# A MATRIX APPROACH FOR GENERAL HIGHER ORDER LINEAR RECURRENCES

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Abstract. We consider $k$ sequences of generalized order-$k$ linear recurrences with arbitrary initial conditions and coefficients, and we give their generalized Binet formulas and generating functions. We also obtain a new matrix method to derive explicit formulas for the sums of terms of the $k$ sequences. Further, some relationships between determinants of certain Hessenberg matrices and the terms of these sequences are obtained.

## 1. Introduction

Linear recurrences have played (and will most certainly play) an important role in many areas of mathematics. A lot of authors have studied various properties of linear recurrences (such as the well-known Fibonacci and Pell sequences).

In [2], Er defined $k$ linear recurring sequences of order at most $k$ as shown: for $n > 0$ and $1 \leq i \leq k$,

$$g_n^i = \sum_{j=1}^{k} g_{n-j}^i$$

with initial conditions

$$g_n^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 - k \leq n \leq 0,$$

where $g_n^i$ is the $n$term of the $i$th generalized order-$k$ Fibonacci sequence.

More generally, in [6], the author gave the generalized order-$k$ Fibonacci and Pell (F-P) sequence as follows: for $m \geq 0,\ n > 0$ and $1 \leq i \leq k$

$$u_n^i = 2^m u_{n-1}^i + u_{n-2}^i + \cdots + u_{n-k}^i$$

with initial conditions

$$u_n^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 - k \leq n \leq 0,$$

where $u_n^i$ is the $n$term of the $i$th generalized order-$k$ F-P sequence.

When $m = 0$, the generalized order-$k$ F-P sequence $\{u_n^i\}$ is reduced to the generalized order-$k$ Fibonacci sequence $\{g_n^i\}$. Also when $m = 1$, the generalized order-$k$ F-P sequence is reduced to the generalized order-$k$ Pell sequence $\{P_n^i\}$ (for more details see [5]).

Define $k$ sequences of $k$-th order linear recurrence relation $\{f_n^i\}$ as shown, for $n > 0$ and $1 \leq i \leq k$

$$f_n^i = c_1 f_{n-1}^i + c_2 f_{n-2}^i + \cdots + c_k f_{n-k}^i \tag{1.1}$$

2000 *Mathematics Subject Classification.* 11B37, 40C05, 15A36, 15A15.
*Key words and phrases.* Higher order recurrence, generating matrix, sum, matrix method.

with initial conditions

$$f_n^i = \begin{cases} 1 & \text{if } n = 1-i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1-k \le n \le 0$$

where $c_j$, $1 \le j \le k$, are real constant coefficients, and $f_n^i$ is the $n$th term of the $i$th sequence. When $k = 2, c_1 = c_2 = 1$, respectively, $k = c_1 = 2$, $c_2 = 1$ the sequence $\{f_n^2\}$ is reduced to the Fibonacci sequence $\{F_n\}$, respectively, the Pell sequence $\{P_n\}$.

Define the $k \times k$ companion matrix $A$ and the matrix $G_n$ as follows:

$$A = \begin{bmatrix} c_1 & c_2 & \dots & c_{k-1} & c_k \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \text{ and } G_n = \begin{bmatrix} f_n^1 & f_n^2 & \dots & f_n^k \\ f_{n-1}^1 & f_{n-1}^2 & \dots & f_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ f_{n-k+1}^1 & f_{n-k+1}^2 & \dots & f_{n-k+1}^k \end{bmatrix} \tag{1.2}$$

Using the approach of Kalman [3], Er [2] showed that

$$G_n = A^n \tag{1.3}$$

and

$$f_{n+1}^i = c_i f_n^1 + f_n^{i+1}, \text{ for } 1 \le i \le k-1 \tag{1.4}$$

$$f_{n+1}^k = c_k f_n^1. \tag{1.5}$$

Matrix methods are helpful and convenient in solving certain problems stemming from linear recursion relations, such as that of finding an explicit expression for the $n$th term of the Fibonacci sequence (see [9]), or of analyzing the vibration of a weighted string [10, pp. 152–154]. Here we will consider a more general case using matrix methods to obtain some explicit formulas for the $n$th term of a general recurrence relation and the sums of terms of the recurrence. The general linear recurrence relations have been considered by many mathematicians (for references, one may see [1, 2, 4, 5]). The authors of [4, 6, 7] give the generalized Binet formula for the generalized order-$k$ Fibonacci, Lucas and Pell numbers by matrix methods.

In this paper, we consider $k$ sequences of general order-$k$ linear recurrences with $k$ arbitrary initial conditions and coefficients. Then we study the properties of $k$ linear recursive sequences and derive many applications to matrices.

## 2. General linear recurrence with $k$ initial conditions

Define a set of $k$ sequences satisfying the generalized order-$k$ linear recurrence $\{t_n^i(r_1, r_2, \dots, r_k)\}$ as shown: for $n > 0$ and $1 \le i \le k$

$$t_n^i = c_1 t_{n-1}^i + c_2 t_{n-1}^i + \dots + c_k t_{n-k}^i$$

with $k$ initial conditions

$$t_n^i = \begin{cases} r_1 & \text{if } n = 1-i, \\ r_2 & \text{if } n = 2-i, \\ \vdots & \vdots \\ r_k & \text{if } n = k-i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1-k \le n \le 0$$

where the coefficients $c_i$ and the initial conditions $r_i$ are arbitrary, for $1 \le i \le k$, and $t_n^i$ is the $n$th term of $i$th sequence. Clearly, $\left\{t_n^i\,(1,0,\ldots,0)\right\} = \left\{f_n^i\right\}$, where $f_n^i$ are given by (1.1).

Next, we define a $k \times k$ matrix $H_n = [h_{ij}]$ by

$$H_n = \begin{bmatrix} t_n^1 & t_n^2 & \cdots & t_n^k \\ t_{n-1}^1 & t_{n-1}^2 & \cdots & t_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ t_{n-k+1}^1 & t_{n-k+1}^2 & \cdots & t_{n-k+1}^k \end{bmatrix}. \tag{2.1}$$

By Kalman's [3] approach, we find that

$$H_n = AH_{n-1} \text{ and so, } \; H_n = A^{n-1}H_1, \tag{2.2}$$

where the matrix $A$ is given by (1.2).

**Theorem 1.** *For $n > 0$,*

$$t_n^i = \sum_{j=1}^{i} r_{i+1-j} f_n^j,$$

*where $f_n^i$ is defined as before.*

*Proof.* From (2.2), we have $H_n = A^{n-1}H_1$. From (2.1) we get

$$H_1 = \begin{bmatrix} t_1^1 & t_1^2 & \cdots & t_1^k \\ t_0^1 & t_0^2 & \cdots & t_0^k \\ \vdots & \vdots & \ddots & \vdots \\ t_{2-k}^1 & t_{2-k}^2 & \cdots & t_{2-k}^k \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{1} c_j r_{2-j} & \sum_{j=1}^{2} c_j r_{3-j} & \cdots & \sum_{j=1}^{k} c_j r_{k+1-j} \\ r_1 & r_2 & \cdots & r_k \\ 0 & r_1 & \cdots & r_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_1 \end{bmatrix},$$

which implies that

$$H_1 = AE, \tag{2.3}$$

where the matrix $E$ is the $k \times k$ upper tridiagonal matrix of the form

$$E = \begin{bmatrix} r_1 & r_2 & r_3 & \cdots & r_k \\ & r_1 & r_2 & \cdots & r_{k-1} \\ & & r_1 & \cdots & r_{k-2} \\ & & & \ddots & \vdots \\ 0 & & & & r_1 \end{bmatrix}.$$

Using Er's approach [2] and (1.3), we obtain $A^n = G_n$. Since $H_n = A^{n-1}H_1$ and $H_1 = AE$, we get

$$H_n = A^n E, \tag{2.4}$$

which can be re-written as

$$t_n^i = \sum_{j=1}^{i} r_{i+1-j} f_n^j, \tag{2.5}$$

and the proof is complete. ∎

Therefore we see that the general recurrence with arbitrary initial conditions can be written as a linear combination of terms of the recurrence $\left\{f_n^i\right\}$. By this result, we can easily derive some properties of the recurrence $\left\{t_n^i\right\}$.

**Corollary 1.** *For* $n \in \mathbb{Z}$,

$$\det \begin{pmatrix} t_n^1 & t_n^2 & \cdots & t_n^k \\ t_{n-1}^1 & t_{n-1}^2 & \cdots & t_{n-1}^k \\ \vdots & \vdots & & \vdots \\ t_{n-k+1}^1 & t_{n-k+1}^2 & \cdots & t_{n-k+1}^k \end{pmatrix} = (-1)^{k+1} c_k r_1^k.$$

*Proof.* Let $H_n, G_n$ and $E$ be the matrices defined in the proof of Theorem 1. It is clear that $\det G_n = (-1)^{k+1} c_k$ and $\det E = r_1^k$. Taking the determinant in $H_n = G_n E$ shows our claim. ∎

Corollary 1 is a vast generalization of the well known Cassini's identity for the Fibonacci numbers, that is, $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1}$.

**Corollary 2.** *Let* $x^k - c_1x^{k-1} - c_2x^{k-2} - \cdots - c_k = (x - \lambda_1)\cdots(x - \lambda_k)$ *and* $e_n = \lambda_1^n + \lambda_2^n + \cdots + \lambda_k^n$. *Then*

$$e_n = \sum_{i=1}^{k} \left( \sum_{m=1}^{i} r_{i+1-m} f_{n+1-t}^m \right).$$

*Proof.* $A$ is the companion matrix from (1.2) and $x^k - c_1x^{k-1} - c_2x^{k-2} - \cdots - c_k$ is its characteristic polynomial, whose roots (also, eigenvalues of $A$) are $\lambda_1, \ldots, \lambda_k$. Thus the eigenvalues of $A^n$ are $\lambda_1^n, \ldots, \lambda_k^n$. Denote the trace of the matrix $W$ by $\operatorname{tr}(W)$. By Theorem 1,

$$\begin{aligned} e_n &= \lambda_1^n + \lambda_2^n + \cdots + \lambda_k^n = \operatorname{tr}(H_n) = \operatorname{tr}(G_n E) \\ &= \sum_{i=1}^{k} \left( \sum_{m=1}^{i} r_{i+1-m} f_{n+1-t}^m \right). \end{aligned}$$

Thus the proof is complete. ∎

## 3. Sums of the terms of recurrence $\{t_n^k\}$

In this section we deal with the sums of the terms of recurrence $\{t_n^k\}$ subscripted from 1 to $n$. By the result of Theorem 1, clearly

$$t_n^k = \sum_{j=1}^{k} r_{k-j+1} f_n^j. \tag{3.1}$$

The characteristic polynomial of both the matrix $A$ and the sequence $\{f_n^k\}$ is $E(x) = x^k - c_1x^{k-1} - c_2x^{k-2} - \cdots - c_{k-1}x - c_k$. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the characteristic roots of the equation.

**Hypothesis 1.** *Throughout this paper, we suppose that the roots* $\lambda_1, \ldots, \lambda_k$ *are distinct (which happens if* $\gcd(E, E') = 1$*) and not equal to 1.*

As special cases, we note that when $c_i = 1$ for $1 \le i \le k$, the equation $x^k - x^{k-1} - \cdots - x - 1 = 0$ does not have multiple roots (see [7]). Also, when $c_1 = 2$ and $c_i = 1$ for $2 \le i \le k$, the equation $x^k - 2x^{k-1} - x^{k-2} - \cdots - x - 1 = 0$ does not have multiple roots (see [5]). For the case $c_1 = 2^m$, $c_i = 1$ for $2 \le i \le k$ and $m \ge 0$, we refer to [6].

Let $V = \Lambda^T$ be a $k \times k$ Vandermonde matrix, where

$$\Lambda = \begin{bmatrix} \lambda_1^{k-1} & \lambda_1^{k-2} & \dots & \lambda_1 & 1 \\ \lambda_2^{k-1} & \lambda_2^{k-2} & \dots & \lambda_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_k^{k-1} & \lambda_k^{k-2} & \dots & \lambda_k & 1 \end{bmatrix}. \tag{3.2}$$

Let $w_k^i$ be the column matrix

$$w_k^i = \begin{bmatrix} \lambda_1^{n+k-i} \\ \lambda_2^{n+k-i} \\ \vdots \\ \lambda_k^{n+k-i} \end{bmatrix}$$

and $\Lambda_j^{(i)}$ be the $k \times k$ matrix obtained from $\Lambda$ by replacing the $j$th column of $\Lambda$ by $w_k^i$.

The generalized Binet formula for the recurrence $\{f_n^i\}$ can be expressed using $V = \Lambda^T$ and $V_j^{(i)} = \Lambda_j^{(i)}$.

**Theorem 2.** *For $n > 0$ and $1 \le i \le k$,*

$$f_{n-i+1}^j = \frac{\det\left(V_j^{(i)}\right)}{\det(V)}.$$

*Proof.* Since the eigenvalues of $A$ are distinct (by our Hypothesis 1), we infer that $A$ is diagonalizable. It is readily seen that $AV = VD$, where $D = diag(\lambda_1, \lambda_2, \dots, \lambda_k)$. Since $V$ is invertible, $V^{-1}AV = D$. Hence, $A$ is similar to $D$. So we obtain $A^n V = VD^n$. Since $A^n = G_n = [g_{ij}]$, we obtain the following linear system of equations:

$$\begin{aligned} g_{i1}\lambda_1^{k-1} + g_{i2}\lambda_1^{k-2} + \dots + g_{ik} &= \lambda_1^{n+k-i} \\ g_{i1}\lambda_2^{k-1} + g_{i2}\lambda_2^{k-2} + \dots + g_{ik} &= \lambda_2^{n+k-i} \\ \vdots \qquad & \qquad \vdots \\ g_{i1}\lambda_k^{k-1} + g_{i2}\lambda_k^{k-2} + \dots + g_{ik} &= \lambda_k^{n+k-i} \end{aligned}$$

Thus, for $j = 1, 2, \dots, k$, we get $g_{ij} = \dfrac{\det\left(\Lambda_j^{(i)}\right)}{\det(\Lambda)}$, where $G_n = [g_{ij}]$ and $g_{ij} = f_{n-i+1}^j$. The proof is complete. ∎

**Corollary 3.** *For $n > 0$, we have* $t_n^i = \dfrac{1}{\det(\Lambda)} \displaystyle\sum_{j=1}^{i} r_{k+1-j} \det\left(\Lambda_j^{(1)}\right).$

For example, when $c_1 = 2$ and $c_i = 1$ for all $2 \le j \le k$, the sequence $\{f_n^i\}$ is reduced to the generalized order-$k$ Pell sequence $\{P_n^i\}$ and so the sums of the generalized order-$k$ Pell numbers is given by

$$\sum_{i=1}^{n} P_i^k = \left(P_n^1 + P_n^2 + \dots + P_n^k - 1\right)/k.$$

When $k=3$, $c_i=1$ for $1\leq i\leq 3$, the sequence $\left\{f_n^i\right\}$ is reduced to the generalized Tribonacci sequence $\left\{T_n^i\right\}$ and so

$$\sum_{i=1}^{n} T_i^3 = \left(T_n^1 + T_n^2 + T_n^3 - 1\right)/2$$

and by the definition of the $\left\{T_n^i\right\}$, we have $T_n^1 = T_{n+1}^3$ and $T_n^2 = T_n^3 + T_{n-1}^3$. For easy writing, we denote $T_n^3$ by $T_n$. Thus we can write

$$\sum_{i=1}^{n} T_i = \left(T_{n+1} + 2T_n + T_{n-1} - 1\right)/2 = \left(T_{n+2} + T_n - 1\right)/2.$$

We expand our matrix method to find all sums of terms of $k$ sequences of generalized order-$k$ recurrences $\left\{f_n^i\right\}$ subscripted 1 to $n$ for all $1\leq i\leq k$.

Define the following two sums: for $1\leq i\leq k$, let $S_n^{(i)} = \sum_{m=1}^{n-1} f_m^i$ and $T_n^{(i)} = \sum_{m=1-i}^{n-i} f_m^i$. Then $T_n^{(i)} = S_{n-i+1}^{(i)} + 1$, since

$$f_n^i = \begin{cases} 1 & \text{if } i = 1-n, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1-k\leq n\leq 0.$$

Further,

$$S_{n+1}^{(i)} = f_n^i + S_n^{(i)} \tag{3.3}$$

$$T_{n+1}^{(i)} = f_{n-i+1}^i + T_n^{(i)} \tag{3.4}$$

We next define two $(k+1)\times(k+1)$ matrices as follows:

$$B_i = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & & \\ 0 & & A & \\ 1 & & & \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \leftarrow (i+1)\,\text{st row} \\ \\ \\ \\ \end{matrix}$$

and

$$Y_{n,i} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ S_n^{(i)} & & & \\ S_{n-1}^{(i)} & & & \\ \vdots & & G_n & \\ S_{n-i+2}^{(i)} & & & \\ T_n^{(i)} & & & \\ T_{n-1}^{(i)} & & & \\ \vdots & & & \\ T_{n-k+i}^{(i)} & & & \end{bmatrix} \begin{matrix} \\ \leftarrow & 1\text{st row} \\ \leftarrow & 2\text{nd row} \\ \vdots & \vdots \\ \leftarrow & (i-1)\,\text{st row} \\ \leftarrow & i\text{th row} \\ \leftarrow & (i+1)\,\text{st row} \\ \vdots & \vdots \\ \leftarrow & k\text{th row} \end{matrix}$$

where the matrices $A$ and $G_n$ were defined before. We have the following result.

**Theorem 3.** *For $n>0$,*

$$Y_{n,i} = B_i^n.$$

*Proof.* Combining the identities (3.3) and (3.4), we obtain

$$Y_{n+1,i} = Y_{n,i}B_i = \cdots = Y_{1,i}B_i^n.$$

From the definitions of $\left\{T_n^{(i)}\right\}$ and $\left\{S_n^{(i)}\right\}$, we can easily check that $Y_{1,i} = B_i$, and the theorem is proven. ∎

Now we are going to derive an explicit expression for every sum $S_n^{(i)}$ for $1 \le i \le k$ by matrix methods.

We first make some observations. If we expand $\det B_i$ with respect to the first row, we get

$$\det B_i = \det A$$

and the characteristic polynomials of $A, B_i$ satisfy

$$C_{B_i}(\lambda) = (1-\lambda)\, C_A(\lambda).$$

Since $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the roots of $C_A(\lambda)$ (distinct and nonequal to 1), the eigenvalues of matrix $B_i$ are $\lambda_1, \lambda_2, \ldots, \lambda_k, 1$. Therefore the eigenvalues of the matrix $B_i$ are distinct, and so $B_i$ is diagonalizable.

For easy writing, let

$$\mu_i = \frac{\sum\limits_{t=i}^{k} c_t}{1 - \sum\limits_{t=1}^{k} c_t} \text{ for } 1 < i \le k \text{ and } \mu_1 = \frac{1}{1 - \sum\limits_{t=1}^{k} c_t}.$$

The following $(k+1)\times(k+1)$ matrix for $1 < i \le k$

$$P = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ \mu_i & \lambda_1^{k-1} & \lambda_2^{k-1} & \ldots & \lambda_k^{k-1} \\ \mu_i & \lambda_1^{k-2} & \lambda_2^{k-2} & \ldots & \lambda_k^{k-2} \\ \vdots & \vdots & \vdots & & \vdots \\ \mu_i & \lambda_1^{k-i+1} & \lambda_2^{k-i+1} & & \lambda_k^{k-i+1} \\ \mu_i+1 & \lambda_1^{k-i} & \lambda_2^{k-i} & & \lambda_k^{k-i} \\ \mu_i+1 & \lambda_1^{k-i-1} & \lambda_2^{k-i-1} & & \lambda_k^{k-i-1} \\ \vdots & \lambda_1 & \lambda_2 & \ldots & \lambda_k \\ \mu_i+1 & 1 & 1 & \ldots & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \; 0 \; \ldots \; 0 \\ \mu_i & \\ \mu_i & \\ \vdots & V \\ \mu_i & \\ \mu_i+1 & \\ \vdots & \\ \mu_i+1 & \end{bmatrix}$$

satisfies $B_iP = PD_1$, where $D_1$ is the $(k+1)\times(k+1)$ diagonal matrix defined previously, $D_1 = diag\,(1, \lambda_1, \lambda_2, \ldots, \lambda_k)$. Here we note that if we expand $\det P$ with respect to the first row, then we get $\det P = \det \Lambda$. Since $\Lambda$ is the Vandermonde matrix, the matrix $P$ is invertible.

**Theorem 4.** *For $n > 0$ and $1 < i < k$,*

$$S_n^{(i)} = \mu_i \left(1 - \sum_{j=1}^{k} f_n^j\right) - \sum_{m=i}^{k} f_n^m$$

*and*

$$S_n^{(1)} = \mu_1 \left(1 - \sum_{j=1}^{k} f_n^j\right).$$

*Proof.* Since $B_iP = PD_1$ for $1 < i \leq k$ and the matrix $P$ is invertible, we write $B_i^nP = PD_1^n$ and so $Y_{n,i}P = PD_1^n$. By equating the $(2,1)$ entries of the equality $Y_{n,i}P = PD_1^n$, we have the conclusion.

For the case $i = 1$, one can see that $BP_1 = P_1D_1$ where the $(k+1)\times(k+1)$ matrices $B$ and $P_1$ are as follows

$$B = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & & & \\ 0 & & A & \\ \vdots & & & \\ 0 & & & \end{bmatrix} \text{ and } P_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \mu_1 & & & \\ \vdots & & V & \\ \mu_1 & & & \end{bmatrix}.$$

By induction on $n$, we see that

$$Y = B^n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ S_n^{(i)} & & & \\ S_{n-1}^{(i)} & & G_n & \\ \vdots & & & \\ S_{n-k+1}^{(i)} & & & \end{bmatrix}.$$

Similar to the cases $1 < i \leq k$, the proof is easily seen for the case $i = 1$. ∎

As a consequence of Theorem 4, we get

$$S_n = \sum_{i=1}^{n} f_i^k = \frac{c_k \left( \sum_{j=1}^{k} f_n^j - 1 \right)}{c_1 + c_2 + \cdots + c_k - 1}.$$

Let $V_{i,j}$ be a $k \times k$ matrix obtained from the Vandermonde matrix $V$ by replacing the $j$th column of $V$ by $e_i$ where $V = \Lambda^T$ is defined as in (3.2) and $e_i$ is the $i$th element of the natural basis for $\mathbb{R}^n$, that is,

$$e_i = (0, \dots, 0, \underset{\substack{\uparrow \\ i\text{th}}}{1}, 0, \dots 0)^T$$

and

$$V_{i,j} = \begin{bmatrix} \lambda_1^{k-1} & \dots & \lambda_{j-1}^{k-1} & 0 & \lambda_{j+1}^{k-1} & \dots & \lambda_k^{k-1} \\ \lambda_1^{k-2} & \dots & \lambda_{j-1}^{k-2} & 0 & \lambda_{j+1}^{k-2} & \dots & \lambda_k^{k-2} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \lambda_1^{k-i+1} & \dots & \lambda_{j-1}^{k-i+1} & 0 & \lambda_{j+1}^{k-i+1} & \dots & \lambda_k^{k-i+1} \\ \lambda_1^{k-i} & \dots & \lambda_{j-1}^{k-i} & 1 & \lambda_{j+1}^{k-i} & \dots & \lambda_k^{k-i} \\ \lambda_1^{k-i-1} & \dots & \lambda_{j-1}^{k-i-1} & 0 & \lambda_{j+1}^{k-i-1} & \dots & \lambda_k^{k-i-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \lambda_1 & \dots & \lambda_{j-1} & 0 & \lambda_{j+1} & \dots & \lambda_k \\ 1 & \dots & 1 & \underset{\substack{\uparrow \\ e_i}}{0} & 1 & \dots & 1 \end{bmatrix}.$$

Let $q_j^{(i)} = \frac{|V_{i,j}|}{|V|}$.

**Theorem 5.** *For any integer $n$ and $1 \leq i \leq k$,*

$$f_n^i = \sum_{j=1}^{k} q_j^{(i)} \lambda_j^{n+k-1}.$$

*Proof.* We consider the following system of $k$ linear equations in $k$ unknowns $x_1, x_2, \ldots, x_k$:

$$\begin{bmatrix} \lambda_1^{k-1} & \lambda_2^{k-1} & \ldots & \lambda_k^{k-1} \\ \vdots & \vdots & & \vdots \\ \lambda_1^{k-i} & \lambda_2^{k-i} & \ldots & \lambda_k^{k-i} \\ \vdots & \vdots & & \vdots \\ \lambda_1 & \lambda_2 & \ldots & \lambda_k \\ 1 & 1 & \ldots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_k \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}}_{e_i}.$$

Using Vandermonde's determinants and Cramer rule, we get

$$q_j^{(i)} = \frac{|V_{i,j}|}{|V|} \ (i = 1, 2, \ldots, k),$$

and so, for $n, k > 0$ and $1 \leq i \leq k$, $f_n^i = \sum_{j=1}^{k} q_j^{(i)} \lambda_j^{n+k-1}$, which completes the proof. ∎

Consequently, we extend the result of Theorem 5 to the general order linear recurrences $\{t_n^i\}$ by the result given by (2.5).

**Corollary 4.** *For any integer $n$ and $1 \leq i \leq k$,*

$$t_n^i = \sum_{j=1}^{i} \sum_{s=1}^{k} r_{i+1-j} q_s^{(j)} \lambda_s^{n+k-1}.$$

As an example, we consider the sequence $\{T_n^i\}$,

$$T_n^i = T_{n-1}^i + 3T_{n-2}^i + T_{n-2}^i, n \geq 2, 1 \leq i \leq 3$$

with

$$T_n^i = \begin{cases} 1 & \text{if } i = 1 - n, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 - k \leq n \leq 0,$$

displayed in the following table

| $i \backslash n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 4 | 8 | 21 | 49 | 120 | 288 | 697 | … | $\{T_n^1\}$ |
| 2 | 3 | 4 | 13 | 28 | 71 | 168 | 409 | 984 | … | $\{T_n^2\}$ |
| 3 | 1 | 1 | 4 | 8 | 21 | 49 | 120 | 288 | … | $\{T_n^3\}$ |

Table 1

Here we note that $\gamma_1 = -1,\ \gamma_2 = 1 + \sqrt{2}, \gamma_3 = 1 - \sqrt{2}$ and

$$q_1^{(1)} = \frac{1}{(\gamma_1-\gamma_3)(\gamma_1-\gamma_2)},\ q_2^{(1)} = \frac{1}{(\gamma_2-\gamma_3)(\gamma_2-\gamma_1)},\ q_3^{(1)} = \frac{1}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)},$$

$$q_1^{(2)} = -\frac{\gamma_2+\gamma_3}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)},\ q_2^{(2)} = \frac{\gamma_1+\gamma_3}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_2)},\ q_3^{(2)} = -\frac{\gamma_1+\gamma_2}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)},$$

$$q_1^{(3)} = \frac{\gamma_2\gamma_3}{(\gamma_1-\gamma_3)(\gamma_1-\gamma_2)},\ q_2^{(3)} = -\frac{\gamma_1\gamma_3}{(\gamma_1-\gamma_2)(\gamma_2-\gamma_3)},\ q_3^{(3)} = \frac{\gamma_1\gamma_2}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)}.$$

Therefore, by Theorem 5, we get

$$T_n^1 = \frac{\gamma_1^{n+2}}{(\gamma_1-\gamma_3)(\gamma_1-\gamma_2)} + \frac{\gamma_2^{n+2}}{(\gamma_2-\gamma_3)(\gamma_2-\gamma_1)} + \frac{\gamma_3^{n+2}}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)},$$

$$T_n^2 = -\frac{(\gamma_2+\gamma_3)\gamma_1^{n+2}}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)} + \frac{(\gamma_1+\gamma_3)\gamma_2^{n+2}}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_2)} - \frac{(\gamma_1+\gamma_2)\gamma_3^{n+2}}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)}$$

and since $\gamma_1\gamma_2\gamma_3 = 1$,

$$\begin{aligned} T_n^3 &= \frac{\gamma_2\gamma_3\gamma_1^{n+2}}{(\gamma_1-\gamma_3)(\gamma_1-\gamma_2)} - \frac{\gamma_1\gamma_3\gamma_2^{n+2}}{(\gamma_1-\gamma_2)(\gamma_2-\gamma_3)} + \frac{\gamma_1\gamma_2\gamma_3^{n+2}}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)} \\ &= \frac{\gamma_1^{n+1}}{(\gamma_1-\gamma_3)(\gamma_1-\gamma_2)} + \frac{\gamma_2^{n+1}}{(\gamma_2-\gamma_1)(\gamma_2-\gamma_3)} + \frac{\gamma_3^{n+1}}{(\gamma_2-\gamma_3)(\gamma_1-\gamma_3)} \\ &= T_{n-1}^1. \end{aligned}$$

Observe (from Table 1) that $T_n^3 = T_{n-1}^1$.

## 4. Generating Functions

In this section we derive the family of generating functions $G(i,x) = \sum_{n=0}^{\infty} f_n^i x^n$ for the generalized order-$k$ recurrences $\{f_n^i\}$ for all $i,\ 1 \le i \le k$.

**Theorem 6.** *For* $1 \le i \le k$,

$$G(i,x) = \frac{f_0^i + \sum_{m=1}^{k-1}\left(\sum_{v=m+1}^{k} c_v f_{m-v}^i\right) x^m}{1 - c_1 x - c_2 x^2 - \cdots - c_k x^k}.$$

*Proof.* Let $G(i,x) = f_0^i x^0 + f_1^i x^1 + f_2^i x^2 + \cdots + f_n^i x^n + \cdots$. Consider

$$\begin{aligned} &\left(1 - c_1 x - c_2 x^2 - \cdots - c_k x^k\right) G(i,x) \\ &= f_0^i + f_1^i x + f_2^i x^2 + \cdots + f_k^i x^k + \cdots + f_n^i x^n + \cdots \\ &- c_1 f_0^i x - c_1 f_1^i x^2 - c_1 f_2^i x^3 - \cdots - c_1 f_{k-1}^i x^k - \cdots - c_1 f_{n-1}^i x^n - \cdots \\ &- c_k f_0^i x^k - c_k f_1^i x^{k+1} - c_k f_2^i x^{k+2} - \cdots - c_k f_{n-k}^i x^n - \cdots \\ &= f_0^i + \left(f_1^i - c_1 f_0^i\right) x + \left(f_2^i - c_1 f_1^i - c_2 f_0^i\right) x^2 + \cdots + \\ &\left(f_{k-1}^i - c_1 f_{k-2}^i - c_2 f_{k-3}^i - \cdots - c_{k-1} f_0^i\right) x^{k-1} \\ &+ \left(f_k^i - c_1 f_{k-1}^i - c_2 f_{k-2}^i - \cdots - c_{k-1} f_0^i - c_k f_1^i\right) x^k + \cdots + \\ &\left(f_n^i - c_1 f_{n-1}^i - c_2 f_{n-2}^i - \cdots - c_k f_{n-k}^i\right) x^n + \cdots . \end{aligned}$$

Now we compute the coefficients of $x^n$ of the equation above. From the definition of $\{f_n^i\}$, we get

$$\begin{aligned} f_1^i &= c_1 f_0^i + c_2 f_{-1}^i + \cdots + c_k f_{1-k}^i \\ &\ \vdots \\ f_{k-1}^i &= c_1 f_{k-2}^i + c_2 f_{k-3}^i + \cdots + c_{k-1} f_0^i + c_k f_{-1}^i \\ &\ \vdots \\ f_n^i &= c_1 f_{n-1}^i + c_2 f_{n-2}^i + \cdots + c_k f_{n-k}^i. \end{aligned}$$

and so

$$\begin{aligned} f_1^i - c_1 f_0^i &= c_2 f_{-1}^i + \cdots + c_k f_{1-k}^i \\ f_2^i - c_1 f_1^i - c_2 f_0^i &= c_3 f_{-1}^i + \cdots + c_k f_{2-k}^i \\ &\vdots \\ f_{k-1}^i - c_1 f_{k-2}^i - c_2 f_{k-3}^i - \cdots - c_{k-1} f_0^i &= c_k f_{-1}^i. \end{aligned}$$

Then for $n \geq k$, by the definition of $\left\{f_n^i\right\}$, the coefficients of $x^n$ are all 0. ∎

For example, for fixed $k$ and $1 \leq i \leq k$, we take $i = 1$. Thus

$$G(1, x) = f_0^1 x^0 + f_1^1 x^1 + f_2^1 x^2 + \cdots + f_n^1 x^n + \cdots .$$

From the definition of $\left\{f_n^i\right\}$, the initial conditions of the recurrence $\left\{f_n^1\right\}$ are given by

$$f_n^1 = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 - k \leq n \leq 0,$$

which implies

$$G(1, x) = \frac{1}{1 - c_1 x - c_2 x^2 - \cdots - c_k x^k}. \tag{4.1}$$

More generally, we derive the generating function of recurrence $\left\{t_n^i\right\}$, namely $g(i, x) = \sum_{k \geq 0} t_k^i x^k$.

**Corollary 5.** *For* $1 \leq i \leq k$,

$$g(i, x) = \frac{t_0^i + \sum_{m=1}^{k-1} \left( \sum_{v=m+1}^{k} c_v t_{m-v}^i \right) x^m}{1 - c_1 x - c_2 x^2 - \cdots - c_k x^k}.$$

As an example, if we take $k = i = 2$, $c_1 = c_2 = 1$ and $r_1 = -1, r_2 = 0$, then the sequence $\left\{t_n^2\right\}$ is

$$1, 3, 4, 7, 11, 18, 29, \ldots$$

which is the well known Lucas sequence $\{L_n\}$. Then by Corollary 5, we obtain

$$g(2, x) = \sum_{n=0}^{\infty} t_n^2 x^n = \sum_{n=0}^{\infty} L_n x^n = \frac{t_0^i - \left(t_{-1}^i\right) x^1}{1 - x - x^2}$$

where $t_0^2 = r_2 = 2$ and $t_{-1}^2 = r_1 = 1$. Thus we have the well known result for the Lucas numbers:

$$\sum_{n=0}^{\infty} L_n x^n = \frac{2 - x}{1 - x - x^2}.$$

## 5. $n$th powers of a companion and $k$-superdiagonal determinants

In [8], the author gave a relationship between determinants of certain $n \times n$ $k$-superdiagonal matrices and the terms of the $n$th power of matrix $A$ given by (1.2). In this section, we derive some new relationships between some Hessenberg determinants and the terms of generalized recurrences $\left\{f_n^i\right\}$ for all $1 \leq i \leq k$.

Here, we recall a result of [8]. Define an $n\times n$ $k$-superdiagonal matrix $M_n$ in the following form:

$$M_n=\begin{bmatrix} c_1 & c_2 & \dots & c_k & & & 0\\ -1 & c_1 & c_2 & \dots & c_k & & \\ & -1 & c_1 & c_2 & \dots & \ddots & \\ & & & & \ddots & \dots & \vdots\\ 0 & & & & & -1 & c_1 \end{bmatrix}.$$

**Lemma 1.** *For* $n>0$,

$$\det M_n = f_n^1.$$

Indeed, expanding $\det M_n$ by the elements of the first row gives us

$$\begin{aligned} \det M_n &= c_1\det M_{n-1}+c_2\det M_{n-2}+\cdots+c_k\det M_{n-k}, && (5.1)\\ &= f_n^1 = c_1 f_{n-1}^1 + c_2 f_{n-2}^1+\cdots+c_k f_{n-k}^1. && (5.2)\end{aligned}$$

Now we extend the above result for the generalized sequences $\{f_n^i\}$ for $1\leq i\leq k$. For this purpose we introduce some new notations: For $1\leq t\leq k$, let $M_n(t,t+1,\dots,k;r)=[\hat{m}_{ij}]$ denote the matrix obtained from $M_n=[m_{ij}]$ with $\hat{m}_{ij}=0$ for $i\leq j\leq r$, $i\in\{t,t+1,\dots,k\}$ and otherwise $\hat{m}_{ij}=m_{ij}$. Clearly $M_n(1,2,\dots,k;0)=M_n$.

Recalling that $G_n=[g_{ij}]=A^n$, we give the following theorem for the diagonal elements $g_{jj}=f_{n-j}^{(j+1)}$.

**Theorem 7.** *For* $n>j$ *and* $1\leq j\leq k-1$,

$$\det M_n(1;j)=f_{n-j}^{j+1}$$

*where* $\det M_n(1;0)=f_n^1$.

*Proof.* First consider the case $j=1$. If we expand the $\det M_n(1;1)$ by the elements of the first row, then

$$\begin{aligned}\det M_n(1;1) &= 0\,(\det M_{n-1})+c_2\det M_{n-2}+\cdots+c_k\det M_{n-k}\\ &= c_2\det M_{n-2}+\cdots+c_k\det M_{n-k}.\end{aligned}$$

By (5.1) and (5.2),

$$\begin{aligned}\det M_n(1;1) =& c_2 f_{n-2}^1+c_3 f_{n-3}^1+\cdots+c_k f_{n-k}^1\\ =& f_n^1-c_1 f_{n-1}^1 = f_{n-1}^2.\end{aligned}$$

Thus the proof is complete for the case $j=1$.

Now, we take the general case for $1\leq j\leq k-1$. By expanding $\det M_n(1;j)$ with respect to the first row, we get

$$\det M_n(1;j)=\det\begin{bmatrix}0 & \dots & 0 & c_{j+1} & c_{j+2} & \dots & c_k & 0 & \dots & 0\end{bmatrix},$$

which, by (5.1) and (5.2), becomes

$$\begin{aligned}\det M_n(1;j) =& c_{j+1}\det M_{n-j-1}+c_{j+2}\det M_{n-j-2}+\cdots+c_k\det M_{n-k}\\ =& c_{j+1}f_{n-j-1}^1+c_{j+2}f_{n-j-2}^1+\cdots+c_k f_{n-k}^1.\end{aligned}$$

From (5.2) and after repeating $j$ times the identity (1.4), we get

$$\begin{aligned}\det M_n(1;j) =& c_{j+1}f^1_{n-j-1} + c_{j+2}f^1_{n-j-2} + \cdots + c_k f^1_{n-k} \\ =& f^1_n - c_1 f^1_{n-1} - c_2 f^1_{n-2} - \cdots - c_j f^1_{n-j} \\ =& f^2_{n-1} - c_2 f^1_{n-2} - c_3 f^1_{n-3} - \cdots - c_j f^1_{n-j} \\ & \ldots \\ =& f^j_{n-j+1} - c_j f^1_{n-j} = f^{j+1}_{n-j},\end{aligned}$$

and the proof is complete. ∎

According to the definition of $M_n(t, t+1, \ldots, k; r)$, the matrix $M_n(2,3;n)$ can be expressed in the compact form

$$M_n(2,3;n) = \begin{bmatrix} c_1 & c_2 & \ldots & c_k & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ -1 & 0 & 0 & \ldots & 0 & 0 & \ldots & \ldots & \ldots & 0 \\ & -1 & 0 & 0 & \ldots & 0 & 0 & \ldots & \ldots & 0 \\ & & -1 & c_1 & c_2 & \ldots & c_k & 0 & \ldots & 0 \\ & & & \ddots & \ddots & \ddots & \ldots & \ddots & \ddots & \vdots \\ & & & & -1 & c_1 & c_2 & \ldots & c_k & 0 \\ & & & & & -1 & c_1 & c_2 & \ldots & c_k \\ & & & & & & \ddots & \ddots & \ldots & \vdots \\ & & & & & & & -1 & c_1 & c_2 \\ 0 & & & & & & & & -1 & c_1 \end{bmatrix}.$$

**Theorem 8.** *For $n > k + 2$,*

$$\det M_{n+1}(2, 3, \ldots, k; n) = f^k_{n-k+2}.$$

*Proof.* First we consider the case of $k = 2$, and $\det M_{n+1}(2;n)$. The matrix $M_n(2;n)$ has the following form:

$$M_n(2;n) = \begin{bmatrix} c_1 & c_2 & \ldots & c_k & 0 & \ldots & \ldots & \ldots & 0 \\ -1 & 0 & 0 & \ldots & 0 & 0 & \ldots & \ldots & 0 \\ & -1 & c_1 & c_2 & \ldots & c_k & 0 & \ldots & 0 \\ & & \ddots & \ddots & \ddots & \ldots & \ddots & \ddots & \vdots \\ & & & -1 & c_1 & c_2 & \cdots & c_k & 0 \\ & & & & -1 & c_1 & c_2 & \ldots & c_k \\ & & & & & \ddots & \ddots & \ldots & \vdots \\ & & & & & & -1 & c_1 & c_2 \\ & & & & & & & -1 & c_1 \end{bmatrix}.$$

Expanding $\det M_{n+1}(2;n)$ with respect to the first row, we obtain

$$\det M_{n+1}(2;n) = c_2 \det M_{n-1} + c_3 \det M_{n-2} + \cdots + c_k \det M_{n-k+1}.$$

Since the first principal subdeterminant include a zero row, by Lemma 1, we write

$$\begin{aligned}\det M_{n+1}(2;n) =& c_2 f^1_{n-1} + c_3 f^1_{n-3} + \cdots + c_k f^1_{n-k+1} \\ =& -c_1 f^1_n + c_1 f^1_n + c_2 f^1_{n-1} + c_3 f^1_{n-3} + \cdots + c_k f^1_{n-k+1} \\ =& -c_1 f^1_n + f^1_{n+1}.\end{aligned}$$

By (1.4), we obtain $\det M_{n+1}(2;n) = -c_1 f_n^1 + f_{n+1}^1 = f_n^2$. Thus the proof is complete for $k = 2$.

Continuing this expanding process with respect to the first row for the $\det M_{n+1}(2,3,\ldots,k;n)$, for $j \geq 2$, we get

$$\det M_{n+1}(2,3,\ldots,j;n) = c_j \det M_{n-j+1} + c_{j+1} \det M_{n-j} + \cdots + c_k \det M_{n-k+1}$$

which, by Lemma 1, gives

$$\begin{aligned}
&\det M_{n+1}(2,3,\ldots,j;n) = c_j f_{n-j+1}^1 + c_3 f_{n-j}^1 + \cdots + c_k f_{n-k+1}^1 \\
&= \left(c_1 f_n^1 + c_2 f_{n-1}^1 + \cdots + c_{j-1} f_{n-j+2}^1\right) - \left(c_1 f_n^1 + c_2 f_{n-1}^1 + \cdots + c_{j-1} f_{n-j+2}^1\right) \\
&+ c_j f_{n-j+1}^1 + c_3 f_{n-j}^1 + \cdots + c_k f_{n-k+1}^1 \\
&= f_{n+1}^1 - \left(c_1 f_n^1 + c_2 f_{n-1}^1 + \cdots + c_{j-1} f_{n-j+2}^1\right).
\end{aligned}$$

By (1.4), we obtain

$$\begin{aligned}
\det M_{n+1}(2,3,\ldots,j;n) &= f_{n+1}^1 - c_1 f_n^1 - c_2 f_{n-1}^1 - \cdots - c_{j-1} f_{n-j+2}^1 \\
&= f_n^2 - c_2 f_{n-1}^1 - \cdots - c_{j-1} f_{n-j+2}^1 \\
&\vdots \\
&= f_{n-j+3}^{j-1} - c_{j-1} f_{n-j+2}^1 = f_{n-j+2}^j,
\end{aligned}$$

and the proof is complete. ∎

Now we present further relations including other entries of $G_n$ and the determinant of certain matrices.

Define the $n \times n$ matrix $M_n(c_{i,k})$ in the compact form:

$$M_n(c_{i,k}) = \begin{bmatrix} c_i & c_{i+1} & \ldots & c_k & 0 & \cdots & 0 \\ -1 & & & & & & \\ 0 & & M_{n-1} & & & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{bmatrix}$$

where $M_n$ is defined as before.

For $2 \leq t \leq r$, let $M_n(c_{i,k}, t, t+1, \ldots, r) = [\check{m}_{ij}]$ denote the $n \times n$ matrix obtained from $M_n(c_{i,k}) = [\tilde{m}_{ij}]$ with taking $\check{m}_{ij} = 0$ for $i \leq j \leq r,\ i \in \{t, t+1, \ldots, n\}$ and otherwise $\check{m}_{ij} = \tilde{m}_{ij}$.

For example, $M_7(c_{2,4}, 3, 4)$ takes the form:

$$M_7(c_{2,4}, 3, 4) = \begin{bmatrix} c_2 & c_3 & c_4 & 0 & 0 & 0 & 0 \\ -1 & c_1 & c_2 & c_3 & c_3 & c_4 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & c_1 & c_2 & c_3 \\ 0 & 0 & 0 & 0 & -1 & c_1 & c_2 \\ 0 & 0 & 0 & 0 & 0 & -1 & c_1 \end{bmatrix}.$$

**Theorem 9.** *For $n > j-1,\ 2 \leq r \leq k-1$ and $2 \leq j \leq k$*

$$\det M_n(c_{r,k}, 2, 3, \ldots, j) = g_{j-1,j+r-1} = f_{n-j+1}^{j+r-1}$$

*where $G_n = [g_{ij}]$.*

*Proof.* First we prove the case $r = 2$ and $2 \leq j \leq k$. If we expand $\det M_n (c_{2,k}, 2, 3, \ldots, j)$ by the Laplace expansion of determinant, then we obtain the following equation by combining (5.1) and (5.2)

$$\begin{aligned}&\det M_n (c_{2,k}, 2, 3, \ldots, j)\\ &= c_{j+1} \det M_{n-j} + c_{j+2} \det M_{n-j-1} + \cdots + c_k \det M_{n-k+1}\\ &= c_{j+1} f^1_{n-j} + c_{j+2} f^1_{n-j-1} + \cdots + c_k f^1_{n-k+1}.\end{aligned}$$

By adding and subtracting $c_1 f^1_n + c_2 f^1_{n-1} + \cdots + c_j f^1_{n-j+1}$ to both sides of the above equation, we get

$$\begin{aligned}&\det M_n (c_{2,k}, 2, 3, \ldots, j)\\ &= \left(c_1 f^1_n + \cdots + c_j f^1_{n-j+1}\right) + c_{j+1} f^1_{n-j} + \cdots + c_k f^1_{n-k+1}\\ &\qquad - \left(c_1 f^1_n + c_2 f^1_{n-1} + \cdots + c_j f^1_{n-j+1}\right)\\ &= f^1_{n+1} - c_1 f^1_n - c_2 f^1_{n-1} - \cdots - c_j f^1_{n-j+1}.\end{aligned}$$

By (1.4), we get

$$\begin{aligned}\det M_n (c_{2,k}, 2, 3, \ldots, j) &= f^2_n - c_2 f^1_{n-1} - c_3 f^1_{n-2} - \cdots - c_j f^1_{n-j+1}\\ &= f^3_{n-1} - c_3 f^1_{n-2} - \cdots - c_j f^1_{n-j+1}\\ &\ \ \vdots\\ &= f^j_{n-j+2} - c_j f^1_{n-j+1} = f^{j+1}_{n-j+1}.\end{aligned}$$

Thus the proof is complete for $r = 2$.

Now we consider the case $r > 2$. If $j$ is greater than $k - 2$, then the matrix $M_n (c_{r,k}, 2, 3, \ldots, j)$ has a zero row and so we ignore this case. For $r > 2$ and $j \leq k - 2$, we obtain, by (1.4), (5.1) and (5.2)

$$\begin{aligned}&\det M_n (c_{r,k}, 2, 3, \ldots, j)\\ &= c_{r+j-1} \det M_{n-j} + \cdots + c_k \det M_{n-k+1}\\ &= c_{r+j-1} f^1_{n-j} + c_{r+j} f^1_{n-j-1} + \cdots + c_k f^1_{n-k+1}\\ &= f^1_{n+1} - c_1 f^1_n - c_2 f^1_{n-1} - \cdots - c_{r+j-2} f^1_{n-j+1}\\ &= f^2_n - c_2 f^1_{n-1} - \cdots - c_{r+j-2} f^1_{n-j+1}\\ &\ \ \vdots\\ &= f^{r+j-2}_{n-j+2} - c_{r+j-2} f^1_{n-j+1} = f^{r+j-1}_{n-j+1},\end{aligned}$$

which completes the proof for all cases. ∎

**Acknowledgements.** The authors would like to thank the referees for constructive comments, which improved the presentation of the paper.

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