On binomial sums for the general second order linear recurrence

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Abstract

In this short paper we establish identities involving sums of products of binomial coefficients and coefficients that satisfy the general second–order linear recurrence. We obtain generalizations of identities of Carlitz, Prodinger and Haukkanen.

1 Introduction

There are many types of identities involving sums of products of binomial coefficients and Fibonacci or Lucas numbers. For example, we recall that (see [1, 4, 12]):

$$\sum_{k=0}^{n}\binom{n}{k}F_k = F_{2n},\ \sum_{k=0}^{n}\binom{n}{k}F_{4k}=3^nF_{2n}, \tag{1}$$

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k}F_{5k} = 5^nF_{2n},\ \sum_{k=0}^{n}\binom{n}{k}3^{n-k}F_{6k}=8^nF_{2n}, \tag{2}$$

Furthermore many additional sums were given in [2, 8].

As more generalizations of the identities given by (1)-(2), Carlitz [1] derived the following nice result by ordinary generating functions: Let s,t be fixed positive integers such that $s\neq t$,

$$\lambda^nG_{sn+r}=\sum_{k=0}^{n}\binom{n}{k}\mu^kG_{tk+r} \tag{3}$$

if and only if

$$\lambda=(-1)^s\frac{F_t}{F_{t-s}} \text{ and } \mu=(-1)^s\frac{F_s}{F_{t-s}} \tag{4}$$

where G_n is either a Fibonacci or Lucas number.

Clearly for positive integers s and t, $s\neq t$,

$$F_t^nG_{sn+r}=\sum_{k=0}^{n}\binom{n}{k}(-1)^{s(n-k)}F_s^kF_{t-s}^{n-k}G_{tk+r}. \tag{5}$$

By using the exponential generating functions (or egf's, see [3, 5, 6, 11]), Prodinger [11] and Haukkanen [7] obtained the same results as Carlitz [1]. Haukkanen obtained similar results for the Pell and Pell-Lucas numbers.

The egf of a sequence $\{a_n\}$ is defined by

$$\hat{a}(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

The product of the egf's of $\{a_n\}$ and $\{b_n\}$ generates the binomial convolution of $\{a_n\}$ and $\{b_n\}$:

$$\hat{a}(x)\,\hat{b}(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a_{n-k} b_k \right) \frac{x^n}{n!} \tag{6}$$

A special case of (5) can be found in [12]. Here the author obtains this special case by the binomial theorem.

The general recurrence $\{W_n(a, b; p, q)\}$ is defined for $n \geq 2$

$$W_n = pW_{n-1} - qW_{n-2}, \tag{7}$$

where $W_0 = a, W_1 = b$.

We write $W_n = W_n(a, b; p, q)$. Let α and β be the roots of $\lambda^2 - p\lambda + q = 0$, assumed distinct. The Binet form of $\{W_n\}$ is as follows:

$$W_n = A\alpha^n + B\beta^n \tag{8}$$

where $A = \frac{b - a\beta}{\alpha - \beta}$ and $B = \frac{a\alpha - b}{\alpha - \beta}$.

Define $U_n = W_n(0, 1; p, q)$ and $V_n = W_n(2, p; p, q)$. The Binet forms of U_n and V_n are given by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n$$

where $\{U_n\}$ and $\{V_n\}$ are the generalized Fibonacci and Lucas-types sequences, respectively.

For more details and properties related to the sequence $\{W_n\}$, we refer to [9, 10].

In this short paper, we derive generalizations of the results of [1, 11, 7] for the sequence $\{W_n\}$. Further some new applications are also given.

2 The results for the sequence $\{W_n\}$

We recall the following result from [7]:

Lemma 1. *Let λ_1 and λ_2 be distinct complex numbers, and let c_1 and c_2 be nonzero distinct complex numbers. Then*

$$c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} = c_1 e^{\mu_1 x} + c_2 e^{\mu_2 x}$$

if and only if

$$\mu_1 = \lambda_1 \text{ and } \mu_2 = \lambda_2.$$

Lemma 2. *Let λ_1 and λ_2 be distinct complex numbers, and let c be a nonzero complex number. Then*

$$ce^{\lambda_1 x} + ce^{\lambda_2 x} = ce^{\mu_1 x} + ce^{\mu_2 x}$$

if only if either

$$\mu_1 = \lambda_1 \text{ and } \mu_2 = \lambda_2 \text{ or } \mu_1 = \lambda_2 \text{ and } \mu_2 = \lambda_1.$$

For the sequence $\{W_n\}$, we can deduce

$$\hat{W}(x) = Ae^{\alpha x} + Be^{\beta x}.$$

Thus we have the following two cases: $r \neq 0$ and $r = 0$.

Theorem 1. *Let c and d be nonzero integers and, let r be a nonzero integer. Then for $n \geq 0$*

$$W_{cn+r} = \sum_{k=0}^{n} \binom{n}{k} t^{n-k} s^k W_{dk+r} \tag{9}$$

if and only if

$$s = \frac{U_c}{U_d} \text{ and } t = q^c \frac{U_{d-c}}{U_d}.$$

Proof. By the egf's, (9) can be rewritten as

$$A\alpha^r e^{\alpha^c x} + B\beta^r e^{\beta^c x} = e^{tx} \left(A\alpha^r e^{\alpha^d sx} + B\beta^r e^{\beta^d sx} \right) \tag{10}$$

where the right hand side comes from (6). Since $\alpha^r \neq \beta^r$ for $r \neq 0$, by Lemma 1, (10) holds if and only if

$$\alpha^c = \alpha^d s + t \text{ and } \beta^c = \beta^d s + t, \tag{11}$$

and clearly,

$$s = \frac{\alpha^c - \beta^c}{\alpha^d - \beta^d} = \frac{U_c}{U_d} \text{ and } t = \alpha^c - \alpha^d \frac{\alpha^c - \beta^c}{\alpha^d - \beta^d} = q^c \frac{U_{d-c}}{U_d}.$$

Thus the proof is complete. □

Theorem 2. *Let c and d be nonzero integers and $p = 2b/a$. Then for $n \geq 0$*

$$W_{cn} = \sum_{k=0}^{n} \binom{n}{k} t^{n-k} s^k W_{dk} \tag{12}$$

if and only if either (11) *holds or*

$$s = \frac{-U_c}{U_d} \text{ and } t = \frac{U_{d+c}}{U_d}.$$

Proof. In terms of the egf's, (12) could be rewritten as

$$Ae^{\alpha^c x} + Be^{\beta^c x} = e^{tx}\left(Ae^{\alpha^d sx} + Be^{\beta^d sx}\right) \tag{13}$$

where the right side is seen from (6). Since $p = 2b/a$, $A = B$, thus (13) takes the form

$$e^{\alpha^c x} + e^{\beta^c x} = e^{tx}\left(e^{\alpha^d sx} + e^{\beta^d sx}\right). \tag{14}$$

By Lemma 2, (14) holds if and only if either (11) holds or

$$\alpha^c = \beta^d s + t \text{ and } \beta^c = \alpha^d s + t,$$

clearly,

$$s = \frac{\alpha^c - \beta^c}{\beta^d - \alpha^d} = \frac{-U_c}{U_d} \text{ and } t = \alpha^c - \beta^d \frac{\alpha^c - \beta^c}{\beta^d - \alpha^d} = \frac{U_{d+c}}{U_d}.$$

Thus the proof is complete. □

From Theorems 1 and 2, we have the following consequence.

Corollary 3. *If c and d are nonzero integers and r is an integer, then*

$$U_d^n W_{cn+r} = \sum_{k=0}^{n} \binom{n}{k} q^{c(n-k)} U_{d-c}^{n-k} U_c^k W_{dk+r}.$$

If c and d are nonzero integers, then

$$U_d^n W_{cn} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k U_{d+c}^{n-k} U_c^k W_{dk}.$$

We note the following some known special cases of $\{W_n\}$:

p	q	a	b	W_n	
1	-1	0	1	F_n	Fibonacci numbers
1	-1	2	1	L_n	Lucas numbers
2	-1	0	1	P_n	Pell numbers
2	-1	2	2	2	Pell-Lucas numbers
1	-2	0	1	J_n	Jacobsthal numbers
1	-2	2	1	j_n	Jacobsthal-Lucas numbers

Thus we have the following examples:

$$F_d^n F_{cn+r} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{c(n-k)} F_{d-c}^{n-k} F_c^k F_{dk+r}$$

and

$$F_d^n L_{cn} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{d+c}^{n-k} F_c^k L_{dk}$$

which are also given in [1, 11, 7].

Similar to the Fibonacci and Lucas numbers, for the Jacobsthal and Jacobsthal-Lucas sequences, we obtain

$$\begin{aligned} J_d^n J_{cn+r} &= \sum_{k=0}^{n} \binom{n}{k} (-2)^{c(n-k)} J_{d-c}^{n-k} J_c^k J_{dk+r}, \\ J_d^n j_{cn} &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k J_{d+c}^{n-k} J_c^k j_{dk}. \end{aligned}$$

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