SECOND ORDER LINEAR RECURSIONS WHOSE SUBSCRIPTS ARE A POWER

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Abstract. We consider two kinds second order linear recurrences whose subscripts are a power and then present certain new identites including these recurrences. Further we derive first-order nonlinear homogeneous recurrence relations for these recurrences so that our results generalize earlier results as well as cover new solutions for some uncompleted cases in the literature.

1. INTRODUCTION

Let p be a nonzero integer such that $\Delta = p^2 + 4 \neq 0$. Define the generalized Fibonacci type $\{u_n\}$ and Lucas type $\{v_n\}$ sequences as follows:

$$u_n = pu_{n-1} + u_{n-2}$$

and

$$v_n = pv_{n-1} + v_{n-2},$$

where $u_0 = 0, u_1 = 1$ and $v_0 = 2, v_1 = p$, respectively.

If the roots of $x^2 - px - 1 = 0$ are α and β, then the Binet forms of $\{u_n\}$ and $\{v_n\}$ are

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } v_n = \alpha^n + \beta^n.$$

If $p = 1$, then $u_n = F_n$ (nth Fibonacci number) and $v_n = L_n$ (nth Lucas number) respectively.

Usiskin [8, 7] suggested the following problems separately: For $n > 0$, show that

$$F_{3^n} = \prod_{k=0}^{n-1} L_{2.3^k} - 1 \tag{1.1}$$

and

$$L_{3^n} = \prod_{k=0}^{n-1} L_{2.3^k} + 1. \tag{1.2}$$

In [4], the author asked for the solution of the first order cubic recurrence relation:

$$a_{n+1} = 5a_n^3 - 3a_n \tag{1.3}$$

with $a_0 = 1$.

Then in [6], the solution is given as $a_n = F_{3^n}$. The same problem appeared as Problem 1809 in Crux Mathematicorum 20 (1994): 19-20.

In the same issue, there was a proposal to solve the recurrence

$$P_{n+1} = 25P_n^5 - 25P_n^3 + 5P_n,\ P_0 = 1.$$

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The solution was given as $P_n = 5^n$. Also the following recurrences and their solutions were commented by Murray S. Klamkin as an addendum to the solution of the problem given in [6]:

$$\begin{aligned} A_{n+1} &= A_n^2 - 2,\ A_1 = 3, \\ B_{n+1} &= B_n^4 - 4B_n^2 + 2,\ B_1 = 7, \\ C_{n+1} &= C_n^6 - 6C_n^4 + 9C_n^2 - 2,\ C_1 = 18, \end{aligned}$$

where $A_n = L_{2^n},\ B_n = L_{4^n}$ and $C_n = L_{6^n}$.

In [1], the author presented some identities involving Fibonacci numbers of the form F_{k^n} for positive odd k and gave a first-order nonlinear homogeneous recurrence relation for F_{k^n}, which generalized (1.3), (1.1) and (1.2).

Recently in [2], *Helmut Prodinger* gave a general expansion formula for a sum of powers of Fibonacci numbers, as considered by Melham, as well as some extensions.

In this paper, we consider two generalized second order recursion sequences and then generalize the results of [1] for the odd k case as well as derive a new first-order nonlinear homogeneous recurrence relation for the sequence $\{u_{k^n}\}$ for possible even k. Further we present that the generalized Lucas numbers $v_{k^{n+1}}$ is a polynomial of generalized Fibonacci numbers u_{k^n} of degree k for even k.

2. Main Results

In this section, in order to derive a recurrence relation for both even and odd subscripted terms u_{k^n}, we start with the following result.

Proposition 1. *For $n \geq 1$ and even k,*

$$u_{k^n} = u_k \prod_{i=1}^{n-1} \left(\sum_{j=1}^{k/2} v_{(2j-1)k^i} \right).$$

Proof. Consider

$$u_{k^n} = u_k \frac{u_{k^2}}{u_k} \frac{u_{k^3}}{u_{k^2}} \dots \frac{u_{k^n}}{u_{k^{n-1}}} = u_k \prod_{i=1}^{n-1} \frac{u_{k^{i+1}}}{u_{k^i}}. \tag{2.1}$$

From [5], we have that for even $k \geq 2$,

$$\frac{m^k - n^k}{m-k} = \sum_{j=0}^{k/2-1} (mn)^i \left(m^{k-2i-1} + n^{k-2i-1} \right) \tag{2.2}$$

When $m = \alpha^n, n = \beta^n$ in (2.2), we get

$$\frac{u_{kn}}{u_n} = \sum_{j=0}^{k/2-1} (\alpha\beta)^{jn} \left(\alpha^{(k-2j-1)n} + \beta^{(k-2j-1)n} \right) = \sum_{j=0}^{k/2-1} (-1)^{jn} v_{(k-2j-1)n}$$

$$\frac{u_{k^{i+1}}}{u_{k^i}} = \sum_{j=0}^{k/2-1} (-1)^{jk^i} v_{(k-2j-1)k^i} = \sum_{j=1}^{k/2} v_{(2j-1)k^i} \tag{2.3}$$

By (2.3), the equation (2.1) equals that

$$u_{k^n} = u_k \prod_{i=1}^{n-1} \frac{u_{k^{i+1}}}{u_{k^i}} = u_k \prod_{i=1}^{n-1} \left(\sum_{j=1}^{k/2} v_{(2j-1)k^i} \right).$$

Thus the proof is complete. □

For the Fibonacci and Lucas case when $p = 1$, we also refer to [9].

For later use, we give the following two identities:

$$\begin{aligned}\sum_{i=1}^{r}(-1)^i v_{ai} &= \sum_{i=1}^{r}(-1)^i \alpha^{ai} + \sum_{i=1}^{r}(-1)^i \beta^{ai} \\ &= \frac{-\alpha^a + (-1)^r (\alpha^a)^{r+1}}{1+\alpha^a} + \frac{-\beta^a + (-1)^r (\beta^a)^{r+1}}{1+\beta^a} \\ &= \frac{(-1)^{a+r} v_{ar} + (-1)^r v_{a(r+1)} - v_a - 2(-1)^a}{v_a + 1 + (-1)^a}\end{aligned} \tag{2.4}$$

and

$$v_{a+b} - (-1)^b v_{a-b} = \Delta u_a u_b. \tag{2.5}$$

Proposition 2. *For odd k and $n \geq 1$,*

$$u_{k^n} = (-1)^{(n-1)(k-1)/2} u_k \prod_{i=1}^{n-1}\left(1 + \sum_{j=1}^{(k-1)/2}(-1)^j v_{2k^i j}\right).$$

Proof. If $(k-1)/2$ is even, we write by (2.4) and (2.5)

$$\begin{aligned}u_{k^n} &= u_k \prod_{i=1}^{n-1}\frac{u_{k^{i+1}}}{u_{k^i}} = u_k \prod_{i=1}^{n-1}\left(1 + \frac{\Delta u_{k^{i+1}} u_{k^i}}{\Delta u_{k^i} u_{k^i}} - 1\right) \\ &= u_k \prod_{i=1}^{n-1}\left(1 + \frac{v_{k^{i+1}+k^i} - (-1)^{k^i} v_{k^{i+1}-k^i}}{v_{k^i+k^i} - (-1)^{k^i} v_{k^i-k^i}} - 1\right) \\ &= u_k \prod_{i=1}^{n-1}\left(1 + \frac{v_{2k^i\left(\frac{k+1}{2}\right)} + v_{2k^i\left(\frac{k-1}{2}\right)} - v_{2k^i} - 2}{v_{2k^i} + 2}\right) \\ &= u_k \prod_{i=1}^{n-1}\left(1 + \sum_{j=1}^{(k-1)/2}(-1)^j v_{2k^i j}\right) \\ &= (-1)^{(n-1)(k-1)/2} u_k \prod_{i=1}^{n-1}\left(1 + \sum_{j=1}^{(k-1)/2}(-1)^j v_{2k^i j}\right).\end{aligned}$$

If $(k-1)/2$ is odd, then we write

$$\begin{aligned}u_{k^n} &= (-1)^{n-1} u_k \prod_{i=1}^{n-1}\frac{u_{k^{i+1}}}{u_{k^i}} = (-1)^{n-1} u_k \prod_{i=1}^{n-1}\left(1 - \frac{\Delta u_{k^{i+1}} u_{k^i}}{\Delta u_{k^i}} - 1\right) \\ &= (-1)^{n-1} u_k \prod_{i=1}^{n-1}\left(1 + \frac{-v_{2k^i\left(\frac{k+1}{2}\right)} - v_{2k^i\left(\frac{k-1}{2}\right)} - v_{2k^i} - 2}{v_{2k^i} + 2}\right) \\ &= (-1)^{n-1} u_k \prod_{i=1}^{n-1}\left(1 + \sum_{k=1}^{(k-1)/2}(-1)^j v_{2k^i j}\right).\end{aligned}$$

So we have the conclusion for all cases. □

3. Recurrence Relation for $\{u_{k^n}\}$

We shall derive recurrence relations for $\{u_{k^n}\}$ or $\{v_{k^n}\}$ for odd or even k. Thus we need the following result:

Lemma 1. *For $n, q \geq 0$,*

$$\begin{aligned} u_{(2q+1)n} &= u_n \sum_{k=0}^{q} (-1)^{n(q+k)} \frac{2q+1}{q+k+1} \left(p^2+4\right)^k \binom{q+k+1}{2k+1} u_n^{2k}, \\ v_{2qn} &= \sum_{k=0}^{q} (-1)^{n(q+k)} \frac{2q}{q+k} \binom{q+k}{2k} \left(p^2+4\right)^k u_n^{2k}, \\ v_{(2q+1)n} &= v_n \sum_{k=0}^{q} (-1)^{(n+1)(q+k)} \frac{2q+1}{q+k+1} \binom{q+k+1}{2k+1} v_n^{2k}, \\ v_{2qn} &= \sum_{k=0}^{q} (-1)^{(n+1)(q+k)} \frac{2q}{q+k} \binom{q+k}{2k} v_n^{2k}. \end{aligned}$$

Proof. The proof can be easily obtained from [5] by considering classical binomial expansion for $a^n - b^n$ and $a^n + b^n$ where a and b are any real numbers. □

We give a recurrence relation to the sequence $\{u_{k^n}\}$ for odd k.

Proposition 3. *For odd $k > 0$ and $n \geq 0$,*

$$u_{k^{n+1}} = \triangle^{(k-1)/2} u_{k^n}^k - \sum_{i=0}^{(k-3)/2} \triangle^i C_{i,k} u_{k^n}^{2i+1}$$

where the coefficients $C_{i,k}$ are given by for $0 \leq i \leq (k-3)/2$

$$C_{i,k} = (-1)^{(k+1)/2+i} \frac{2k}{k+2i+1} \binom{(k+1)/2+i}{2i+1}.$$

Proof. Consider

$$\begin{aligned} u_{k^n}^k &= \frac{1}{\triangle^{k/2}} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \beta^{jk^n} \alpha^{(k-j)k^n} \\ &= \frac{1}{\triangle^{(k-1)/2}} \left(u_{k^{n+1}} + \sum_{j=1}^{(k-1)/2} \binom{k}{j} u_{(k-2j)k^n} \right), \end{aligned} \tag{3.1}$$

where Δ is defined as before. By (3.1), we obtain for odd k,

$$u_{k^{n+1}} = \triangle^{(k-1)/2} u_{k^n}^k - \sum_{j=1}^{(k-1)/2} \binom{k}{j} u_{(k-2j)k^n}. \tag{3.2}$$

Then by (3.2) and Lemma 1, we write

$$\begin{aligned} u_{k^{n+1}} &= \triangle^{(k-1)/2} u_{k^n}^k \\ &\quad - \sum_{j=1}^{\frac{k-1}{2}} \sum_{i=0}^{\frac{k-1}{2}-j} (-1)^{\frac{k-1}{2}+i-j} \triangle^i \binom{k}{j} \binom{\frac{k+1}{2}+i-j}{2i+1} \frac{k-2j}{\frac{k+1}{2}+i-j} u_{k^n}^{2i+1} \end{aligned}$$

which, after reversing the summation order, can be rewritten as

$$u_{k^{n+1}} = \triangle^{(k-1)/2} u_{k^n}^k - \sum_{i=0}^{(k-1)/2} \triangle^i A_{i,k} u_{k^n}^{2i+1}, \tag{3.3}$$

where

$$A_{i,k} = \sum_{j=1}^{(k-1)/2-i} (-1)^{(k-1)/2+i-j} \binom{k}{j} \binom{(k+1)/2+i-j}{2i+1} \frac{k-2j}{(k+1)/2+i-j}.$$

Since $A_{(k-1)/2,k} = 0$, the equality (3.3) becomes

$$u_{k^{n+1}} = \triangle^{(k-1)/2} u_{k^n}^k - \sum_{i=0}^{(k-3)/2} \triangle^i A_{i,k} u_{k^n}^{2i+1}.$$

From (pp. 58, [3]), we have the combinatorial identity: for $1 \le m \le (k-3)/2$

$$\sum_{j=1}^{m} (-1)^j \frac{k-2j}{k-m-j} \binom{k}{j} \binom{k-m-j}{m-j} = (-1)^m \frac{k}{k-m} \binom{k-m}{m}. \tag{3.4}$$

If we replace m by $\frac{k-1}{2} - i$ in (3.4), then we obtain $C_{i,k} = A_{i,k}$. Thus the proof is complete. □

In a similar manner , we may give the following result:

Proposition 4. *For $n > 0$ and odd $k > 1$,*

$$u_{k^{n+1}} = \Delta^{\frac{k-1}{2}} u_{k^n}^k - \sum_{i=0}^{\frac{k-3}{2}} (-1)^{\frac{k+1}{2}+i} \frac{2k}{k-2i-1} \binom{\frac{k-1}{2}+i}{2i+1} \Delta^i u_{k^n}^{2i+1}.$$

Proof. For odd k, we get

$$\begin{aligned} u_{k^n}^k &= \frac{1}{\Delta^k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \beta^{jk^n} \alpha^{(k-j)k^n} \\ &= \frac{1}{\Delta^{\frac{k-1}{2}}} \left(u_{k^{n+1}} + \sum_{j=1}^{\frac{k-1}{2}} \binom{k}{j} u_{(k-2j)k^n} \right). \end{aligned}$$

So we write

$$u_{k^{n+1}} = \Delta^{\frac{k-1}{2}} u_{k^n}^k - \sum_{j=1}^{\frac{k-1}{2}} \binom{k}{j} u_{(k-2j)k^n}.$$

Using Lemma 1 and reversing the summation order, we write

$$\begin{aligned} u_{k^{n+1}} &= \Delta^{\frac{k-1}{2}} u_{k^n}^k - u_{k^n} \sum_{j=1}^{\frac{k-1}{2}} \sum_{i=0}^{\frac{k-1}{2}-j} (-1)^{\frac{k-1}{2}+i-j} \\ &\quad \times \frac{2(k-2j)}{k-2j+2i+1} \binom{k}{j} \binom{\frac{k-1}{2}+i-j+1}{2i+1} \Delta^i u_{k^n}^{2i} \\ &= \Delta^{\frac{k-1}{2}} u_{k^n}^k - u_{k^n} \sum_{i=0}^{\frac{k-3}{2}} \sum_{j=1}^{\frac{k-1}{2}-i} (-1)^{\frac{k-1}{2}+i-j} \\ &\quad \times \frac{2(k-2j)}{k-2j+2i+1} \binom{k}{j} \binom{\frac{k-1}{2}+i-j+1}{2i+1} \Delta^i u_{k^n}^{2i}. \end{aligned}$$

By simplifying, we derive

$$u_{k^{n+1}} = \Delta^{\frac{k-1}{2}} u_{k^n}^k - u_{k^n} \sum_{i=0}^{\frac{k-3}{2}} (-1)^{\frac{k+1}{2}+i} \frac{2k}{k-2i-1} \binom{\frac{k-1}{2}+i}{2i+1} \Delta^i u_{k^n}^{2i}.$$

□

We give a recurrence relation to the sequence $\{v_{k^n}\}$ for odd k.

Proposition 5. *For $n > 0$ and odd $k > 1$,*

$$v_{k^{n+1}} = v_{k^n}^k - \sum_{i=0}^{\frac{k-1}{2}} E_{i,k} v_{k^n}^{2i+1},$$

where for $0 \leq i < (k-1)/2$

$$E_{i,k} = \frac{2k}{-k+2i+1} \binom{\frac{k-1}{2}+i}{2i+1}.$$

Proof. By the Binet formula of $\{v_n\}$ and the binomial expansion, we write

$$\begin{aligned} v_{k^n}^k &= \sum_{j=0}^{k} \binom{k}{j} \beta^{jk^n} \alpha^{(k-j)k^n} \\ &= v_{k^{n+1}} + \sum_{j=1}^{\frac{k-1}{2}} (-1)^j \binom{k}{j} v_{(k-2j)k^n}. \end{aligned}$$

By Lemma 1, we write

$$v_{k^{n+1}} = v_{k^n}^k - \sum_{j=1}^{\frac{k-1}{2}} \sum_{i=0}^{\frac{k-1}{2}-j} (-1)^j \binom{k}{j} \binom{\frac{k+1}{2}+i-j}{2i+1} \frac{k-2j}{\frac{k+1}{2}+i-j} v_{k^n}^{2i+1}$$

and by reversing the summation order, we get

$$v_{k^{n+1}} = v_{k^n}^k - \sum_{i=0}^{\frac{k-1}{2}} \sum_{j=1}^{\frac{k-1}{2}-i} (-1)^j \binom{k}{j} \binom{\frac{k+1}{2}+i-j}{2i+1} \frac{k-2j}{\frac{k+1}{2}+i-j} v_{k^n}^{2i+1}$$

which, by the definition of $C_{i,k}$, gives

$$v_{k^{n+1}} = v_{k^n}^k - \sum_{i=0}^{\frac{k-3}{2}} \sum_{j=1}^{\frac{k-1}{2}-i} (-1)^j \binom{k}{j} \binom{\frac{k+1}{2}+i-j}{2i+1} \frac{k-2j}{\frac{k+1}{2}+i-j} v_{k^n}^{2i+1}.$$

If we take $m = \frac{k-1}{2} - i$ in (3.4), we get

$$v_{k^{n+1}} = v_{k^n}^k - \sum_{i=0}^{\frac{k-3}{2}} E_{i,k} v_{k^n}^{2i+1},$$

where

$$E_{i,k} = \frac{2k}{-k+2i+1} \binom{\frac{k-1}{2}+i}{2i+1}.$$

□

For example, when $k = 5$,

$$v_{5^{n+1}} = v_{5^n}^5 + 5v_{5^n}^3 + 5v_{5^n}^1. \tag{3.5}$$

We give a recurrence relation to the sequence $\{v_{k^n}\}$ for even k.

Proposition 6. *For n and even $k > 0$,*

$$v_{k^{n+1}} = v_{k^n}^k - \sum_{i=0}^{\frac{k-2}{2}} H_{i,k} v_{k^n}^{2i},$$

where for $1 \le m \le k/2$

$$H_{i,k} = (-1)^{\left(\frac{k}{2}+i\right)} \binom{\frac{k}{2}+i-1}{2i} \frac{2k}{-k+2i}.$$

Proof. It is easy to see that

$$\begin{aligned} v_{k^n}^k &= \sum_{j=0}^{k} \binom{k}{j} \beta^{jk^n} \alpha^{(k-j)k^n} \\ &= v_{k^{n+1}} - \binom{k}{\frac{k}{2}} + \sum_{j=1}^{\frac{k}{2}} \binom{k}{j} v_{(k-2j)k^n}, \end{aligned}$$

Then by Lemma 1 and reverting the summation order, we get

$$\begin{aligned}
& v_{k^n}^k \\
= \;& v_{k^{n+1}} - \binom{k}{\frac{k}{2}} + \sum_{j=1}^{\frac{k}{2}} \binom{k}{j} v_{(k-2j)k^n} \\
= \;& v_{k^{n+1}} + \binom{k}{\frac{k}{2}} + \sum_{j=1}^{\frac{k}{2}} \sum_{i=0}^{\frac{k}{2}-j} (-1)^{\left(\frac{k}{2}-j+i\right)} \binom{k}{j} \binom{\frac{k}{2}-j+i}{2i} \frac{2\left(\frac{k}{2}-j\right)}{\frac{k}{2}-j+i} v_{k^n}^{2i} \\
= \;& v_{k^{n+1}} + \sum_{i=0}^{\frac{k-2}{2}} \sum_{j=1}^{\frac{k}{2}-i} (-1)^{\left(\frac{k}{2}-j+i\right)} \binom{k}{j} \binom{\frac{k}{2}-j+i}{2i} \frac{2\left(\frac{k}{2}-j\right)}{\frac{k}{2}-j+i} v_{k^n}^{2i} \\
= \;& v_{k^{n+1}} + \sum_{i=0}^{\frac{k-2}{2}} (-1)^{\left(\frac{k}{2}+i\right)} \binom{\frac{k}{2}+i-1}{2i} \frac{2k}{-k+2i} v_{k^n}^{2i}.
\end{aligned}$$

Thus the proof is complete. □

When $k = 6$, we get

$$v_{6^{n+1}} = v_{6^n}^6 - 6v_{6^n}^4 + 9v_{6^n}^2 - 2v_{6^n}^0. \tag{3.6}$$

Here we note that the coefficients of the formulas in (3.5) and (3.6) with adjusted sings appears to be the terms of the sequence *A034807* in the OEIS.

4. A Polynomial Representations for $v_{k^{n+1}}$

In this section, we show that the generalized Lucas numbers $v_{k^{n+1}}$ is a polynomial of generalized Fibonacci numbers u_{k^n} of degree k for even k.

Proposition 7. *For even $k > 0$ and $n \geq 0$,*

$$v_{k^{n+1}} = \sum_{i=0}^{\frac{k-2}{2}} D_{i,k} \triangle^i u_{k^n}^{2i},$$

where $D_{i,k}$ is given by for $1 \leq m < k/2$

$$D_{i,k} = \frac{2k}{k+2i} \binom{i+\frac{k}{2}}{2i}.$$

Proof. Consider

$$\begin{aligned}
& u_{k^n}^k \\
= \;& \frac{1}{\triangle^{\frac{k}{2}}} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \beta^{jk^n} \alpha^{(k-j)k^n} \\
= \;& \frac{1}{\triangle^{\frac{k}{2}}} \left((-1)^{\frac{k}{2}} \binom{k}{k/2} + v_{k^{n+1}} - (-1)^{\frac{k}{2}} \binom{k}{k/2} v_0 + \sum_{j=1}^{\frac{k}{2}} (-1)^j \binom{k}{j} v_{(k-2j)k^n} \right) \\
= \;& \frac{1}{\triangle^{\frac{k}{2}}} \left(v_{k^{n+1}} - (-1)^{\frac{k}{2}} \binom{k}{k/2} + \sum_{j=1}^{\frac{k}{2}} (-1)^j \binom{k}{j} v_{(k-2j)k^n} \right)
\end{aligned}$$

After using by Lemma 1 and reversing the summation order, we get for even k,

$$\begin{aligned} u_{k^n}^k &= \frac{1}{\triangle^{\frac{k}{2}}}\Bigg(v_{k^{n+1}} + (-1)^{\frac{k}{2}}\binom{k}{k/2} \\ &+ \sum_{j=1}^{\frac{k-2}{2}}\sum_{i=0}^{\frac{k}{2}-j}\left((-1)^j \frac{2\left(\frac{k}{2}-j\right)}{\left(\frac{k}{2}-j\right)+i}\binom{k}{j}\binom{\left(\frac{k}{2}-j\right)+i}{2i}\triangle^i u_{k^n}^{2i}\right)\Bigg) \end{aligned}$$

which becomes

$$u_{k^n}^k = \frac{1}{\triangle^{\frac{k}{2}}}\left(v_{k^{n+1}} + \sum_{i=0}^{\frac{k-2}{2}}\sum_{j=1}^{\frac{k}{2}-i}\left((-1)^j \frac{2\left(\frac{k}{2}-j\right)}{\left(\frac{k}{2}-j\right)+i}\binom{k}{j}\binom{\left(\frac{k}{2}-j\right)+i}{2i}\triangle^i u_{k^n}^{2i}\right)\right).$$

If we take $m = \frac{k}{2} - i$ in (3.4) for $1 \le m \le k/2$, the last equation takes the form:

$$u_{k^n}^k = \frac{1}{\triangle^{\frac{k}{2}}}\left(v_{k^{n+1}} + \sum_{i=0}^{\frac{k-2}{2}} D_{i,k}\triangle^i u_{k^n}^{2i}\right)$$

where $D_{i,k}$ is the right hand side of (3.4) for $m = \frac{k}{2} - i$, that is,

$$D_{i,k} = \left(\frac{-2k}{k+2i}\right)\binom{i+\frac{1}{2}k}{2i}.$$

Thus we have the conclusion. $\square$

When $k = 6$, we have that

$$v_{6^{n+1}} = 125u_{6^n}^6 + 150u_{6^n}^4 + 45u_{6^n}^2 + 2. \tag{4.1}$$

Note that the coefficients of the formula in (4.1) with adjusted sings appears to be the terms of the sequence *A104064* in the OEIS.

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