SUMS OF PRODUCTS OF THE TERMS OF GENERALIZED LUCAS SEQUENCE $\{V_{kn}\}$

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ABSTRACT. In this study we consider the generalized Lucas sequence $\{V_n\}$ with indices in arithmetic progression. We also compute the sums of products of the terms of the Lucas sequence $\{V_{kn}\}$ for positive odd integer k.

1. INTRODUCTION

The binary linear recurrence $W_n = W_n(a, b; p, q)$ is defined as follows for $n > 1$,

$$W_n = pW_{n-1} + qW_{n-2},$$

where $W_0 = a, W_1 = b$.

The Binet formula for $\{W_n\}$ is

$$W_n = A\alpha^n + B\beta^n, \tag{1.1}$$

where $A = \frac{b-a\beta}{\alpha-\beta}$, $B = \frac{a\alpha-b}{\alpha-\beta}$ and $\alpha, \beta = \left(p \pm \sqrt{p^2+4q}\right)/2$.

For $n > 1$ and a fixed positive integer k, the terms of $\{W_{kn}\}$ satisfies the recursion [6, 7]:

$$W_{kn} = V_k W_{k(n-1)} - (-q)^k W_{k(n-2)},$$

where $V_k = \alpha^k + \beta^k$.

Specifically, define the generalized Fibonacci $\{U_n\}$ and Lucas $\{V_n\}$ sequences as $U_n = W_n(0, 1; p, 1)$, $V_n = W_n(2, p; p, 1)$, respectively. Thus

$$U_{kn} = V_k U_{k(n-1)} - (-1)^k U_{k(n-2)}, \tag{1.2}$$

$$V_{kn} = V_k V_{k(n-1)} - (-1)^k V_{k(n-2)}. \tag{1.3}$$

The Fibonomial coefficients $\begin{bmatrix} n \\ m \end{bmatrix}_F$ are defined by the relation for $n \geq m \geq 1$

$$\begin{bmatrix} n \\ m \end{bmatrix}_F = \frac{F_1F_2...F_n}{(F_1F_2...F_{n-m})(F_1F_2...F_m)}$$

with $\begin{bmatrix} n \\ 0 \end{bmatrix}_F = \begin{bmatrix} n \\ n \end{bmatrix}_F = 1$, where F_n is the nth Fibonacci number. These coefficients satisfy the relation:

$$\begin{bmatrix} n \\ m \end{bmatrix}_F = F_{m+1}\begin{bmatrix} n-1 \\ m \end{bmatrix}_F + F_{n-m-1}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_F.$$

Hoggatt [4] defined the following generalization by taking F_{kn} instead of F_n for a fixed positive integer k :

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$$\begin{bmatrix} n \\ m \end{bmatrix}_{F_k} = \frac{F_k F_{2k} ... F_{kn}}{\left(F_k F_{2k} ... F_{k(n-m)}\right)\left(F_k F_{2k} ... F_{km}\right)}.$$

Jarden and Motzkin were the first to study the generalized Fibonomial coefficients formed by terms of sequence $\{U_n\}$ as follows: for $n \geq m \geq 1$,

$$\begin{bmatrix} n \\ m \end{bmatrix}_U = \frac{U_1 U_2 ... U_n}{\left(U_1 U_2 ... U_{n-m}\right)\left(U_1 U_2 ... U_m\right)},$$

with $\begin{bmatrix} n \\ 0 \end{bmatrix}_U = \begin{bmatrix} n \\ n \end{bmatrix}_U = 1$.

When $p = 1$, the generalized Fibonomial coefficients $\begin{bmatrix} n \\ m \end{bmatrix}_U$ are reduced to the Fibonomial coefficients $\begin{bmatrix} n \\ m \end{bmatrix}_F$.

By taking U_{kn} instead of U_n for a fixed positive integer k, one can get

$$\begin{bmatrix} n \\ m \end{bmatrix}_{U_k} = \frac{U_k U_{2k} ... U_{kn}}{\left(U_k U_{2k} ... U_{k(n-m)}\right)\left(U_k U_{2k} ... U_{km}\right)}.$$

These coefficients satisfy the relation:

$$\begin{bmatrix} n \\ m \end{bmatrix}_{U_k} = U_{km+1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_{U_k} + U_{k(n-m)-1} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_{U_k}$$

and

$$\begin{bmatrix} n \\ m \end{bmatrix}_{U_k} = U_{km-1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_{U_k} + U_{k(n-m)+1} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_{U_k}.$$

Golomb [3] found the generating function for the numbers F_n^2 and this result started the effort to find a recurrences or a closed form for the generating function

$$f_m(x) = \sum_{n=0}^{\infty} F_n^m x^n$$

of the mth powers of the Fibonacci numbers.

In [8], Riordan found the general recurrence for $f_m(x)$ (see also [2]). Carlitz [1] and Horadam [5] generalized the result of Riordan and found similar recurrences for the generating functions of different types of generalized Fibonacci numbers. They also found closed form for the polynomial $N_m(x)$ in the numerator and the polynomial $D_m(x)$ in the denominator of the generating functions. As a special case of Horadam's result, it is possible to get the following formula for the generating function of an odd integer powers of Fibonacci numbers

$$f_m(x) = \frac{\sum\limits_{i=0}^{m} \sum\limits_{j=0}^{i} (-1)^{\frac{j(j+1)}{2}} \begin{bmatrix} m+1 \\ j \end{bmatrix}_F F_{i-j}^m x^i}{\sum\limits_{i=0}^{m+1} (-1)^{\frac{i(i+1)}{2}} \begin{bmatrix} m+1 \\ i \end{bmatrix}_F x^i}. \quad (1.4)$$

In [13], applying Carlitz's approach, Shannon obtained some special results for the numerator and the denominator in the expression of the generating function $f_m(x)$. Using the $q-$analogue of the terminating binomial theorem, he obtained the relation

$$\prod_{i=0}^{m} \left(1 - q^i x\right) = \sum_{i=0}^{m+1} (-1)^i q^{\frac{i}{2}(i-1)} \begin{Bmatrix} m+1 \\ i \end{Bmatrix} x^i,$$

where $\left\{ {m \atop i} \right\} = \frac{(1-q^m)(1-q^{m-1})\dots(1-q^{m-i+1})}{(1-q)(1-q^2)\dots(1-q^i)}$ is the Gaussian q-binomial coefficient for $i \geq 1$ any complex numbers q, x and any positive integer m with $\left\{ {m \atop 0} \right\} = 1$. Replacing q by β/α and x by $\alpha^m x$, one can get

$$\prod_{i=0}^{m} \left(1 - \alpha^{m-i}\beta^i x\right) = \sum_{i=0}^{m+1} (-1)^{\frac{i}{2}(i+1)} \begin{bmatrix} m+1 \\ i \end{bmatrix}_F x^i.$$

It is easy to obtain for any odd integer m that

$$f_m(x) = 5^{-\frac{m-1}{2}} \sum_{i=0}^{\frac{m-1}{2}} \binom{m}{j} \frac{F_{m-2j}x}{1 - (-1)^j L_{m-2j}x - x^2} \tag{1.5}$$

after simplifications of one of Shannon's results. Seibert and Trojovsky [11] gave certain generalizations of the well-known formulas for the Fibonacci and Lucas numbers. For example,

$$\sum_{i=0}^{n} (-1)^i L_{n-2i} = 2F_{n+1}.$$

For odd positive integer m, authors concentrated on the sums

$$\sum_{i_n=0}^{\frac{m-1}{2}} \sum_{i_{n-1}=i_n+1}^{\frac{m-1}{2}} \dots \sum_{i_1=i_2+1}^{\frac{m-1}{2}} (-1)^{i_1+i_2+\dots+i_n} \prod_{j=1}^{n} L_{m-2i_j}.$$

Combining (1.4) and (1.5), they gave some new results about these sums with the help of the Fibonomial coefficients.

In [12], for arbitrary positive integer m, taking the sums

$$\sum_{i_n=0}^{\lfloor\frac{m-1}{2}\rfloor} \sum_{i_{n-1}=i_n+1}^{\lfloor\frac{m-1}{2}\rfloor} \dots \sum_{i_1=i_2+1}^{\lfloor\frac{m-1}{2}\rfloor} (-1)^{i_1+i_2+\dots+i_n} \prod_{j=1}^{n} L_{m-2i_j}, \tag{1.6}$$

the authors gave analogous new results for even integer m using the method from [11].

We consider the generalized Lucas sequence $\{V_n\}$ with indices in arithmetic progression and then compute the sums of products of terms of the sequence $\{V_{kn}\}$ for positive odd integer k.

2. Some Identities Including the Terms of $\{U_{kn}\}$ and $\{V_{kn}\}$

We shall give some results for later use. Throughout this study, we will denote $W_n(a, b, p, 1)$ by H_n.

Lemma 1. *For $m, n > 0$,*

$$V_{k(m+n)} + V_{k(m-n)} = \begin{cases} \frac{V_k^2+4}{U_k^2} U_{km}U_{kn} & \text{if } n \text{ is odd,} \\ V_{km}V_{kn} & \text{if } n \text{ is even,} \end{cases} \tag{2.1}$$

$$V_{k(m+n)} - V_{k(m-n)} = \begin{cases} V_{km}V_{kn} & \text{if } n \text{ is odd,} \\ \frac{V_k^2+4}{U_k^2} U_{km}U_{kn} & \text{if } n \text{ is even,} \end{cases} \tag{2.2}$$

$$U_{k(m+n)} + U_{k(m-n)} = \begin{cases} V_{km}U_{kn} & \text{if } n \text{ is odd,} \\ U_{km}V_{kn} & \text{if } n \text{ is even,} \end{cases} \tag{2.3}$$

$$U_{k(m+n)} - U_{k(m-n)} = \begin{cases} U_{km}V_{kn} & \text{if } n \text{ is odd,} \\ V_{km}U_{kn} & \text{if } n \text{ is even,} \end{cases} \tag{2.4}$$

$$V_{k(m-2n)}V_{k(2(m-2n)-3)} = \begin{cases} V_{3k(m-2n-1)} - V_{k(m-2n-3)} & \text{if } m \text{ is odd,} \\ V_{3k(m-2n-1)} + V_{k(m-2n-3)} & \text{if } m \text{ is even,} \end{cases} \tag{2.5}$$

$$V_{k(m-2n)}U_{k(m-2n-1)} = \begin{cases} U_{k(2m-4n-1)} - U_k & \text{if } m \text{ is odd,} \\ U_{k(2m-4n-1)} + U_k & \text{if } m \text{ is even,} \end{cases} \tag{2.6}$$

$$V_{3k(m+1)} + V_{3k(m-1)} = U_{3km}U_{3k}\left(V_k^2 + 4\right) \tag{2.7}$$

$$V_{kn}U_{k(n-1)} - U_{k(2n-1)} = (-1)^{kn} U_k \tag{2.8}$$

$$U_{k(m+n)} = U_{km}V_{kn} + (-1)^{n+1} U_{k(m-n)} \tag{2.9}$$

$$U_{k(m+n)}U_{k(m+t)} - U_{km}U_{k(m+t+n)} = (-1)^m U_{kn}U_{kt} \tag{2.10}$$

Proof. By the Binet formulas for $\{U_{kn}\}$ and $\{V_{kn}\}$, the proof follows. □

Theorem 1. *For any integers r, c, d with $c \neq 0$ and $n \geq 0$,*

$$i) \sum_{i=r}^{n} H_{k(ci+d)} = [H_{k(cr+d)} - H_{k(c(n+1)+d)} - (-1)^c H_{k(c(r-1)+d)} + (-1)^c H_{k(cn+d)}]/1 - V_{kc} + (-1)^c \tag{2.11}$$

$$ii) \sum_{i=r}^{n} (-1)^i H_{k(ci+d)} = [(-1)^r H_{k(cr+d)} + (-1)^n H_{k(c(n+1)+d)} + (-1)^{c+r} H_{k(c(r-1)+d)} + (-1)^{c+n} H_{k(cn+d)}\Big] /1 + V_{kc} + (-1)^c \tag{2.12}$$

Proof. $i)$ By the Binet formula for $\{H_n\}$, we have

$$\begin{aligned} \sum_{i=r}^{n} H_{k(ci+d)} &= H_{k(cr+d)} + H_{k(c(r+1)+d)} + ... + H_{k(cn+d)} \\ &= A\alpha^{k(cr+d)} + B\beta^{k(cr+d)} + A\alpha^{k(c(r+1)+d)} + B\beta^{k(c(r+1)+d)} \\ &\quad + ... + A\alpha^{k(cn+d)} + B\beta^{k(cn+d)} \\ &= A\alpha^{k(cr+d)}\left[1 + \alpha^{kc} + \alpha^{2kc} + ... + \alpha^{kc(n-r)}\right] \\ &\quad + B\beta^{k(cr+d)}\left[1 + \beta^{kc} + \beta^{2kc} + ... + \beta^{kc(n-r)}\right] \\ &= A\alpha^{k(cr+d)}\frac{1 - \alpha^{kc(n-r+1)}}{1 - \alpha^{kc}} + B\beta^{k(cr+d)}\frac{1 - \beta^{kc(n-r+1)}}{1 - \beta^{kc}}, \end{aligned}$$

which, by $(\alpha\beta)^k = -1$, gives us

$$\begin{aligned} &= \Big[A\alpha^{k(cr+d)} + B\beta^{k(cr+d)} - \left(A\alpha^{k(c(n+1)+d)} + B\beta^{k(c(n+1)+d)}\right) \\ &\quad - (-1)^c \left(A\alpha^{k(c(r-1)+d)} + B\beta^{k(c(r-1)+d)}\right) \\ &\quad + (-1)^c \left(A\alpha^{k(cn+d)} + B\beta^{k(cn+d)}\right)\Big] /1 - \alpha^{kc} - \beta^{kc} + (-1)^c \\ &= (H_{k(cr+d)} - H_{k(c(n+1)+d)} - (-1)^c H_{k(c(r-1)+d)} \\ &\quad + (-1)^c H_{k(cn+d)})/1 - V_{kc} + (-1)^c. \end{aligned}$$

Thus the proof of (i) is complete. The equation (2.12) can be similarly proven. □

Theorem 2. *For any integers $r, c\,(c \neq 0)$ and d and $n \geq 0$,*

$$
\begin{aligned}
i)\ \sum_{i=r}^{n} iH_{k(ci+d)} \quad = \quad & \big(nH_{k(c(n+2)+d)} - \left(n+1+2n\,(-1)^{c}\right) H_{k(c(n+1)+d)} \\
& + \left(n+2\,(n+1)\,(-1)^{c}\right) H_{k(cn+d)} - (n+1)\, H_{k(c(n-1)+d)} \\
& - \left(r+1+2r\,(-1)^{c}\right) H_{k(c(r-1)+d)} + rH_{k(c(r-2)+d)} \\
& + \left(r+2\,(r-1)\,(-1)^{c}\right) H_{k(cr+d)} - (r-1)\, H_{k(c(r+1)+d)}\big) \\
& / \left(1 - V_{kc} + (-1)^{c}\right)^{2}, \qquad (2.13)
\end{aligned}
$$

$$
\begin{aligned}
& ii)\ \sum_{i=r}^{n} (-1)^{i-1}\, iH_{k(ci+d)} = \Big[\Big(n\,(-1)^{n+1} - 2\,(n+1)\,(-1)^{c+n}\Big)\ H_{k(cn+d)} \\
& - (n+1)\,(-1)^{n}\, H_{k(c(n-1)+d)} + \Big(2n\,(-1)^{c+n+1} - (n+1)\,(-1)^{n}\Big)\, H_{k(c(n+1)+d)} \\
& + n\,(-1)^{n+1}\, H_{k(c(n+2)+d)} - \Big((r-1)\,(-1)^{r} - 2r\,(-1)^{c+r-1}\Big)\, H_{k(c(r-1)+d)} + \\
& \Big(r\,(-1)^{r-1} - 2\,(r-1)\,(-1)^{c+r}\Big)\, H_{kcr+d} \\
& + \Big(r\,(-1)^{r-1}\Big)\, H_{k(c(r-2)+d)} - (r-1)\,(-1)^{r}\, H_{k(c(r+1)+d)}\big] \\
& / \left(1 + V_{kc} + (-1)^{c}\right)^{2}. \qquad (2.14)
\end{aligned}
$$

Proof. Considering

$$\sum_{i=r}^{n} ix^{i-1} = \frac{nx^{n+1} - (n+1)\,x^{n} - (r-1)\,x^{r} + rx^{r-1}}{(x-1)^{2}},$$

and using the Binet formula, Theorem 2 is proven. □

Corollary 1. *For odd positive integer m,*

$$\sum_{i=j+1}^{\frac{m-1}{2}} U_{k(2m-4i-1)} = \frac{U_k}{V_k\,(V_k^2+4)} \left[V_{k(2m-4j-3)} + V_k\right], \qquad (2.15)$$

$$\sum_{i=r}^{\frac{m-1}{2}} (-1)^{i}\, V_{k(m-2i)} = (-1)^{r}\, \frac{U_{k(m-2r+1)}}{U_k}, \qquad (2.16)$$

$$\sum_{i=0}^{\frac{m-1}{2}} (-1)^{i}\, V_{k(3(m-2i-1))} = (-1)^{\frac{m-1}{2}} + \frac{U_{3km}U_{3k}}{(V_k^2+1)^{2}}, \qquad (2.17)$$

$$\sum_{i=0}^{\frac{m-1}{2}} (-1)^{i-1}\, iV_{k(m-2i)} = \frac{(-1)^{\frac{m-1}{2}}\, V_k - V_{km}}{V_k^2+4}, \qquad (2.18)$$

Proof. Substituting $n = \frac{m-1}{2}$, $c = -4$, $d = 2m-1$ and $r = j+1$ in (2.11) gives us

$$\sum_{i=j+1}^{\frac{m-1}{2}} U_{k(2m-4i-1)} = \frac{-U_{k(2m-4j-5)} + U_{k(2m-4j-1)} + U_{3k} - U_k}{V_k^2\,(V_k^2+4)},$$

which, by (2.4), equals to

$$\frac{U_{2k}V_{k(2m-4j-3)}+U_kV_k^2}{V_k^2\left(V_k^2+4\right)}=\frac{U_k}{V_k\left(V_k^2+4\right)}\left[V_{k(2m-4j-3)}+V_k\right],$$

as claimed in (2.15). Using Lemma 1, the identities (2.16)-(2.18) can be proved in a similar way as the proof of (2.15). □

Corollary 2. *For even positive integer m,*

$$\begin{aligned}\sum_{i=j+1}^{\frac{m-2}{2}}U_{k(2m-4i-1)} &= \frac{U_k}{V_k\left(V_k^2+4\right)}\left[V_{k(2m-4j-3)}-V_k\right],\\ \sum_{i=r}^{\frac{m-2}{2}}(-1)^iV_{k(m-2i)} &= (-1)^r\frac{U_{k(m-2r+1)}}{U_k}-(-1)^{\frac{m}{2}},\\ \sum_{i=0}^{\frac{m-2}{2}}(-1)^iV_{k(3(m-2i-1))} &= \frac{U_{3km}}{U_k\left(V_k^2+1\right)},\\ \sum_{i=0}^{\frac{m-2}{2}}(-1)^{i-1}iV_{k(m-2i)} &= (-1)^{\frac{m}{2}}\frac{m}{2}+\frac{V_{km}-2(-1)^{\frac{m}{2}}}{V_k^2+4}.\end{aligned}$$

3. Sums of the Products of the Terms of $\{V_{kn}\}$

Define the sequence $\left\{S_n^k(m)\right\}_{n=0}^{\infty}$ in the following way: for $m>0$

$$S_0^k(m)=U_k,\quad S_1^k(m)=\sum_{i_1=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}(-1)^{i_1}V_{k(m-2i_1)} \tag{3.1}$$

and for $n>1$,

$$S_n^k(m)=\sum_{i_n=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\sum_{i_{n-1}=i_n+1}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\cdots\sum_{i_1=i_2+1}^{\left\lfloor\frac{m-1}{2}\right\rfloor}(-1)^{i_1+i_2+\ldots+i_n}\prod_{j=1}^{n}V_{k(m-2i_j)}. \tag{3.2}$$

Throughout this section, we shall frequently follow the organization of the work [11] while giving our results.

Theorem 3. *For odd positive integer m,*

$$S_1^k(m)=\sum_{i_1=0}^{\frac{m-1}{2}}(-1)^{i_1}V_{k(m-2i_1)}=\frac{U_{k(m+1)}}{U_k},$$

$$\begin{aligned}S_2^k(m) &= \sum_{i_2=0}^{\frac{m-1}{2}}\sum_{i_1=i_2+1}^{\frac{m-1}{2}}(-1)^{i_1+i_2}V_{k(m-2i_2)}V_{k(m-2i_1)}\\ &= \frac{m+1}{2}-\frac{1}{U_kU_{2k}}U_{km}U_{k(m+1)},\end{aligned}$$

and

$$\begin{aligned} S_3^k(m) &= \sum_{i_3=0}^{\frac{m-1}{2}} \sum_{i_2=i_3+1}^{\frac{m-1}{2}} \sum_{i_1=i_2+1}^{\frac{m-1}{2}} (-1)^{i_1+i_2+i_3} V_{k(m-2i_3)} V_{k(m-2i_2)} V_{k(m-2i_1)} \\ &= \binom{m-1}{2} \frac{U_{k(m+1)}}{U_k} - \frac{1}{U_k U_{2k} U_{3k}} U_{k(m+1)} U_{km} U_{k(m-1)}. \end{aligned}$$

Proof. *i*) Substituting $n = \frac{m-1}{2}$, $c = -2$, $d = m$ and $r = 0$ in (2.12), we get

$$S_1^k(m) = \sum_{i_1=0}^{\frac{m-1}{2}} (-1)^{i_1} V_{k(m-2i_1)} = \frac{V_{km} + (-1)^{\frac{m-1}{2}} V_k + V_{k(m+2)} + (-1)^{\frac{m-1}{2}} V_k}{(V_k^2+4)}.$$

Since $V_{-k} = V_k$ and by (2.1), we get

$$S_1^k(m) = \frac{V_{km} + V_{k(m+2)}}{(V_k^2+4)} = \sum_{i_1=0}^{\frac{m-1}{2}} (-1)^{i_1} V_{k(m-2i_1)} = \frac{U_{k(m+1)}}{U_k}.$$

ii) Using (2.6) and (2.16), we have

$$\begin{aligned} S_2^k(m) &= \sum_{i_2=0}^{\frac{m-1}{2}} \sum_{i_1=i_2+1}^{\frac{m-1}{2}} (-1)^{i_1+i_2} V_{k(m-2i_2)} V_{k(m-2i_1)} \\ &= -\frac{1}{U_k} \sum_{i_2=0}^{\frac{m-1}{2}} V_{k(m-2i_2)} U_{k(m-2i_2-1)} \\ &= \frac{1}{U_k} \sum_{i_2=0}^{\frac{m-1}{2}} \left(U_k - U_{k(2m-4i_2-1)} \right). \end{aligned}$$

Taking $j = -1$ in (2.15), we get

$$\sum_{i=0}^{\frac{m-1}{2}} U_{k(2m-4i-1)} = \frac{U_{k(m+1)} U_{km}}{U_{2k}}.$$

Then

$$S_2^k(m) = \frac{m+1}{2} - \frac{1}{U_k U_{2k}} U_{km} U_{k(m+1)}.$$

iii)Using (2.16) and (2.6), we write

$$\begin{aligned} S_3^k(m) &= \sum_{i_3=0}^{\frac{m-1}{2}} \sum_{i_2=i_3+1}^{\frac{m-1}{2}} \sum_{i_1=i_2+1}^{\frac{m-1}{2}} (-1)^{i_1+i_2+i_3} V_{k(m-2i_3)} V_{k(m-2i_2)} V_{k(m-2i_1)} \\ &= \frac{1}{U_k} \sum_{i_3=0}^{\frac{m-1}{2}} \sum_{i_2=i_3+1}^{\frac{m-1}{2}} (-1)^{i_3+1} V_{k(m-2i_3)} V_{k(m-2i_2)} U_{k(m-2i_2-1)} \\ &= \frac{1}{U_k} \sum_{i_3=0}^{\frac{m-1}{2}} \sum_{i_2=i_3+1}^{\frac{m-1}{2}} (-1)^{i_3+1} V_{k(m-2i_3)} \left(U_{k(2m-4i_2-1)} - U_k \right) \\ &= \frac{1}{U_k} \sum_{i_3=0}^{\frac{m-1}{2}} (-1)^{i_3+1} V_{k(m-2i_3)} \sum_{i_2=i_3+1}^{\frac{m-1}{2}} \left(U_{k(2m-4i_2-1)} - U_k \right). \end{aligned}$$

From (2.15), we have

$$\begin{aligned}
S_3^k(m) &= \frac{1}{U_k}\sum_{i_3=0}^{\frac{m-1}{2}}(-1)^{i_3+1}V_{k(m-2i_3)}\times \\
&\left(\frac{U_k}{V_k(V_k^2+4)}\left(V_{k(2m-4i_3-3)}+V_k\right)+\left(i_3-\frac{m-1}{2}\right)U_k\right) \\
&= \frac{1}{U_k}\sum_{i_3=0}^{\frac{m-1}{2}}(-1)^{i_3}V_{k(m-2i_3)}\times \\
&\left(-\frac{U_k}{V_k(V_k^2+4)}\left(V_{k(2m-4i_3-3)}+V_k\right)+\left(\frac{m-1}{2}-i_3\right)U_k\right) \\
&= -\frac{1}{V_k(V_k^2+4)}\sum_{i_3=0}^{\frac{m-1}{2}}(-1)^{i_3}V_{k(m-2i_3)}V_{k(2m-4i_3-3)} \\
&+\left(\frac{m-1}{2}-\frac{1}{V_k^2+4}\right)\sum_{i_3=0}^{\frac{m-1}{2}}(-1)^{i_3}V_{k(m-2i_3)} \\
&-\sum_{i_3=0}^{\frac{m-1}{2}}(-1)^{i_3}i_3V_{k(m-2i_3)}.
\end{aligned}$$

From (2.16)- (2.18) and (2.1), we get

$$\begin{aligned}
&S_3^k(m) \\
&= \frac{1}{V_k(V_k^2+4)}\left(\frac{U_{k(m-2)}}{U_k}+(-1)^{\frac{m-1}{2}}\left(V_k^2+1\right)-\frac{U_{3km}U_{3k}}{U_k^2(V_k^2+1)^2}-(-1)^{\frac{m-1}{2}}\right) \\
&+\left(\frac{m-1}{2}-\frac{1}{V_k^2+4}\right)\left(\frac{V_{k(m+2)}+V_{km}}{V_k^2+4}\right)-\frac{1}{V_k^2+4}\left((-1)^{\frac{m-1}{2}}V_k-V_{km}\right) \\
&= -\frac{U_{3km}}{U_{2k}(V_k^2+4)(V_k^2+1)}+\left(\frac{m-1}{2}-\frac{1}{V_k^2+4}\right)\frac{U_{k(m+1)}}{U_k} \\
&+\frac{1}{V_k^2+4}\left(\frac{U_{k(m-2)}}{U_kV_k}+\frac{V_{km}U_{2k}}{U_kV_k}\right) \\
&= -\frac{U_{3km}}{U_{2k}(V_k^2+4)(V_k^2+1)}+\left(\frac{m-1}{2}-\frac{1}{V_k^2+4}\right)\frac{U_{k(m+1)}}{U_k}+\frac{U_{k(m+2)}}{(V_k^2+4)U_{2k}} \\
&= \left(\frac{m-1}{2}\right)\frac{U_{k(m+1)}}{U_k}-\frac{U_{3km}}{U_{2k}(V_k^2+4)(V_k^2+1)}+\frac{U_{km}}{(V_k^2+4)U_{2k}} \\
&= \left(\frac{m-1}{2}\right)\frac{U_{k(m+1)}}{U_k}-\frac{1}{U_kU_{2k}U_{3k}}U_{k(m+1)}U_{km}U_{k(m-1)}.
\end{aligned}$$

Thus we have the conclusion. □

Theorem 4. *For even positive integer m,*

$$S_1^k(m)=\sum_{i_1=0}^{\frac{m-2}{2}}(-1)^{i_1}V_{k(m-2i_1)}=\frac{U_{k(m+1)}}{U_k}-(-1)^{\frac{m}{2}},$$

$$\begin{aligned} S_2^k(m) &= \sum_{i_2=0}^{\frac{m-2}{2}} \sum_{i_1=i_2+1}^{\frac{m-2}{2}} (-1)^{i_1+i_2} V_{k(m-2i_2)} V_{k(m-2i_1)} \\ &= \frac{m-2}{2} + (-1)^{\frac{m}{2}} \frac{U_{k(m+1)}}{U_k} + \frac{U_{km}U_{k(m+1)}}{U_k U_{2k}}, \end{aligned}$$

$$\begin{aligned} S_3^k(m) &= \sum_{i_3=0}^{\frac{m-2}{2}} \sum_{i_2=i_3+1}^{\frac{m-2}{2}} \sum_{i_1=i_2+1}^{\frac{m-2}{2}} (-1)^{i_1+i_2+i_3} V_{k(m-2i_3)} V_{k(m-2i_2)} V_{k(m-2i_1)} \\ &= \left(\frac{m-4}{2}\right)\left((-1)^{\frac{m}{2}} - \frac{U_{k(m+1)}}{U_k}\right) \\ &\quad + U_{km}U_{k(m+1)}\left((-1)^{\frac{m}{2}} - \frac{U_{k(m-1)}}{U_k U_{2k} U_{3k}}\right). \end{aligned}$$

Proof. The proof is similar as the proof of Theorem 3. □

In [14], Stanica gave the generating function for powers of the terms of sequence $\{W_n\}$, $W(m,x) = \sum_{i=0}^{\infty} W_i^m x^i$, as follows:

Theorem 5. *For $n \geq 0$ and odd positive integer m,*

$$\begin{aligned} W(m,x) &= \sum_{i=0}^{\frac{m-1}{2}} (-AB)^i \binom{m}{i} \times \\ &\quad \frac{A^{m-2i} - B^{m-2i} + (-q)^i \left(B^{m-2i}\alpha^{m-2i} - A^{m-2i}\beta^{m-2i}\right) x}{1 - (-q)^i V_{m-2i} x - q^m x^2} \end{aligned} \tag{3.3}$$

and for even positive integer m,

$$\begin{aligned} W(m,x) &= \sum_{i=0}^{\frac{m}{2}-1} (-AB)^i \binom{m}{i} \times \\ &\quad \frac{B^{m-2i} + A^{m-2i} - (-q)^i \left(B^{m-2i}\alpha^{m-2i} + A^{m-2i}\beta^{m-2i}\right) x}{1 - (-q)^i V_{m-2i} x + q^m x^2} \\ &\quad + \binom{m}{\frac{m}{2}} \frac{(-AB)^{\frac{m}{2}}}{1 - (-q)^{\frac{m}{2}} x}. \end{aligned} \tag{3.4}$$

Theorem 6. *For $n \geq 0$ and odd positive integer m,*

$$S_n^k(m) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\lfloor \frac{n}{2} \rfloor - i} \theta(i,m,n) \begin{bmatrix} m+1 \\ n-2i \end{bmatrix}_{U_k}, \tag{3.5}$$

and for even positive integer m,

$$S_n^k(m) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-2i} (-1)^{i+n\left(\frac{m}{2}+1\right)+\frac{j}{2}(j+m+1)} \theta(i,m,n) \begin{bmatrix} m+1 \\ j \end{bmatrix}_{U_k},$$

where $\theta(i,m,n) = \binom{\lfloor \frac{m+1}{2} \rfloor - n + i}{i} + \binom{\lfloor \frac{m+1}{2} \rfloor - n + i - 1}{i-1}$.

Proof. We give the proof for odd integer m. From (3.3), we write

$$U_{kn}(m,x) = \left(\frac{U_k}{\sqrt{V_k^2+4}}\right)^{\frac{m-1}{2}} \sum_{j=0}^{\frac{m-1}{2}} \binom{m}{j} \frac{U_{k(m-2j)}x}{1-(-1)^j V_{k(m-2j)}x - x^2}. \tag{3.6}$$

Relation (3.6), which hold for an odd m lead to

$$D_{m+1}^k(x) = \prod_{j=0}^{\frac{m-1}{2}} \left(1-(-1)^j V_{k(m-2j)}x - x^2\right) = \sum_{i=0}^{m+1} d_{m+1,i}x^i,$$

where $d_{m+1,i} = (-1)^{\frac{i(i+1)}{2}} \begin{bmatrix} m+1 \\ i \end{bmatrix}_{U_k}$. After multiplication of all factors in $D_{m+1}^k(x)$, it follows that

$$d_{m+1,0} = S_0^k(m), \; d_{m+1,i} = \sum_{l=0}^{\lfloor \frac{i}{2} \rfloor} \binom{\frac{m+1}{2}-(i-2l)}{l} (-1)^{i+l} S_{i-2l}^k(m),$$

where $i = 1, 2, ..., m+1$. Since $\binom{n}{m} = (-1)^m \binom{m-n-1}{m}$, we rewrite the last identity for $n > 0$ as follows:

$$\begin{aligned} d_{m+1,2n-1} &= -\sum_{l=1}^{n} \binom{n+i-\frac{m+5}{2}}{n-i} S_{2i-1}^k(m) \\ d_{m+1,2(n-1)} &= \sum_{l=1}^{n} \binom{n+i-\frac{m+7}{2}}{n-i} S_{2(i-1)}^k(m). \end{aligned}$$

By the binomial inversion theorem (for more details, see [10]),

$$a_n = \sum_{l=1}^{n} \binom{n+i+r}{n-i} b_i \tag{3.7}$$

holds if and only if

$$b_n = \sum_{l=1}^{n} (-1)^{i+n} \left(\binom{2n+r}{n-i} - \binom{2n+r}{n-i-1}\right) a_i,$$

where r is any integer. After this, by taking $a_n = d_{m+1,2n-1}$, $b_i = -S_{2i-1}^k$, $r = -\frac{m+5}{2}$ in (3.7), we obtain

$$\begin{aligned} S_{2n-1}^k(m) &= \sum_{i=1}^{n} (-1)^{-i+n+1} \left[\binom{2n-\frac{m+5}{2}}{n-i} - \binom{2n-\frac{m+5}{2}}{n-i-1}\right] d_{m+1,2i-1} \\ &= \sum_{i=1}^{n} (-1) \left[\binom{-n-i+\frac{m+3}{2}}{n-i} - \binom{-n-i+\frac{m+1}{2}}{n-i-1}\right] d_{m+1,2i-1} \\ &= \sum_{i=1}^{n} (-1)^{(2i-1)i+1} \left[\binom{-n-i+\frac{m+3}{2}}{n-i} - \binom{-n-i+\frac{m+1}{2}}{n-i-1}\right] \begin{bmatrix} m+2 \\ 2i-1 \end{bmatrix}_{U_k} \end{aligned} \tag{3.8}$$

Similarly if we take $a_n = d_{m+1,2(n-1)}$, $b_i = -S_{2(i-1)}^k$, $r = -\frac{m+7}{2}$ in (3.7), then we get

$$S_{2(n-1)}^k(m) = \sum_{i=1}^{n} (-1)^{(2i-1)i+1} \left[\binom{-n-i+\frac{m+5}{2}}{n-i} - \binom{-n-i+\frac{m+5}{2}}{n-i-1}\right] \begin{bmatrix} m+1 \\ 2(i-1) \end{bmatrix}_{U_k}. \tag{3.9}$$

Combining (3.8) and (3.9), we obtain (3.5).

For the case m is even, the proof is complete by considering (3.4) similar to the proof of $S_n^k(m)$ for odd m. $\square$

Lemma 2. *For positive integer s and positive even integer m, it holds*

$$\begin{bmatrix} m+1 \\ s \end{bmatrix}_{U_k} + (-1)^{\frac{m}{2}+s} \begin{bmatrix} m+1 \\ s-1 \end{bmatrix}_{U_k} = \frac{U_{k\left(\frac{m}{2}+1-s\right)}}{U_{k\left(\frac{m}{2}+1\right)}} \begin{bmatrix} m+2 \\ s \end{bmatrix}_{U_k}. \tag{3.10}$$

For positive integer s and positive odd integer m, it holds

$$\begin{bmatrix} m \\ s \end{bmatrix}_{U_k} + (-1)^{\frac{m-1}{2}+s} \begin{bmatrix} m \\ s-1 \end{bmatrix}_{U_k} = \frac{U_{k\left(\frac{m+1}{2}-s\right)}}{U_{k\left(\frac{m+1}{2}\right)}} \begin{bmatrix} m+1 \\ s \end{bmatrix}_{U_k}.$$

Proof. For even positive integer m and positive integer s, substituting $m = \frac{m}{2}-s+1$, $n = \frac{m}{2}+1$ in (2.9), we get

$$U_{k\left(\frac{m}{2}-s+1\right)} V_{k\left(\frac{m}{2}+1\right)} = U_{k(m-s+2)} + (-1)^{\frac{m}{2}+s} U_{ks}. \tag{3.11}$$

From (3.11), we get

$$\begin{aligned}
&\begin{bmatrix} m+1 \\ s \end{bmatrix}_{U_k} + (-1)^{\frac{m}{2}+s} \begin{bmatrix} m+1 \\ s-1 \end{bmatrix}_{U_k} \\
= \; & \frac{U_{k(m+1)}U_{km}...U_{k(m-s+2)}}{(U_kU_{2k}...U_{ks})} + (-1)^{\frac{m}{2}+s} \frac{U_{k(m+1)}U_{km}...U_{k(m-s+3)}}{\left(U_kU_{2k}...U_{k(s-1)}\right)} \\
= \; & \frac{U_{k(m+2)}U_{k(m+1)}...U_{k(m-s+3)}}{(U_kU_{2k}...U_{ks})} \left[\frac{U_{k(m-s+2)} + (-1)^{\frac{m}{2}+s} U_{ks}}{U_{k(m+2)}} \right] \\
= \; & \frac{U_{k(m+2)}U_{k(m+1)}...U_{k(m-s+3)}}{(U_kU_{2k}...U_{ks})} \left[\frac{U_{k\left(\frac{m}{2}-s+1\right)} V_{k\left(\frac{m}{2}+1\right)}}{U_{k(m+2)}} \right] \\
= \; & \frac{U_{k(m+2)}U_{k(m+1)}...U_{k(m-s+3)}}{(U_kU_{2k}...U_{ks})} \frac{U_{k\left(\frac{m}{2}-s+1\right)}}{U_{k\left(\frac{m}{2}+1\right)}} \\
= \; & \frac{U_{k\left(\frac{m}{2}+1-s\right)}}{U_{k\left(\frac{m}{2}+1\right)}} \begin{bmatrix} m+2 \\ s \end{bmatrix}_{U_k}.
\end{aligned}$$

Thus the proof is complete. $\square$

Define

$$\sigma_m^k(t) := \sum_{j=0}^{m-t} (-1)^{\frac{j}{2}(j+m+1)} \begin{bmatrix} m+1 \\ j \end{bmatrix}_{U_k}, \tag{3.12}$$

where m is even positive integer and t is any integer.

Lemma 3. *For even positive integer m and any integer t,*

i) $\sigma_m^k(t) = 0,$ *for* $t \leq -1$ *or* $t \geq m+1$

ii) $\sigma_m^k(m-t) = \sigma^k(t)$

iii) $\sigma_m^k(0) = 1,\ \sigma_m^k(1) = 1 + \frac{1}{U_k}(-1)^{\frac{m-2}{2}} U_{k(m+1)}$

$\sigma_m^k(2) = 1 - \frac{1}{U_kU_{2k}} V_{k\left(\frac{m+2}{2}\right)} U_{k(m+1)} U_{k\left(\frac{m-2}{2}\right)}$

$\sigma_m^k(3) = 1 - \frac{1}{U_kU_{2k}U_{3k}} (-1)^{\frac{m}{2}} U_{k(m+1)}$

$\times \left(U_{2k}U_{3k} - V_{k\left(\frac{m+2}{2}\right)} U_{km} U_{k\left(\frac{m-4}{2}\right)}\right).$

Proof. In order to get the proof of (i) and (ii), one can follow the proof way of Lemma 16 i) and ii) in [12] since they are similar statements.

(iii) Identities for $\sigma_m^k(0)$ and $\sigma_m^k(1)$ are directly implied by $\sigma_m^k(-1) = 0$. Using $ii)$ and (3.12), we have

$$\begin{aligned}
\sigma_m^k(2) &= \sum_{j=0}^{2} (-1)^{\frac{j}{2}(j+m+1)} \begin{bmatrix} m+1 \\ j \end{bmatrix}_{U_k} \\
&= 1 + (-1)^{\frac{(m+2)}{2}} \frac{U_{k(m+1)}}{U_k} - (-1)^m \frac{U_{k(m+1)}U_{km}}{U_k U_{2k}} \\
&= 1 - \frac{U_{k(m+1)}}{U_k U_{2k}} \left(U_{km} + (-1)^{\frac{m}{2}} \right) \\
&= 1 - \frac{U_{k(m+1)}}{U_k U_{2k}} \left(U_{k\left(\frac{m}{2}-1\right)} V_{k\left(\frac{m}{2}+1\right)} \right),
\end{aligned}$$

and

$$\begin{aligned}
\sigma_m^k(3) &= \sigma_m^k(2) - (-1)^{\frac{(m-2)}{2}} \frac{U_{k(m-1)}U_{km}U_{k(m+1)}}{U_k U_{2k} U_{3k}} \\
&= 1 - (-1)^{\frac{m}{2}} \frac{U_{k(m+1)}}{U_k} - \frac{U_{k(m+1)}U_{km}}{U_k U_{2k}} + (-1)^{\frac{m}{2}} \frac{U_{k(m-1)}U_{km}U_{k(m+1)}}{U_k U_{2k} U_{3k}} \\
&= 1 - (-1)^{\frac{m}{2}} \frac{U_{k(m+1)}}{U_k} \left(U_{2k}U_{3k} - U_{k(m-1)}U_{km} + (-1)^{\frac{m}{2}} U_{3k}U_{km} \right) \\
&= 1 - (-1)^{\frac{m}{2}} \frac{U_{k(m+1)}}{U_k} \left(U_{2k}U_{3k} - U_{km} \left(U_{k(m-1)} - (-1)^{\frac{m}{2}} U_{3k} \right) \right) \\
&= 1 - (-1)^{\frac{m}{2}} \frac{U_{k(m+1)}}{U_k} \left(U_{2k}U_{3k} - U_{km} U_{k\left(\frac{m-4}{2}\right)} V_{k\left(\frac{m+2}{2}\right)} \right),
\end{aligned}$$

as desired. □

Lemma 4. *For even positive integer m and any integer t,*

$$\sigma_m^k(t) - \sigma_m^k(t-2) = (-1)^{\frac{t}{2}(t+m+1)} \begin{bmatrix} m+2 \\ t \end{bmatrix}_{U_k} \frac{U_{k\left(\frac{m}{2}-t+1\right)}}{U_{k\left(\frac{m}{2}+1\right)}}.$$

Proof. For $t < 2$, the claim follows from the definition of Fibonomial coefficients and Lemma 3. For $m \geq 2$, we have

$$\begin{aligned}
& \sigma_m^k(t) - \sigma_m^k(t-2) \\
={}& \sigma_m^k(m-t) - \sigma_m^k(m-t+2) \\
={}& \sum_{j=0}^{t} (-1)^{\frac{j}{2}(j+m+1)} \begin{bmatrix} m+1 \\ j \end{bmatrix}_{U_k} - \sum_{j=0}^{t-2} (-1)^{\frac{j}{2}(j+m+1)} \begin{bmatrix} m+1 \\ j \end{bmatrix}_{U_k} \\
={}& (-1)^{\frac{t}{2}(t+m+1)} \left(\begin{bmatrix} m+1 \\ t \end{bmatrix}_{U_k} + (-1)^{\frac{m}{2}+t} \begin{bmatrix} m+1 \\ t-1 \end{bmatrix}_{U_k} \right).
\end{aligned}$$

By Lemma 2, the proof is complete. □

Lemma 5. *For even positive integer m and any integer t,*

$$\sigma_m^k(t)-\sigma_m^k(t-4) = (-1)^{\frac{t}{2}(t+m+1)}\begin{bmatrix} m+4 \\ t \end{bmatrix}_{U_k}\times \frac{U_{k\left(\frac{m}{2}-t+2\right)}}{U_{k\left(\frac{m}{2}+1\right)}U_{k(m+3)}U_{k(m+4)}}\omega(t,m),$$

where $\omega(t,m)=U_{k\left(\frac{m}{2}+1-t\right)}V_{k\left(\frac{m}{2}+2-t\right)}U_{k(m+3)}-V_kU_{km}U_{k(m-1)}$.

Proof. By Lemma 4, we have for any integer t,

$$\begin{aligned}
& \sigma_m^k(t)-\sigma_m^k(t-4) \\
= {} & \sigma_m^k(t)-\sigma_m^k(t-2)+\sigma_m^k(t-2)-\sigma_m^k(t-4) \\
= {} & (-1)^{\frac{t}{2}(t+m+1)}\begin{bmatrix} m+2 \\ t \end{bmatrix}_{U_k}\frac{U_{k\left(\frac{m}{2}-t+1\right)}}{U_{k\left(\frac{m}{2}+1\right)}} \\
& +(-1)^{\frac{t-2}{2}(t+m-1)}\begin{bmatrix} m+2 \\ t-2 \end{bmatrix}_{U_k}\frac{U_{k\left(\frac{m}{2}-t+3\right)}}{U_{k\left(\frac{m}{2}+1\right)}} \\
= {} & (-1)^{\frac{t}{2}(t+m+1)}\frac{1}{U_{k\left(\frac{m}{2}+1\right)}}\left(U_{k\left(\frac{m}{2}-t+1\right)}\begin{bmatrix} m+2 \\ t \end{bmatrix}_{U_k}-U_{k\left(\frac{m}{2}-t+3\right)}\begin{bmatrix} m+2 \\ t-2 \end{bmatrix}_{U_k}\right) \\
= {} & (-1)^{\frac{t}{2}(t+m+1)}\begin{bmatrix} m+4 \\ t \end{bmatrix}_{U_k}\frac{1}{U_{k(m+3)}U_{k(m+4)}}\times \\
& \left(U_{k\left(\frac{m}{2}-t+1\right)}U_{k(m+3-t)}-U_{k\left(\frac{m}{2}-t+3\right)}U_{kt}U_{k(t-1)}\right).
\end{aligned}$$

From (2.10), we get

$$\begin{aligned}
& \sigma_m^k(t)-\sigma_m^k(t-4) \\
= {} & (-1)^{\frac{t}{2}(t+m+1)}\begin{bmatrix} m+4 \\ t \end{bmatrix}_{U_k}\frac{1}{U_{k(m+3)}U_{k(m+4)}}\times \\
& \left(U_{k\left(\frac{m}{2}-t+1\right)}\left(U_{k(m+4-2t)}U_{k(m+3)}+U_{kt}U_{k(t-1)}\right)-U_{k\left(\frac{m+6}{2}-t\right)}U_{kt}U_{k(t-1)}\right) \\
= {} & (-1)^{\frac{t}{2}(t+m+1)}\begin{bmatrix} m+4 \\ t \end{bmatrix}_{U_k}\frac{1}{U_{k(m+3)}U_{k(m+4)}}\times \\
& \left(U_{k\left(\frac{m}{2}-t+1\right)}U_{k(m+4-2t)}U_{k(m+3)}-V_kU_{k\left(\frac{m}{2}+2-t\right)}U_{kt}U_{k(t-1)}\right) \\
= {} & (-1)^{\frac{t}{2}(t+m+1)}\begin{bmatrix} m+4 \\ t \end{bmatrix}_{U_k}\frac{U_{k\left(\frac{m}{2}-t+2\right)}}{U_{k(m+3)}U_{k(m+4)}}\times \\
& \left(V_{k\left(\frac{m}{2}+2-t\right)}U_{k(m+4-2t)}U_{k(m+3)}-V_kU_{kt}U_{k(t-1)}\right),
\end{aligned}$$

as claimed. □

Theorem 7. *For any integer m,*

$$\begin{aligned}
&\sum_{j=0}^{t}(-1)^{\frac{j}{2}(j+m+1)}\begin{bmatrix} m+1 \\ j \end{bmatrix}_{U_k} \\
= \; &\frac{(-1)^{\frac{t}{2}(t+m+1)}}{U_{k\left(\frac{m}{2}+1\right)}U_{k(m+3)}U_{k(m+4)}}\sum_{i=0}^{\left\lfloor\frac{t}{4}\right\rfloor}\begin{bmatrix} m+4 \\ t-4i \end{bmatrix}_{U_k} U_{k\left(\frac{m}{2}+2-t+4i\right)} \times \\
&\left(U_{k\left(\frac{m}{2}+1-t+4i\right)}V_{k\left(\frac{m}{2}+2-t+4i\right)}U_{k(m+3)} - V_k U_{k(t-4i)}U_{k(t-4i-1)}\right).
\end{aligned}$$

Proof. By Lemma 5, we can write

$$\begin{aligned}
&\sum_{i=0}^{\left\lfloor\frac{t}{4}\right\rfloor}\left[\sigma_m^k(t-4i)-\sigma_m^k(t-4(i+1))\right] \\
= \; &\sigma_m^k(t)-\sigma_m^k\left(t-4\left(\left\lfloor\frac{t}{4}\right\rfloor+1\right)\right).
\end{aligned}$$

From Lemma 3, we have

$$\sigma_m^k(t)=\sum_{i=0}^{\left\lfloor\frac{t}{4}\right\rfloor}\left[\sigma_m^k(t-4i)-\sigma_m^k(t-4(i+1))\right].$$

Again by Lemma 5, we get

$$\begin{aligned}
&\sigma_m^k(t)=\sum_{i=0}^{\left\lfloor\frac{t}{4}\right\rfloor}(-1)^{\frac{t-4i}{2}(t-4i+m+1)}\begin{bmatrix} m+4 \\ t-4i \end{bmatrix}_{U_k}\frac{U_{k\left(\frac{m}{2}+2-t+4i\right)}}{U_{k\left(\frac{m}{2}+1\right)}U_{k(m+3)}U_{k(m+4)}} \times \\
&\left(U_{k\left(\frac{m}{2}+1-t+4i\right)}V_{k\left(\frac{m}{2}+2-t+4i\right)}U_{k(m+3)} - V_k U_{k(t-4i)}U_{k(t-4i-1)}\right) \\
= \; &\frac{(-1)^{\frac{t}{2}(t+m+1)}}{U_{k\left(\frac{m}{2}+1\right)}U_{k(m+3)}U_{k(m+4)}}\sum_{i=0}^{\left\lfloor\frac{t}{4}\right\rfloor}\begin{bmatrix} m+4 \\ t-4i \end{bmatrix}_{U_k} \times \\
&U_{k\left(\frac{m}{2}+2-t+4i\right)}\left(U_{k\left(\frac{m}{2}+1-t+4i\right)}V_{k\left(\frac{m}{2}+2-t+4i\right)}U_{k(m+3)} - V_k U_{k(t-4i)}U_{k(t-4i-1)}\right),
\end{aligned}$$

as claimed. □

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