 5^nF_{2n} \tag{1.1}$$

$$n \geq m,\ \sum_{k=0}^{n}(-1)^k\binom{n}{k}F_{n+k-m} = F_{n-m},\quad \sum_{k=0}^{2n}\binom{2n}{k}F_{2k}^2 = 5^{n-1}L_{2n}, \tag{1.2}$$

$$\sum_{k=0}^{2n}\binom{2n}{k}L_{2k} = 5^nL_{2n},\quad \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}2^{k-1}L_k = 5^n, \tag{1.3}$$

where F_n and L_n stand as usual for the n^{th} Fibonacci and respectively the n^{th} Lucas number. We remind the reader that $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, $L_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$, $L_{n+1} = L_n + L_{n-1}$ for $n \in \mathbb{N}$. As a more sophisticated example the following identity was asked by Hoggatt as an advanced problem in [4]:

$$\sum_{k=0}^{n}\binom{n}{k}F_{4mk} = L_{2m}^nF_{2mn}. \tag{1.4}$$

Generalizations of the above identities appeared in [1, 3]. For instance, if u, v and r are integers, $uv(u-v) \neq 0$, then

$$F_v^nF_{un+r} = \sum_{k=0}^{n}(-1)^{(n-k)u}\binom{n}{k}F_{v-u}^{n-k}F_u^kF_{vk+r},$$

which generalizes the identities (1.1-1.2). Similar identities to (1.1-1.3) for the Pell numbers can be derived. Our interest here is for identities in which only half of the binomial coefficients are used. Three of such identities are

$$\sum_{k=0}^{n}\binom{2n}{n+k}F_k^2 = 5^{n-1}, \quad n\in\mathbb{N}, \tag{1.5}$$

$$\sum_{k=0}^{n}\binom{2n}{n+k}L_k^2 = 5^{n}+2\binom{2n}{n}, \quad n\in\mathbb{N}, \tag{1.6}$$

$$\sum_{k=0}^{n}\binom{2n}{n+k}P_k^2 = 8^{n-1}, \quad n\in\mathbb{N}, \tag{1.7}$$

where P_n is the the n^{th} Pell number ($P_0=0$, $P_1=1$, and $P_{n+1}=2P_n+P_{n-1}$, $n\in\mathbb{N}$). We are also going to work with generalized Fibonacci ($\{u_n\}$) and Lucas ($\{v_n\}$) sequences defined by

$$u_{n+1}=pu_n+u_{n-1},\ v_{n+1}=pv_n+v_{n-1}\ \ n\in\mathbb{N}, \tag{1.8}$$

where $u_0=0,\ u_1=1$ or $v_0=2,\ v_1=p$ for any complex number p.

One can derive easily the Binet formulae for $\{u_n\}$ and $\{v_n\}$:

$$u_n=\frac{\alpha^n-\beta^n}{\alpha-\beta} \text{ and } v_n=\alpha^n+\beta^n,\ n\in\mathbb{N}\cup\{0\},$$

where $\alpha=\left(p+\sqrt{p^2+4}\right)/2$ and $\beta=\left(p-\sqrt{p^2+4}\right)/2$, with the principal branch of the square root $\sqrt{re^{i\theta}}=\sqrt{r}e^{\frac{\theta}{2}}$, $\theta\in(-\pi,\pi]$, $r\geq 0$. From these formulae, we can easily get the following identity:

$$v_{n+m}-(-1)^m v_{n-m}=\left(p^2+4\right)u_nu_m, \qquad m,n\in\mathbb{N}\cup\{0\}. \tag{1.9}$$

As aims of this note we will establish generalizations of the formulae (1.5), (1.6), (1.7) and a version of (1.9) with combinatorial coefficients involved. Our techniques are definitely pure computational and very much in line with the standard ones used to show (1.1)-(1.3). One way of generalizing the identities in (1.5)-(1.7) is to use different powers for the recursive function term:

$$\sum_{k=0}^{n}\binom{2n}{n+k}F_k^4 = \frac{1}{25}\left(3^{2n}-4\left(-1\right)^n+3\times 2^{2n}\right), \quad n\in\mathbb{N}, \tag{1.10}$$

$$\sum_{k=0}^{n}\binom{2n}{n+k}L_k^4 = 3^{2n}+3\times 2^{2n}+8\binom{2n}{n}+4\left(-1\right)^n, \quad n\in\mathbb{N}, \tag{1.11}$$

$$\sum_{k=0}^{n}\binom{2n}{n+k}P_k^4 = \frac{1}{64}\left(6^{2n}-2^{2n+2}-4\left(-1\right)^n+3\times 2^{2n}\right), \quad n\in\mathbb{N}, \tag{1.12}$$

An interesting question at this point is whether or not one can arrange so that for some Fibonacci type recurrent sequence, powers of any positive integer could be represented as in (1.5) and (1.7). We will address this question in the last section of this note.

2. Half of the binomial formula

In this section we build up the main ingredients for our calculations. Let us define the function f of $a \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$:

$$f(n,a) = \sum_{k=0}^{n} \binom{2n}{n+k} \left(a^k + a^{-k}\right).$$

We have the following lemma.

Lemma 2.1. *For every non-negative integer n and $a \in \mathbb{C} \setminus \{0\}$ the function f satisfies*

$$f(n,a) = \frac{1}{a^n}(a+1)^{2n} + \tbinom{2n}{n}.$$

Proof. We can write

$$\begin{aligned}
a^n f(n,a) &= \tbinom{2n}{n}a^n + \tbinom{2n}{n+1}a^{n+1} + \ldots + \tbinom{2n}{2n}a^{2n} \\
&\quad + \tbinom{2n}{n}a^n + \tbinom{2n}{n+1}a^{n-1} + \cdots + \tbinom{2n}{2n}a^0 \\
&= \left(\tbinom{2n}{n}a^n + \tbinom{2n}{n+1}a^{n+1} + \ldots + \tbinom{2n}{2n}a^{2n}\right) \\
&\quad + \left(\tbinom{2n}{n-1}a^{n-1} + \cdots + \tbinom{2n}{0}a^0\right) + \tbinom{2n}{n}a^n \\
&= (a+1)^{2n} + \tbinom{2n}{n}a^n.
\end{aligned}$$

Hence the identity claimed follows by dividing by a^n. □

An observation here is necessary. We formulate this as a proposition.

Proposition 2.2. *For every non-negative integer n and $a \in \mathbb{C} \setminus \{0\}$ the following formulae yield true:*

$$\sum_{k=0}^{n} \binom{2n}{n+k} = 2^{2n-1} + \frac{1}{2}\tbinom{2n}{n}, \quad \sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} = \frac{1}{2}\tbinom{2n}{n},$$

$$f(n,(-1)^r) = (1+(-1)^r)2^{2n-1} + \tbinom{2n}{n}, \quad f(n,-a^2) = (-1)^n \left(a - \frac{1}{a}\right)^{2n} + \tbinom{2n}{n}.$$

Proof. To obtain the first two identities we set $a = 1$ and then $a = -1$. The last claim is obtained by substituting a with $-a$ in Lemma 2.1. □

3. Proof of the claimed identities

We will work with the generalized Fibonacci type sequences $\{u_k\}$ and $\{v_k\}$ defined in the introduction. The formulae (1.5), (1.10), (1.7), and (1.12) are contained in the next theorem.

Theorem 3.1. *For $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{N}$, and $\{u_k\}$ and $\{v_k\}$ defined as before by (1.8), we have*

$$\sum_{k=0}^{n} \binom{2n}{n+k} u_k^{2r} = \begin{cases} \frac{1}{(p^2+4)^r} \left(\binom{2r}{r} 2^{2n-2} + \sum_{i=0}^{r-1} (-1)^{i(n+1)} \binom{2r}{i} v_{r-i}^{2n} \right) & if\ r\ is\ even. \\ (p^2+4)^{n-r} \left(\sum_{i=0}^{r-1} (-1)^{i(n+1)} \binom{2r}{i} u_{r-i}^{2n} \right) & if\ r\ is\ odd. \end{cases}$$

Proof. We expand first u_k to the power $2r$ using the binomial formula:

$$\sum_{k=0}^{n} \binom{2n}{n+k} u_k^{2r} = \frac{1}{(\alpha-\beta)^{2r}} \sum_{k=0}^{n} \binom{2n}{n+k} \left[\sum_{i=0}^{2r} (-1)^i \binom{2r}{i} \alpha^{(2r-i)k} \beta^{ik} \right].$$

Taking into account that $\alpha^k \beta^k = (-1)^k$ and the facts that $\binom{2r}{i} = \binom{2r}{2r-i}$, $(-1)^i = (-1)^{2r-i}$, for $i = 0, 1, 2, \ldots, 2r$, we can turn the above into

$$\sum_{k=0}^{n} \binom{2n}{n+k} u_k^{2r} = \frac{1}{(p^2+4)^r} \sum_{k=0}^{n} \binom{2n}{n+k} \left[(-1)^{r(1+k)} \binom{2r}{r} + \sum_{i=0}^{r-1} (-1)^{i+ik} \binom{2r}{i} (\alpha^{2(r-i)k} + \alpha^{-2(r-i)k}) \right].$$

Commuting the two summations and using Lemma 2.1 (Proposition 2.2) the calculation can be continued to

$$\sum_{k=0}^{n} \binom{2n}{n+k} u_k^{2r} =$$
$$\frac{1}{(p^2+4)^r} \left(\tfrac{(-1)^r}{2} \tbinom{2r}{r} f(n,(-1)^r) + \textstyle\sum_{i=0}^{r-1} (-1)^i \tbinom{2r}{i} f(n,(-1)^i \alpha^{2(r-i)}) \right) =$$
$$\frac{1}{(p^2+4)^r} \left(\frac{(-1)^r}{2} \binom{2r}{r} f(n,(-1)^r) + \sum_{i=0}^{r-1} (-1)^i \binom{2r}{i} [\tbinom{2n}{n} + (-1)^{in} \left((-1)^{r-i}\beta^{r-i} + (-1)^i \alpha^{r-i} \right)^{2n}] \right) =$$
$$\begin{cases} \frac{1}{(p^2+4)^r} \left(\binom{2r}{r} 2^{2n-1} + \sum_{i=0}^{r-1} (-1)^{i(n+1)} \binom{2r}{i} v_{r-i}^{2n} \right) & if\ r\ is\ even. \\ (p^2+4)^{n-r} \left(\sum_{i=0}^{r-1} (-1)^{i(n+1)} \binom{2r}{i} u_{r-i}^{2n} \right) & if\ r\ is\ odd. \end{cases}$$

□

The similar result to Theorem 3.1 but for $\{v_n\}$ is stated next.

Theorem 3.2. *For $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{N}$, and $\{u_k\}$ and $\{v_k\}$ defined as before by (1.8), we have*

$$\sum_{k=0}^{n} \binom{2n}{n+k} v_k^{2r} = \begin{cases} \binom{2r}{r} 2^{2n-1} + 2^{2r-1}\binom{2n}{n} + \sum_{i=0}^{r-1} (-1)^{in} \binom{2r}{i} v_{r-i}^{2n} \ \textit{if } r \textit{ is even.} \\ \\ 2^{2r-1}\binom{2n}{n} + (p^2+4)^n \left(\sum_{i=0}^{r-1} (-1)^{in} \binom{2r}{i} u_{r-i}^{2n} \right) \ \textit{if } r \textit{ is odd.} \end{cases}$$

Proof. In this case after we expand v_k to the power $2r$, we get:

$$\sum_{k=0}^{n} \binom{2n}{n+k} v_k^{2r} = \sum_{k=0}^{n} \binom{2n}{n+k} \left[\sum_{i=0}^{2r} \binom{2r}{i} \alpha^{(2r-i)k} \beta^{ik} \right].$$

Again using that $\alpha^k \beta^k = (-1)^k$ and the facts that $\binom{2r}{i} = \binom{2r}{2r-i}$, $(-1)^i = (-1)^{2r-i}$, for $i = 0, 1, 2, \ldots, 2r$, we can turn the above into

$$\sum_{k=0}^{n} \binom{2n}{n+k} v_k^{2r} = \sum_{k=0}^{n} \binom{2n}{n+k} \left[(-1)^{rk} \binom{2r}{r} + \sum_{i=0}^{r-1} (-1)^{ik} \binom{2r}{i} (\alpha^{2(r-i)k} + \alpha^{-2(r-i)k}) \right].$$

Commuting the two summations and using Proposition 2.2 the above can be continued into

$$\sum_{k=0}^{n} \binom{2n}{n+k} v_k^{2r} = \left(\frac{1}{2} \binom{2r}{r} f(n, (-1)^r) + \sum_{i=0}^{r-1} \binom{2r}{i} f(n, (-1)^i \alpha^{2(r-i)}) \right) =$$

$$\left(\frac{1}{2} \binom{2r}{r} f(n, (-1)^r) + \sum_{i=0}^{r-1} \binom{2r}{i} [\binom{2n}{n} + (-1)^{in} \left((-1)^{r-i} \beta^{r-i} + (-1)^i \alpha^{r-i} \right)^{2n}] \right) =$$

$$\begin{cases} \binom{2r}{r} 2^{2n-1} + 2^{2r-1}\binom{2n}{n} + \sum_{i=0}^{r-1} (-1)^{in} \binom{2r}{i} v_{r-i}^{2n} \ \textit{if } r \textit{ is even.} \\ \\ 2^{2r-1}\binom{2n}{n} + (p^2+4)^n \left(\sum_{i=0}^{r-1} (-1)^{in} \binom{2r}{i} u_{r-i}^{2n} \right) \ \textit{if } r \textit{ is odd.} \end{cases}$$

□

As another application of Lemma 2.1 to the second order recurrence $\{u_n\}$, we have the following consequence.

Proposition 3.3. *For $n \geq 0$ and all even integers $m \geq 2$ and $t \geq 0$,*

$$\sum_{k=0}^{n} \binom{2n}{n+k} (v_{km} - v_{kt}) = (p^2+4)^n \left(u_{m/2}^{2n} - u_{t/2}^{2n} \right) \tag{3.1}$$

where p is as before.

Proof. By Lemma 2.1 and the Binet formulas of the sequences $\{u_n\}$ and $\{v_n\}$, we write the right side of (3.1) as

$$\begin{aligned}
&\sum_{k=0}^{n} \tbinom{2n}{n+k} \left(\alpha^{km} + \beta^{km} - \alpha^{kt} - \beta^{kt}\right) \\
= \;& f(n, \alpha^m) - f\left(n, \alpha^t\right) \\
= \;& \frac{1}{\alpha^{mn}} (\alpha^m + 1)^{2n} - \frac{1}{\alpha^{tn}} \left(\alpha^t + 1\right)^{2n} \\
= \;& \left(\alpha^{m/2} - \beta^{m/2}\right)^{2n} - \left(\alpha^{t/2} - \beta^{t/2}\right)^{2n} \\
= \;& (\alpha - \beta)^{2n} \left(u_{m/2}^{2n} - u_{t/2}^{2n}\right)
\end{aligned}$$

which completes the proof. □

One interesting consequence of Proposition 3.3 is the following.

Proposition 3.4. *For* $n \geq 0$,

$$\sum_{k=0}^{n} \tbinom{2n}{n+k} v_{2k} = \left(p^2 + 4\right)^n.$$

Proof. We take $m = 2$, $t = 0$ in Proposition 3.3, and the proof follows. □

4. Representations of the powers of every integer and other comments

To address the question we have raised in the introduction we notice that as a result of Theorem 3.1 for $r = 1$ we obtain

$$\sum_{k=0}^{n} \binom{2n}{n+k} u_k^2 = (p^2 + 4)^{n-1}, \quad n \in \mathbb{N}, \; p \in \mathbb{C}.$$

Suppose we would like to have a power of 7 in the above equality, i.e. $p^2 + 4 = 7$. This can be accomplished if, for instance, $p = \sqrt{3}$. This turns $\{u_k\}$ and $\{v_k\}$ into sequences of the form $u_k = a_k + b_k\sqrt{3}$ and $v_k = c_k + d_k\sqrt{3}$. Then the sequences $\{a_k\}$, $\{b_k\}$, $\{c_k\}$, and $\{d_k\}$ are uniquely determined by the recurrences

$$a_0 = b_0 = 0, a_1 = 1, b_1 = 0, \; a_{n+1} = 3b_n + a_{n-1}, \; b_{n+1} = a_n + b_{n-1}, \quad \textit{and} \tag{4.1}$$

$$\begin{aligned}
&c_0 = 2, d_0 = 0, c_1 = 1, d_1 = 0, \\
&c_{n+1} = 3d_n + c_{n-1}, \; d_{n+1} = c_n + d_{n-1}, \; n \in \mathbb{N}.
\end{aligned} \tag{4.2}$$

Hence, for a generalized Fibonacci double sequence defined as in (4.1) we obtain a similar identity to (1.5):

$$\sum_{k=0}^{n} \binom{2n}{n+k} (a_k^2 + 3b_k^2) = 7^{n-1}, \quad n \in \mathbb{N}.$$

Let us observe that $\sum_{k=0}^{n}\binom{2n}{n+k}a_kb_k = 0$ which implies that $a_kb_k = 0$ for every $k \in \mathbb{N}$. This is a little surprising since the two sequences $\{a_k\}$ and $\{b_k\}$ seem to be increasing. Theorem 3.1 for $r = 4$ gives

$$\sum_{k=0}^{n}\binom{2n}{n+k}F_k^8 = \frac{1}{625}\left(70 \times 2^{2n-1} + 7^{2n} + 8(-1)^{n+1}4^{2n} + 28 \times 3^{2n} + 56(-1)^{n+1}\right).$$

Certain congruences can be derived by use of these identites. For example:

$$\forall n \in \mathbb{N},\quad 3^{2n} - 4\,(-1)^n + 3 \times 2^{2n} \equiv 0\ (mod\ 25)$$

and

$$\forall n \in \mathbb{N},\quad 70 \times 2^{2n-1} + 7^{2n} + 8(-1)^{n+1}4^{2n} + 28 \times 3^{2n} + 56(-1)^{n+1} \equiv 0\ (mod\ 625).$$

Other such identites might be derived by the interested reader.

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