# BINOMIAL SUMS WHOSE COEFFICIENTS ARE PRODUCTS OF TERMS OF BINARY SEQUENCES

EMRAH KILIÇ[1], NEŞE ÖMÜR[2], AND Y.TÜRKER ULUTAŞ[3]

Abstract. In this study, we introduce sums and alternating sums of products of terms of sequences $\{U_{kn}\}$ and $\{V_{kn}\}$ with binomial coefficents. These sums have a nice representation for the terms of sequences $\{U_{kn}\}$ and $\{V_{kn}\}$.

## 1. INTRODUCTION

The second order linear recurrence $X_n(p,q;a,b)$ is defined by for $n>1$,

$$X_n = pX_{n-1} - qX_{n-2},$$

where $X_0 = a,\ X_1 = b$.

We denote $X_n(p,1;0,1)$,$X_n(p,1;2,p)$,$X_n(p,q;0,1)$ and $X_n(p,q;2,p)$ by $\{U_n\}$, $\{V_n\}$,$\{u_n\}$ and $\{v_n\}$.

The Binet formulas of $\{U_n\}$ and $\{V_n\}$ are as follows:

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n \tag{1.1}$$

where $\alpha, \beta = \left(p \pm \sqrt{p^2+4}\right)/2$.

From [2, 3], we have that for $k>0$

$$H_n = V_k H_{n-k} - (-1)^k H_{n-2k},$$

where $V_k$ is as before and $H_n$ is either $U_{kn}$ or $V_{kn}$.

For the alternating sums of squares of the terms of the Fibonacci, Lucas, Pell and Pell-Lucas sequences as well as the alternating sums, we refer to [5, 7, 8, 9]. The author [7] eliminated all restrictions about the initial values of the recurrence relation given in [5] and obtained formulae for the sums given by

$$\sum_{j=1}^{n} u_{a+bj}u_{c+dj}, \sum_{j=1}^{n} u_{a+bj}v_{c+dj} \text{ and } \textstyle\sum_{j=1}^{n} v_{a+bj}v_{c+dj}$$

as well as their general cases which include the binomial coefficients $\binom{n}{j}$ as coefficient.

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The author [6] used the finite differences method and then computed the following combinatorial sums including Fibonacci numbers: for $n > 0$

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\Phi^k F_k &= \Phi\left(\Phi+2\right)^{n-1}, \ \Phi=\left(1+\sqrt{5}\right)/2,\\ \sum_{k=0}^{n}\binom{n}{k}\left(-2\right)^k F_k &= \left(-2\right)5^{(n-1)/2} \text{ for odd } n,\end{aligned}$$

In this study, we consider two kinds of combinatorial sums whose coefficients are certain products of terms of the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$ as well as their alternate analogues. We will study the sums have the form:

$$\sum_{i=0}^{n}\binom{n}{i}T_{k(a+bi)}T_{k(c+di)} \text{ and } \sum_{i=0}^{n}\binom{n}{i}\left(-1\right)^i T_{k(a+bi)}T_{k(c+di)},$$

where $T_n$ is either $U_n$ or $V_n$.

Such or similar sums could be computed by using different methods. One of the well known methods is to use the Binet formula the binary sequences. The second one is may the induction method. But here we should note that it would not be useful for ours. As third optional, one may consider to use finite differences (for more detailed results and their applications, we refer to [1, 6]), but it is not again useful for ours.

## 2. Sums of certain Products of the Terms of $\{U_{kn}\}$ and $\{V_{kn}\}$

In this section, we start with some lemmas for further use. We shall denote $\left(V_k^2+4\right)/U_k^2$ by $\Delta$.

**Lemma 1.** *For any integers $m$ and $n$,*

$$V_{k(m+n)}+V_{k(m-n)} = \begin{cases}\Delta U_{km}U_{kn} & \text{if } n \text{ is odd,}\\ V_{km}V_{kn} & \text{if } n \text{ is even,}\end{cases} \tag{2.1}$$

$$V_{k(m+n)}-V_{k(m-n)} = \begin{cases}V_{km}V_{kn} & \text{if } n \text{ is odd,}\\ \Delta U_{km}U_{kn} & \text{if } n \text{ is even,}\end{cases} \tag{2.2}$$

$$U_{k(m+n)}+U_{k(m-n)} = \begin{cases}V_{km}U_{kn} & \text{if } n \text{ is odd,}\\ U_{km}V_{kn} & \text{if } n \text{ is even,}\end{cases} \tag{2.3}$$

$$U_{k(m+n)}-U_{k(m-n)} = \begin{cases}U_{km}V_{kn} & \text{if } n \text{ is odd,}\\ V_{km}U_{kn} & \text{if } n \text{ is even.}\end{cases} \tag{2.4}$$

*Proof.* By the Binet formulas of $\{U_{kn}\}$ and $\{V_{kn}\}$, the claimed equalities (2.1)-(2.4) are obtained. □

We recall some facts for the readers convenience: For any real numbers $m$ and $n$,

$$(m+n)^t = \begin{cases} \sum_{i=0}^{(t-1)/2} \binom{t}{i} (mn)^i \left(m^{t-2i} + n^{t-2i}\right) & \text{if } t \text{ is odd,} \\ \sum_{i=0}^{t/2-1} \binom{t}{i} (mn)^i \left(m^{t-2i} + n^{t-2i}\right) + \binom{t}{t/2} (mn)^{t/2} & \text{if } t \text{ is even,} \end{cases} \tag{2.5}$$

and

$$(m-n)^t = \begin{cases} \sum_{i=0}^{(t-1)/2} \binom{t}{i} (mn)^i (-1)^i \left(m^{t-2i} - n^{t-2i}\right) & \text{if } t \text{ is odd,} \\ \sum_{i=0}^{t/2-1} \binom{t}{i} (mn)^i (-1)^i \left(m^{t-2i} + n^{t-2i}\right) \\ \qquad + \binom{t}{t/2} (mn)^{t/2} (-1)^{t/2} & \text{if } t \text{ is even.} \end{cases} . \tag{2.6}$$

**Lemma 2.** *For any integers $r$ and $s$,*

*i) for odd $s$*

$$\sum_{i=0}^{n} \binom{n}{i} U_{k(r+2si)} = \begin{cases} \Delta^{\lfloor n/2 \rfloor} U_{ks}^n V_{k(sn+r)} & \text{if } n \text{ is odd,} \\ \Delta^{\lfloor n/2 \rfloor} U_{ks}^n U_{k(sn+r)} & \text{if } n \text{ is even,} \end{cases} \tag{2.7}$$

*and for even $s$*

$$\sum_{i=0}^{n} \binom{n}{i} U_{k(r+2si)} = V_{ks}^n U_{k(sn+r)}.$$

*ii) For odd $s$,*

$$\sum_{i=0}^{n} \binom{n}{i} V_{k(r+2si)} = \begin{cases} \Delta^{(n+1)/2} U_{ks}^n U_{k(sn+r)} & \text{if } n \text{ is odd,} \\ \Delta^{n/2} U_{ks}^n V_{k(sn+r)} & \text{if } n \text{ is even,} \end{cases} \tag{2.8}$$

*and for even $s$,*

$$\sum_{i=0}^{n} \binom{n}{i} V_{k(r+2si)} = V_{ks}^n V_{k(sn+r)}.$$

*iii) For even $s$,*

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{k(r+2si)} = \begin{cases} \Delta^{\lfloor n/2 \rfloor} U_{ks}^n U_{k(sn+r)} & \text{if } n \text{ is even,} \\ -\Delta^{\lfloor n/2 \rfloor} U_{ks}^n V_{k(sn+r)} & \text{if } n \text{ is odd,} \end{cases} \tag{2.9}$$

*and for odd $s$,*

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{k(r+2si)} = (-1)^n V_{ks}^n U_{k(sn+r)}.$$

*iii) For even s,*

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}V_{k(r+2si)}=\begin{cases}\Delta^{n/2}U_{ks}^{n}V_{k(sn+r)} & \text{if } n \text{ is even,}\\ -\Delta^{(n+1)/2}U_{ks}^{n}U_{k(sn+r)} & \text{if } n \text{ is odd,}\end{cases} \tag{2.10}$$

*and for odd s,*

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}V_{k(r+2si)}=(-1)^{n}V_{ks}^{n}V_{k(sn+r)}.$$

*Proof.* From (2.5), (2.6) and Lemma 1 , the proof is obtained. ☐

**Theorem 1.** *Let $a,b,c$ and $d$ be any integers. If $b+d\equiv 2 \pmod 4$, then*

$$\sum_{i=0}^{n}\binom{n}{i}U_{k(a+bi)}U_{k(c+di)} = \begin{cases}\Delta^{(n-1)/2}\Big(U_{k(b+d)/2}^{n}U_{k(n(b+d)/2+a+c)} \\ \quad +(-1)^{c+(b-d)/2}U_{k(b-d)/2}^{n}U_{k(n(b-d)/2+a-c)}\Big) & \text{if } n \text{ is odd,}\\ \Delta^{(n-2)/2}\Big(U_{k(b+d)/2}^{n}V_{k(n(b+d)/2+a+c)} \\ \quad -(-1)^{c}U_{k(b-d)/2}^{n}V_{k(n(b-d)/2+a-c)}\Big) & \text{if } n \text{ is even,}\end{cases}$$

*and if $b+d\equiv 0 \pmod 4$, then*

$$\sum_{i=0}^{n}\binom{n}{i}U_{k(a+bi)}U_{k(c+di)} = \frac{1}{\Delta}\Big(V_{k(b+d)/2}^{n}V_{k(n(b+d)/2+a+c)} - (-1)^{c+n(b-d)/2}V_{k(b-d)/2}^{n}V_{k(n(b-d)/2+a-c)}\Big).$$

*If $b+d\equiv 2 \pmod 4$, then*

$$\sum_{i=0}^{n}\binom{n}{i}V_{k(a+bi)}V_{k(c+di)} = \begin{cases}\Delta^{(n+1)/2}\Big(U_{k(b+d)/2}^{n}U_{k(n(b+d)/2+a+c)}-(-1)^{c+(b-d)/2} \\ \quad \times U_{k(b-d)/2}^{n}U_{k(n(b-d)/2+a-c)}\Big) & \text{if } n \text{ is odd,}\\ \Delta^{n/2}\Big(U_{k(b+d)/2}^{n}V_{k(n(b+d)/2+a+c)}+(-1)^{c}U_{k(b-d)/2}^{n} \\ \quad \times V_{k(n(b-d)/2+a-c)}\Big) & \text{if } n \text{ is even,}\end{cases}$$

and if $b+d\equiv 0 \pmod 4$, then

$$\sum_{i=0}^{n}\binom{n}{i}V_{k(a+bi)}V_{k(c+di)} = V_{k(b+d)/2}^{n}V_{k(n(b+d)/2+a+c)} + (-1)^{c+n(b-d)/2}V_{k(b-d)/2}^{n}V_{k(n(b-d)/2+a-c)}.$$

If $b+d\equiv 2 \pmod 4$, then

$$\sum_{i=0}^{n}\binom{n}{i}U_{k(a+bi)}V_{k(c+di)}$$
$$= \begin{cases} \Delta^{(n-1)/2}\Big(U^n_{k(b+d)/2}\,V_{k(n(b+d)/2+a+c)} - (-1)^{c+\frac{b-d}{2}} \\ \qquad \times U^n_{k(b-d)/2}\,V_{k(n(b-d)/2+a-c)}\Big) & \text{if } n \text{ is odd,} \\ \Delta^{n/2}\Big(U^n_{k(b+d)/2}\,U_{k(n(b+d)/2+a+c)} + (-1)^c\,U^n_{k(b-d)/2} \\ \qquad \times U_{k(n(b-d)/2+a-c)}\Big) & \text{if } n \text{ is even,} \end{cases}$$

and if $b+d \equiv 0 \pmod 4$, then

$$\sum_{i=0}^{n}\binom{n}{i}U_{k(a+bi)}V_{k(c+di)} = V^n_{k(b+d)/2}U_{k(n(b+d)/2+a+c)} + (-1)^{c+n(b-d)/2}\,V^n_{k(b-d)/2}U_{k(n(b-d)/2+a-c)}.$$

*Proof.* Consider

$$\begin{aligned} &\sum_{i=0}^{n}\binom{n}{i}U_{k(a+bi)}U_{k(c+di)} \\ &= \sum_{i=0}^{n}\binom{n}{i}\left(\tfrac{\alpha^{k(a+bi)}-\beta^{k(a+bi)}}{\alpha-\beta}\right)\left(\tfrac{\alpha^{k(c+di)}-\beta^{k(c+di)}}{\alpha-\beta}\right) \\ &= \frac{1}{(\alpha-\beta)^2}\sum_{i=0}^{n}\binom{n}{i}\Big(\alpha^{k(a+c+(b+d)i)}+\beta^{k(a+c+(b+d)i)} \\ &\qquad - (-1)^{c+di}\left(\alpha^{k(a-c+(b-d)i)}+\beta^{k(a-c+(b-d)i)}\right)\Big) \\ &= \Delta^{-1}\sum_{i=0}^{n}\binom{n}{i}\left(V_{k(a+c+(b+d)i)} - (-1)^{c+di}\,V_{k(a-c+(b-d)i)}\right). \end{aligned}$$

For even integers $b$ and $d$, we get

$$\begin{aligned} &\sum_{i=0}^{n}\tbinom{n}{i}U_{k(a+bi)}U_{k(c+di)} \\ &= \tfrac{U_k^2}{V_k^2+4}\left(\sum_{i=0}^{n}\tbinom{n}{i}V_{k(a+c+(b+d)i)} - (-1)^c\,V_{k(a-c+(b-d)i)}\right). \end{aligned}$$

When $b+d \equiv 2 \pmod 4$ and $n$ is an odd integer, using (2.8), we obtain

$$\sum_{i=0}^{n}\binom{n}{i}U_{k(a+bi)}U_{k(c+di)} = \Delta^{\frac{n-1}{2}}(U^n_{k(b+d)/2}U_{k(n(b+d)/2+a+c)} - (-1)^c\,U^n_{k(b-d)/2}U_{k(n(b-d)/2+a-c)}). \quad (2.11)$$

For odd integers $b$ and $d$, we write

$$\begin{aligned} &\sum_{i=0}^{n}\binom{n}{i}U_{k(a+bi)}U_{k(c+di)} \\ &= \Delta^{-1}\left(\sum_{i=0}^{n}\binom{n}{i}V_{k(a+c+(b+d)i)} - (-1)^{c+di}\,V_{k(a-c+(b-d)i)}\right). \end{aligned}$$

When $b+d\equiv 2 \pmod 4$ and $n$ is an odd integer, using (2.8) and (2.10), we derive

$$\sum_{i=0}^{n}\binom{n}{i}U_{k(a+bi)}U_{k(c+di)} = \Delta^{\frac{n-1}{2}}(U^n_{k(b+d)/2}U_{k(n(b+d)/2+a+c)} + (-1)^c U^n_{k(b-d)/2}U_{k(n(b-d)/2+a-c)}). \quad (2.12)$$

From (2.11) and (2.12), for $b+d\equiv 2 \pmod 4$ and $n$ is odd integer, the desired result is obtained.

Using Lemma 2, the other cases are similarly obtained. □

For example, when $k=1, p=1, a=d=3, b=5, c=2$ in Theorem 1, we obtain

$$\sum_{i=0}^{n}\binom{n}{i}L_{3+5i}L_{2+3i} = 7^nL_{4n+5}+(-1)^n L_{n+1}.$$

**Theorem 2.** *Let $a,b,c,\ d$ be any integers. If $b+d\equiv 0 \pmod 4$, then*

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i U_{k(a+bi)}U_{k(c+di)} = \begin{cases} \Delta^{\frac{n-1}{2}}\left(-U^n_{k(b+d)/2}\, U_{k(n(b+d)/2+a+c)} + (-1)^{c+\frac{b-d}{2}} \times U^n_{k(b-d)/2}\, U_{k(n(b-d)/2+a-c)}\right) & \text{if } n \text{ is odd,} \\ \Delta^{\frac{n-2}{2}}\left(U^n_{k(b+d)/2}\, V_{k(n(b+d)/2+a+c)} - (-1)^c U^n_{k(b-d)/2} \times V_{k(n(b-d)/2+a-c)}\right) & \text{if } n \text{ is even,} \end{cases}$$

and if $b+d\equiv 2 \pmod 4$, then

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i U_{k(a+bi)}U_{k(c+di)} = \frac{1}{\Delta}\left((-1)^n V^n_{k(b+d)/2}\, V_{k(n(b+d)/2+a+c)} - (-1)^{c+\frac{n(b-d)}{2}} V^n_{k(b-d)/2}\, V_{k(n(b-d)/2+a-c)}\right).$$

If $b+d\equiv 0 \pmod 4$, then

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i V_{k(a+bi)}V_{k(c+di)} = \begin{cases} \Delta^{(n+1)/2}\left(-U^n_{k(b+d)/2}\, U_{k(n(b+d)/2+a+c)} - (-1)^{c+\frac{b-d}{2}} \times U^n_{k(b-d)/2}\, U_{k(n(b-d)/2+a-c)}\right) & \text{if } n \text{ is odd,} \\ \Delta^{n/2}\left(U^n_{k(b+d)/2}\, V_{k(n(b+d)/2+a+c)} + (-1)^c U^n_{k(b-d)/2} \times V_{k(n(b-d)/2+a-c)}\right) & \text{if } n \text{ is even,} \end{cases}$$

and if $b+d\equiv 2 \pmod 4$, then

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i V_{k(a+bi)}V_{k(c+di)} = (-1)^n V^n_{k(b+d)/2}V_{k(n(b+d)/2+a+c)} + (-1)^{c+\frac{n(b-d)}{2}} V^n_{k(b-d)/2}V_{k(n(b-d)/2+a-c)}.$$

If $b+d\equiv 0 \pmod 4$, then

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}U_{k(a+bi)}V_{k(c+di)}$$
$$=\begin{cases}\Delta^{(n-1)/2}\Big(-U^{n}_{k(b+d)/2}\,V_{k(n(b+d)/2+a+c)}-(-1)^{c+\frac{b-d}{2}} \\ \qquad \times U^{n}_{k(b-d)/2}\,V_{k(n(b-d)/2+a-c)}\Big) & \text{if } n \text{ is odd,}\\ \\ \Delta^{n/2}\Big(U^{n}_{k(b+d)/2}\,U_{k(n(b+d)/2+a+c)}+(-1)^{c}\,U^{n}_{k(b-d)/2} \\ \qquad \times U_{k(n(b-d)/2+a-c)}\Big) & \text{if } n \text{ is even,}\end{cases}$$

and if $b+d\equiv 2 \pmod 4$, then

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}U_{k(a+bi)}V_{k(c+di)} = (-1)^{n}V^{n}_{k(b+d)/2}U_{k(n(b+d)/2+a+c)}$$
$$+(-1)^{c+\frac{n(b-d)}{2}}V^{n}_{k(b-d)/2}U_{k(n(b-d)/2+a-c)}.$$

*Proof.* Consider

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}U_{k(a+bi)}U_{k(c+di)}$$
$$=\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\left(\tfrac{\alpha^{k(a+bi)}-\beta^{k(a+bi)}}{\alpha-\beta}\right)\left(\tfrac{\alpha^{k(c+di)}-\beta^{k(c+di)}}{\alpha-\beta}\right)$$
$$=\frac{1}{(\alpha-\beta)^{2}}\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\Big(\alpha^{k(a+c+(b+d)i)}+\beta^{k(a+c+(b+d)i)}$$
$$-(-1)^{c+di}\left(\alpha^{k(a-c+(b-d)i)}+\beta^{k(a-c+(b-d)i)}\right)\Big)$$
$$=\Delta^{-1}\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\left(V_{k(a+c+(b+d)i)}-(-1)^{c+di}V_{k(a-c+(b-d)i)}\right).$$

For even integers $b$ and $d$, we see that

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}U_{k(a+bi)}U_{k(c+di)}$$
$$=\Delta^{-1}\left(\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\left(V_{k(a+c+(b+d)i)}-(-1)^{c}V_{k(a-c+(b-d)i)}\right)\right).$$

When $b+d\equiv 0\,(\mathrm{mod}\,4)$ and $n$ is an odd integer, using (2.10), we reach at the result:

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}U_{k(a+bi)}U_{k(c+di)}$$
$$=\frac{\left(V_k^2+4\right)^{\frac{n+1}{2}}}{\Delta U_k^{n+1}}\left(-U^{n}_{k(b+d)/2}U_{k(n(b+d)/2+a+c)}+(-1)^{c}U^{n}_{k(b-d)/2}U_{k(n(b-d)/2+a-c)}\right)$$
$$=\Delta^{\frac{n-1}{2}}\left(-U^{n}_{k(b+d)/2}U_{k(n(b+d)/2+a+c)}+(-1)^{c}U^{n}_{k(b-d)/2}U_{k(n(b-d)/'+a-c)}\right)\quad(2.13)$$

For odd integers $b$ and $d$, we conclude

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}U_{k(a+bi)}U_{k(c+di)}\\=\;&\Delta^{-1}\left(\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}V_{k(a+c+(b+d)i)}-(-1)^{c}\sum_{i=0}^{n}\binom{n}{i}V_{k(a-c+(b-d)i)}\right).\end{aligned}$$

When $b+d\equiv 0\,(\bmod 4)$ and $n$ is an odd integer, using (2.8) and (2.10), we have

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}U_{k(a+bi)}U_{k(c+di)}\\=\;&\Delta^{(n-1)/2}(-U_{k(b+d)/2}^{n}U_{k(n(b+d)/2+a+c)}\\&\quad-(-1)^{c}U_{k(b-d)/2}^{n}U_{k(n(b-d)/2+a-c)}).\end{aligned}\tag{2.14}$$

From (2.13) and (2.14), for $b+d\equiv 0\,(\bmod 4)$ and $n$ is an odd integer, the desired result is obtained.

Using Lemma 2, we get the other claims. □

When $k=p=1, a=3, b=8, c=5$ and $d=4$ in Theorem 2, we get

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}F_{3+8i}L_{5+4i}=\begin{cases}5^{(n-1)/2}\left(-8^{n}L_{6n+8}+L_{2n-2}\right) & \text{if } n \text{ is odd},\\ 5^{n/2}\left(8^{n}F_{6n+8}-F_{2n-2}\right) & \text{if } n \text{ is even}.\end{cases}$$

## 3. Concluding Remarks

This paper consider the sums including various products of terms $U_{k(a+bj)}$, $V_{k(c+dj)}$ and alternating analogues of them, where $U_n=X_n\left(p,1;0,1\right)$ and $V_n=X_n\left(p,1;2,1\right)$ for a positive $k$.

In [7], the same kind of sums given in this paper with using more general sequences $u_n=X_n\left(p,q;1,0\right)$ and $v_n=X_n\left(p,q;2,p\right)$ without adding a new parameter $k$ in indices of the sequences are considered.

The results of this paper are represented by the generalized Fibonacci and Lucas numbers, but the results of [7] are given in closed forms. Similar results in closed forms for Horadam sequence, $\{X_n\left(p,q;a,b\right)\}$, can be found in [10].

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[1]TOBB University of Economics and Technology Mathematics Department 06560 Ankara Turkey
*E-mail address*: ekilic@etu.edu.tr[1]
*URL*: http://www.authorone.oneuniv.edu

[2−3]Kocaeli University Mathematics Department 41380 İzmit Kocaeli Turkey
*E-mail address*: neseomur@kocaeli.edu.tr[2],turkery@kocaeli.edu.tr[3]