

BINOMIAL IDENTITIES INVOLVING THE GENERALIZED FIBONACCI TYPE POLYNOMIALS

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ABSTRACT. We present some binomial identities for sums of the bivariate Fibonacci polynomials and for weighted sums of the usual Fibonacci polynomials with indices in arithmetic progression.

1. INTRODUCTION

The Fibonacci polynomials $F_n(x)$ and Lucas polynomials $L_n(x)$ are satisfied the recursion

$$u_{n+1}(x) = xu_n(x) + u_{n-1}(x)$$

with $F_0(x) = 0$, $F_1(x) = 1$, and, $L_0(x) = 2$, $L_1(x) = x$, for $n > 0$.

The bivariate Fibonacci $\{F_n(x, y)\}$ and Lucas $\{L_n(x, y)\}$ polynomial sequences are satisfied the recursion

$$U_{n+1}(x, y) = xU_n(x, y) + yU_{n-1}(x, y)$$

with $F_0(x, y) = 0$, $F_1(x, y) = 1$ and $L_0(x, y) = 2$, $L_1(x, y) = x$, for $n > 0$.

The sequences $\{F_n(x, y)\}$ and $\{L_n(x, y)\}$ can be expressed as

$$F_n(x, y) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n(x, y) = \alpha^n + \beta^n,$$

where α and β are roots of $\lambda^2 - x\lambda - y = 0$.

When $x = y = 1$, $L_n(1, 1) = l_n$ (n th Lucas number) and $F_n(1, 1) = f_n$ (n th Fibonacci number). General background material on Fibonacci type polynomials can be found in [2, 4].

In [3], the authors obtained various results for these polynomials sequences. Thus we recall that for any positive integer r , we have

$$F_{r(n+1)}(x, y) = L_r(x, y)F_{rn}(x, y) - z^r F_{r(n-1)}(x, y) \tag{1.1}$$

where $L_r(x, y) = \alpha^r + \beta^r$, $z^r = (\alpha\beta)^r = (-y)^r$.

In this paper we use F_n and L_n instead of $F_n(x, y)$ and $L_n(x, y)$, respectively.

Let $H(t, x, y) = \sum_{n=0}^{\infty} F_{r(n+1)}t^n$. Clearly,

$$H(t, x, y) = \sum_{n=0}^{\infty} F_{r(n+1)}t^n = \frac{F_r}{1 - L_r t + z^r t^2}. \tag{1.2}$$

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and further

$$\begin{aligned}
\frac{1}{1 - L_r t + z^r t^2} &= \sum_{m=0}^{\infty} (L_r - z^r t)^m t^m \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^m \binom{m}{n} (-z^r)^n L_r^{m-n} t^{m+n} \\
&= \sum_{m=0}^{\infty} \sum_{n=m}^{2m} \binom{m}{n-m} L_r^{2m-n} (-z^r)^{n-m} t^n \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n-m}{m} L_r^{n-2m} (-z^r)^m t^n. \tag{1.3}
\end{aligned}$$

By (1.2-1.3), clearly

$$F_{r(n+1)} = F_r \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n-m}{m} L_r^{n-2m} y^{rm} (-1)^{m(r+1)}. \tag{1.4}$$

The following sums can be found in [10]:

$$\sum_{a_1+a_2+\dots+a_k=n} \prod_{l=1}^k F_{a_l+1}(x) = \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n+k-1-m}{m} \binom{n+k-1-2m}{k-1} x^{n-2m}.$$

In [11], the author generalized the sums above as for $k, m > 0$ and $n \geq 0$,

$$\sum_{a_1+a_2+\dots+a_{k+1}=n} f_{m(a_1+1)} \cdots f_{m(a_{k+1}+1)} = (-i)^{mn} \frac{f_m^{k+1}}{2^k k!} U_{n+k}^{(k)} \left(\frac{i^m}{2} l_m \right).$$

where $U_n^{(k)}(x)$ denotes the k th derivation of the Chebyshev polynomials of second kind $U_n(x)$ with respect to x .

Also in [9], the authors gave a generalization of (??) as

$$\sum_{a_1+a_2+\dots+a_k=n} \prod_{l=1}^k F_{a_l+1}(x, y) = \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n+k-1-m}{m} \binom{n+k-1-2m}{k-1} x^{n-2m} y^m.$$

In similar meaning, some authors also derived various identities including Fibonacci and Lucas numbers (see [5, 8, 12]). In this paper, we consider both the Fibonacci polynomials and the bivariate generalized Fibonacci polynomials with indices in arithmetic progress. We generalize the results of [9].

2. SUMS OF BIVARIATE GENERALIZED FIBONACCI POLYNOMIALS

Theorem 1. For $k, n, r > 0$,

$$\begin{aligned}
&\sum_{a_1+\dots+a_k=n} \prod_{l=1}^k F_{r(a_l+1)} \\
&= F_r^k \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n+k-1-m}{m} \binom{n+k-1-2m}{k-1} L_r^{(n-2m)} y^{rm} (-1)^{m(r+1)}
\end{aligned}$$

where $a_l \geq 0$, for $l = 1, 2, \dots, k$.

Proof. Denote the l_1 th derivation of $H(t, x, y)$ according to the bivariate Lucas polynomials L_r , and, the l_2 th derivation of $H(t, x, y)$ according to $(-z)$ by $\frac{\partial^{l_1+l_2} H(t, x, y)}{\partial L_r^{l_1} \partial (-z)^{l_2}}$. By (1.2) and induction, we write

$$\begin{aligned} \frac{\partial^{2l+1} H(t, x, y)}{\partial L_r^{l+1} \partial (-z^r)^l} &= \frac{\partial^{2l+1}}{\partial L_r^{l+1} \partial (-z^r)^l} \left(\frac{F_r}{1 - L_r t + z^r t^2} \right) \\ &= \frac{F_r (2l+1)! t^{3l+1}}{(1 - L_r t + z^r t^2)^{2l+2}} \\ &= F_r \sum_{n=0}^{\infty} F_{r(n+1)}^{(l+1, l)} t^n \\ &= F_r \sum_{n=l}^{\infty} F_{r(n+2l+2)}^{(l+1, l)} t^{n+2l+1} \\ &= F_r \sum_{j=0}^{\infty} F_{r(j+3l+2)}^{(l+1, l)} t^{j+3l+1}. \end{aligned}$$

And

$$\begin{aligned} \frac{\partial^{2l} H(t, x, y)}{\partial L_r^l \partial (-z^r)^l} &= \frac{\partial^{2l}}{\partial L_r^l \partial (-z^r)^l} \left(\frac{F_r}{1 - L_r t + z^r t^2} \right) \\ &= \frac{F_r (2l+1)! t^{3l}}{(1 - L_r t + z^r t^2)^{2l+1}} \\ &= F_r \sum_{n=0}^{\infty} F_{r(n+1)}^{(l, l)} t^n \\ &= F_r \sum_{n=l}^{\infty} F_{r(n+2l+1)}^{(l, l)} t^{n+2l} \\ &= F_r \sum_{j=0}^{\infty} F_{r(j+3l+1)}^{(l, l)} t^{j+3l}. \end{aligned}$$

Let

$$\sum_{n=0}^{\infty} t^n \sum_{a_1+a_2+\dots+a_k=n} \prod_{l=1}^k F_{r(a_l+1)} = \left(\sum_{n=0}^{\infty} F_{r(n+1)} t^n \right)^k = I. \quad (2.1)$$

For $k = 2l + 2$, we obtain

$$\begin{aligned} I &= \frac{F_r^{2l+1}}{(2l+1)! t^{1+3l}} \frac{\partial^{2l+1} H(t, x, y)}{\partial L_r^{(l+1)} \partial (-z^r)^l} \\ &= \frac{F_r^{2l+1}}{(2l+1)!} \sum_{n=0}^{\infty} F_{r(n+3l+2)}^{(l+1, l)} t^n. \end{aligned} \quad (2.2)$$

From (2.1) and (2.2), we get

$$\sum_{n=0}^{\infty} \sum_{a_1+a_2+\dots+a_k=n} \prod_{l=1}^k F_{r(a_l+1)} t^n = \frac{F_r^{2l+1}}{(2l+1)!} \sum_{n=0}^{\infty} F_{r(n+3l+2)}^{(l+1, l)} t^n. \quad (2.3)$$

By (1.4), we write

$$\begin{aligned}
& F_{r(n+3l+2)}^{(l+1,l)} \\
&= F_r \frac{\partial^{2l+1}}{\partial L_r^{(l+1)} \partial (-z^r)^l} \left(\sum_{m=0}^{\lfloor (n+3l+1)/2 \rfloor} \binom{n+3l+1-m}{m} L_r^{n+3l+1-2m} (-z^r)^m \right) \\
&= F_r \sum_{m=l}^{\lfloor (n+2l)/2 \rfloor} \frac{(n+3l+1-m)!}{(m-l)!(n+2l-2m)!} L_r^{(n+2l-2m)} (-z^r)^{m-l} \\
&= F_r \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(n+2l+1-m)! L_r^{(n-2m)} (-z^r)^m}{m!(n-2m)!} \\
&= F_r \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(n+k-m)! L_r^{(n-2m)} (-z^r)^m}{m!(n-2m)!}.
\end{aligned}$$

So

$$\begin{aligned}
& \frac{F_r^{2l+1} F_{r(n+3l+2)}^{(l+1,l)}}{(2l+1)!} \\
&= \frac{F_r^k}{k!} \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(n+k-m)! L_r^{(n-2m)} (-z^r)^m}{m!(n-2m)!} \\
&= F_r^k \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n+k-1-m}{m} \binom{n+k-1-2m}{k-1} L_r^{(n-2m)} z^{rm} (-1)^m. \quad (2.4)
\end{aligned}$$

For $k = 2l + 1$, from (2.3-2.4), we have the conclusion. For $k = 2l + 1$, the proof is similarly obtained. \square

When $r = x = y = 1$ in Theorem 1, we obtain the result of [9].

3. WEIGHTED SUMS FOR THE FIBONACCI POLYNOMIALS

We derive some identities involving the terms of sequence $\{F_{rn}(x)\}$.

Theorem 2. For $k, n > 0$,

$$\begin{aligned}
& \sum_{a_1+a_2+\dots+a_k=n} \prod_{l=1}^k (a_l+1) F_{r(a_l+1)}(x) \\
&= F_r^k \sum_{m=0}^{\min\{\lfloor n/2 \rfloor, k-1\}} \sum_{j=0}^{\lfloor (n-2m)/2 \rfloor} \frac{n-2m+2k-1}{n-2m-j} \binom{k-1}{m} \\
& \quad \times \binom{n-2m-j}{j} \binom{n-2m+2k-2-j}{2k-1} (L_r(x))^{n-2m-2j}.
\end{aligned}$$

where $a_l \geq 0$, for $l = 1, 2, \dots, k$.

Proof. Let $G(t, x) = \sum_{n=0}^{\infty} (n+1) F_{r(n+1)} t^n$. Thus

$$G(t, x) = \frac{F_r(1 - (-1)^r t^2)}{(1 - L_r(x)t + (-1)^r t^2)^2}. \quad (3.1)$$

By induction on (3.1), we obtain

$$\frac{\partial^l G(t, x)}{\partial L_r^l} = \frac{F_r \left(1 + (-1)^{r+1} t^2\right) (l+1)! t^l}{(1 - L_r(x) t + (-1)^r t^2)^{l+2}}. \quad (3.2)$$

Since

$$\left(1 + (-1)^{r+1} t^2\right)^{k-1} = \sum_{m=0}^{k-1} \binom{k-1}{m} (-1)^{m(r+1)} t^{2m}$$

and from (3.1-3.2), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} t^n \sum_{a_1+a_2+\dots+a_k=n} \prod_{l=1}^k (a_l+1) F_{a_l+1}(x) \\ &= \frac{F_r^k \left(1 + (-1)^{r+1} t^2\right)^k}{(1 - L_r(x) t + (-1)^r t^2)^{2k}} \\ &= \frac{F_r^{k-1} \left(1 + (-1)^{r+1} t^2\right)^{k-1}}{(2k-1)!} \sum_{n=0}^{\infty} (n+2k-1) F_{r(n+2k-1)}^{(2k-2)}(x) t^n \\ &= \frac{F_r^{k-1}}{(2k-1)!} \sum_{n=0}^{\infty} \sum_{m=0}^{\min\{\lfloor n/2 \rfloor, k-1\}} (-1)^{m(r+1)} \binom{k-1}{m} \\ & \quad \times (n-2m+2k-1) F_{r(n-2m+2k-1)}^{(2k-2)}(x) t^n. \end{aligned} \quad (3.3)$$

From (3.3) and (2.4), we derive

$$\begin{aligned} & \sum_{a_1+a_2+\dots+a_k=n} \prod_{l=1}^k (a_l+1) F_{r(a_l+1)}(x) \\ &= \frac{F_r^{k-1}}{(2k-1)!} \sum_{m=0}^{\min\{\lfloor n/2 \rfloor, k-1\}} (-1)^{m(r+1)} \binom{k-1}{m} (n-2m+2k-1) \\ & \quad \times F_{r(n-2m+2k-1)}^{(2k-2)}(x) \\ &= F_r^k \sum_{m=0}^{\min\{\lfloor n/2 \rfloor, k-1\}} (-1)^{m(r+1)} \binom{k-1}{m} \frac{(n-2m+2k-1)}{(2k-1)!} F_{r(n-2m+2k-1)}^{(2k-2)}(x) \\ &= F_r \sum_{m=0}^{\min\{\lfloor n/2 \rfloor, k-1\}} \sum_{j=0}^{\lfloor \frac{n-2m}{2} \rfloor} \frac{n-2m+2k-1}{n-2m-j} \binom{k-1}{m} \\ & \quad \times \binom{n-2m-j}{j} \binom{n-2m+2k-2-j}{2k-1} (L_r(x))^{n-2m-2j} \end{aligned}$$

□

Theorem 3. For $k, n, r > 0$,

$$\begin{aligned} & \sum_{a_1+a_2+\dots+a_k=n} \prod_{l=1}^k (a_l + 2) F_{r(a_l+1)}(x) \\ = & F_r^k \sum_{j=0}^{\min\{k,n\}} \sum_{i=0}^{\lfloor (n-j)/2 \rfloor} \sum_{m=0}^{\lfloor (n-j-2i)/2 \rfloor} (-1)^{i+j} \frac{n-j-2(i-k)-1}{n-j-2i-m} 2^{k-j} \binom{k}{j} \\ & \times \binom{n-j-2i-m}{m} \binom{n-j-2i+2k-2-m}{2k-1} (L_r(x))^{n-2i-m} (-1)^{m(r+1)}. \end{aligned}$$

Proof. By taking $y = 1$ in (1.2), we write

$$\frac{\partial}{\partial t} \left(\sum_{n=0}^{\infty} F_{r(n+1)}(x) t^{n+2} \right) = \frac{\partial}{\partial t} \left(\frac{F_r t^2}{1 - L_r(x)t + (-1)^r t^2} \right).$$

Then

$$\sum_{n=0}^{\infty} (n+2) F_{r(n+1)}(x) t^n = \frac{F_r (2 - L_r(x)t)}{(1 - L_r(x)t + (-1)^r t^2)^2}.$$

Since

$$(2 - L_r(x)t)^k = \sum_{j=0}^k \binom{k}{j} (-L_r(x)t)^j 2^{k-j} \quad \text{and} \quad \frac{1}{1+t^2} = \sum_{i=0}^{\infty} (-t^2)^i,$$

using the same method given in the previous theorem gives us

$$\begin{aligned} & \sum_{n=0}^{\infty} t^n \sum_{a_1+\dots+a_k=n} \prod_{l=1}^k (a_l + 2) F_{r(a_l+1)}(x) = \frac{F_r^k (2 - L_r(x)t)^k}{(1 - L_r(x)t + (-1)^r t^2)^{2k}} \\ = & \frac{F_r^k (2 - L_r(x)t)^k}{(1+t^2)(2k-1)!} \sum_{n=0}^{\infty} (n+2k-1) F_{r(n+2k-1)}^{(2k-2)}(x) t^n \\ = & \frac{F_r^k}{(2k-1)!} \sum_{n=0}^{\infty} \sum_{j=0}^{\min\{k,n\}} \sum_{i=0}^{\lfloor (n-j)/2 \rfloor} (-1)^{i+j} (n-j-2(i+k)-1) \\ & \times F_{r(n-j-2i+2k-1)}^{(2k-2)}(x) t^n. \end{aligned}$$

Considering the proof of previous theorem, we have the conclusion. \square

Identities for the bivariate Fibonacci polynomials $F_n(x, y)$ generate identities for specially multiplicative functions at prime powers $f_{p^{n-1}}$ with $x = f_p$ and $y = -f_p^{-1}$, and vice versa, see [1, 6, 7].

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