A FORMULA FOR THE GENERATING FUNCTIONS OF POWERS OF HORADAM'S SEQUENCE WITH ADDITIONAL 2 PARAMETERS

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Abstract. Recently Mansour [3] give a formula for the generating functions of powers of Horadam's sequence. In this short paper, we give a generalization of this formula by adding two parameters.

1. Introduction

The second-order linear recurrence sequence $\{W_n(a,b;p,q)\}$, or briefly $\{W_n\}$, is defined by

$$W_{n+1} = pW_n + qW_{n-1}, \qquad W_0 = a, W_1 = b \tag{1.1}$$

where a, b and p, q are arbitrary real numbers for $n > 0$ [1, 2]. The Binet formula for the sequence $\{W_n\}$ is

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},$$

where $A = b - a\beta$ and $B = b - a\alpha$.

When $a = 0$, $b = 1$, and, $a = 2$, $b = 1$, we denote W_n by U_n and V_n, respectively. If we take $p = 1, q = 1$, then $U_n = F_n$ (nth Fibonacci number) and $V_n = L_n$ (nth Lucas number).

From [8], it is known that for $r > 0, n > 0$, the sequence $\{W_n\}$ satisfies the following recursion

$$W_{r(n+2)} = V_r W_{r(n+1)} - (-q)^r W_{rn}.$$

In [4], Riordan found the generating function for powers of Fibonacci numbers. He proved that the generating function $S_k(x) = \sum_{n\geq 0} F_n^k x^n$ satisfies the recurrence relation

$$\left(1 - a_k x + (-1)^k x^2\right) S_k(x) = 1 + kx \sum_{j=1}^{[k/2]} (-1)^j \frac{a_{kj}}{j} S_{k-2j}\left((-1)^j x\right),$$

for $k \geq 1$, where $a_1 = 1,\ a_2 = 3,\ a_s = a_{s-1} + a_{s-2}$ for $s \geq 3$, and $\left(1 - x - x^2\right)^{-j} = \sum_{k\geq 0} a_{kj} x^{k-2j}$. In [2], Horadam gave a recurrence relation for $H_k(x)$ (see also [5]). Recently, Haukkanen [6] studied linear combinations of Horadam's sequences and the generating function of the ordinary product of two of Horadam's sequences.

Recently, Mansour [3] studied about the generating function for powers of Horadam's sequence given by $H_k(x; a, b, p, q) = H_k(x) = \sum_{n\geq 0} W_n^k x^n$. Then he

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showed that the generating function $H_k(x)$ can be expressed the ratio of two k by k determinants as well as he gave some applications for it.

In this study, we consider the generating function for powers of Horadam's sequence defined by

$$\Re_{k,t,r}(x;a,b,p,q) = \Re_{k,t,r}(x) = \sum_{n\geq t} W_{rn}^k x^n.$$

Then we follow and use the method of Mansour and then we shall give a ratio to express the generating function $\Re_{k,t,r}(x)$. Moreover we give applications of our results.

2. The Main Result

In order to express the $\Re_{k,t,r}(x)$ as a ratio of two determinants, first we define two k by k matrices. Let $\Delta_{k,r} = (\Delta_{k,r}(i,j))_{1\leq i,j\leq k} = \Delta_{k,r}(p,q)$ be the $k\times k$ matrix have the form

$$\Delta_{k,r}(p,q) = \begin{bmatrix} 1-xv_r^k - x^2(-(-q)^r)^k & -xv_r^{k-1}(-(-q)^r)\binom{k}{1} & \cdots & -xv_r(-(-q)^r)^{k-1}\binom{k}{k-1} \\ -v_r^{k-1}x & 1-xv_r^{k-2}(-(-q)^r)\binom{k-1}{1} & \cdots & -x(-(-q)^r)^{k-1}\binom{k-1}{k-1} \\ -v_r^{k-2}x & -xv_r^{k-3}(-(-q)^r)\binom{k-2}{1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -v_r^2 x & -xv_r(-(-q)^r)\binom{2}{1} & \cdots & 0 \\ -v_r x & -x(-(-q)^r)\binom{1}{1} & \cdots. & 1 \end{bmatrix}$$

and let $\delta_{k,t,r} = \delta_{k,t,r}(a,b,p,q)$ be the $k\times k$ matrix have the form

$$\delta_{k,t,r}(a,b,p,q) = \begin{bmatrix} w_{rt}^k + x\ g_k & -xv_r^{k-1}(-(-q)^r)\binom{k}{1} & \cdots & -xv_r(-(-q)^r)^{k-1}\binom{k}{k-1} \\ x\ g_{k-1} & 1-xv_r^{k-2}(-(-q)^r)\binom{k-1}{1} & \cdots & -x(-(-q)^r)^{k-1}\binom{k-1}{k-1} \\ x\ g_{k-2} & -xv_r^{k-3}(-(-q)^r)\binom{k-2}{1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ x\ g_2 & -xv_r(-(-q)^r)\binom{2}{1} & \cdots & 0 \\ x\ g_1 & -x(-(-q)^r)\binom{1}{1} & \cdots & 1 \end{bmatrix},$$

where $g_j = \left(w_{r(t+1)}^j - v_r^j w_{rt}^j\right) w_{rt}^{k-j}$ for all $j = 1, 2, \ldots k$.

Stanica [7] found the generating function of powers of terms of $\{W_n\}$ given by (1.1), $\sum_{n=0}^{\infty} W_n^k x^n$. Considering this result, we give the following result for the generating function

$$\Re_{k,t,r}(x) = \sum_{n=t}^{\infty} W_{rn}^k x^n$$

as the following Lemma 1.

Lemma 1. *For odd k,*

$$\begin{aligned}\Re_{k,t,r}(x) &= \frac{1}{(\alpha-\beta)^k}\sum_{j=0}^{\frac{k-1}{2}}(-AB)^j\binom{k}{j}\\ &\times\frac{A^{k-2j}-B^{k-2j}+(-q)^{rj}\left(B^{k-2j}\alpha^{r(k-2j)}-A^{k-2j}\beta^{r(k-2j)}\right)x}{1-(-q)^{rj}V_{r(k-2j)}x-q^{rk}x^2}\\ &-\sum_{n=0}^{t-1}W_{rn}^k x^n\end{aligned}$$

and for even k,

$$\begin{aligned}\Re_{k,t,r}(x) &= \frac{1}{(\alpha-\beta)^k}\sum_{j=0}^{\frac{k}{2}-1}(-AB)^j\binom{k}{j}\\ &\times\frac{A^{k-2j}+B^{k-2j}-(-q)^{rj}\left(B^{k-2j}\alpha^{r(k-2j)}+A^{k-2j}\beta^{r(k-2j)}\right)x}{1-(-q)^{rj}V_{r(k-2j)}x+q^{rk}x^2}\\ &+\binom{k}{k/2}\frac{(-AB)^{k/2}}{1-(-q)^{k/2}x}-\sum_{n=0}^{t-1}W_{rn}^k x^n.\end{aligned}$$

Proof. The proof easily follows from [7]. □

For further use, we define a family $\{A_{k,d,t,r}\}_{d=1}^{k}$ of generating functions by

$$A_{k,d,t,r}(x)=\sum_{n=t}^{\infty}W_{rn}^{k-d}W_{r(n+1)}^{d}x^{n+1}. \tag{2.1}$$

Now we give two relations between the generating functions $A_{k,d,t,r}(x)$ and $\Re_{k,t,r}(x)$.

Lemma 2. *For $k\geq 1$, positive integer r and non-negative integer t,*

$$\begin{aligned}&\left(1-V_r^k x+(-(-q)^r)^k x^2\right)\Re_{k,t,r}(x)-x\sum_{j=1}^{k-1}\binom{k}{j}(-(-q)^r)^j V_r^{k-j}A_{k,k-j,t,r}(x)\\ &\qquad= W_{rt}^k x^t+\left(W_{r(t+1)}^k-V_r^k W_{rt}^k\right)x^{t+1}.\end{aligned}$$

Proof. Using the binomial theorem, we get

$$\begin{aligned}W_{r(n+2)}^k &= \left(V_rW_{r(n+1)}-(-q)^r W_{rn}\right)^k\\ &= V_r^k W_{r(n+1)}^k+\sum_{j=1}^{k-1}\binom{k}{j}(-(-q)^r)^j V_r^{k-j}W_{r(n+1)}^{k-j}W_{rn}^j+(-(-q)^r)^k W_{rn}^k.\end{aligned}$$

Multiplying by x^{n+2} and summing over all $n\geq t$, using definition (2.1), we get

$$\begin{aligned}x^{n+2}W_{r(n+2)}^k &= x^{n+2}V_r^k W_{r(n+1)}^k+x^{n+2}\sum_{j=1}^{k-1}\binom{k}{j}(-(-q)^r)^j V_r^{k-j}W_{r(n+1)}^{k-j}W_{rn}^j\\ &+x^{n+2}(-(-q)^r)^k W_{rn}^k\end{aligned}$$

and so

$$\begin{aligned}
&\Re_{k,t,r}(x) - W_{rt}^k x^t - W_{r(t+1)}^k x^{t+1} \\
= \; & x V_r^k \Re_{k,t,r}(x) - x^{t+1} V_r^k W_{rt}^k \\
& + x \sum_{j=1}^{k-1} \binom{k}{j} \left(-(-q)^r\right)^j V_r^{k-j} A_{k,k-j,r}(x) + x^2 \left(-(-q)^r\right)^k \Re_{k,t,r}(x),
\end{aligned}$$

which, by a simple arrangement, completes the proof. □

Lemma 3. *For any $k \geq 1$, positive integer r, non-negative integer t, and $d \geq t+1$,*

$$\begin{aligned}
A_{k,d,t,r}(x) - W_{rt}^{k-d} W_{r(t+1)}^{k-d} &= V_r^d \left(\Re_{k,t,r}(x) - W_{rt}^k\right) \\
&+ x \sum_{j=1}^{d} \binom{d}{j} \left(-(-q)^r\right)^j V_r^{d-j} A_{k,k-j,t,r}(x).
\end{aligned}$$

Proof. Using the binomial theorem, we have

$$\begin{aligned}
W_{rn}^{k-d} W_{r(n+1)}^d &= W_{rn}^{k-d} \left(V_r W_{rn} - (-q)^r W_{r(n-1)}\right)^d \\
&= W_{rn}^{k-d} \sum_{j=0}^{d} \binom{d}{j} V_r^{d-j} \left(-(-q)^r\right)^j W_{rn}^{d-j} W_{r(n-1)}^j.
\end{aligned}$$

Multiplying by x^{n+1} and summing over all $n \geq t+1$, we obtain the claimed result:

$$\begin{aligned}
A_{k,d,t,r}(x) - W_{rt}^{k-d} W_{r(t+1)}^{k-d} &= V_r^d \left(\Re_{k,t,r}(x) - W_{rt}^k\right) \\
&\quad + x \sum_{j=1}^{d} \binom{d}{j} \left(-(-q)^r\right)^j V_r^{d-j} A_{k,k-j,t,r}(x).
\end{aligned}$$

□

Now we shall mention our main result:

Theorem 1. *For any $k \geq 1$, positive integer r, non-negative integer t, the generating function $\Re_{k,t,r}(x)$ is given by*

$$\frac{\det(\delta_{k,t,r})}{\det(\Delta_{k,r})}. \tag{2.2}$$

Proof. By using the above lemmas together with definitions, we obtain

$$\Delta_{k,r} \left[\Re_{k,t,r}(x), A_{k,k-1,t,r}(x), A_{k,k-2,t,r}(x), \ldots A_{k,1,t,r}(x)\right]^T = \upsilon_{k,t,r}$$

where $\upsilon_{k,t,r}$ is given by

$$\begin{aligned}
&\Big[W_{rt}^k + \left(W_{r(t+1)}^k - V_r^k W_{rt}^k\right) x, \left(V_r W_{r(t+1)}^{k-1} - V_r^k W_{rt}^{k-1}\right) x, \\
&\left(V_r^2 W_{r(t+1)}^{k-2} - V_r^k W_{rt}^{k-2}\right) x, \ldots, \left(V_r^{k-1} W_{r(t+1)} - V_r^k W_{rt}\right) x\Big].
\end{aligned}$$

Hence the solution of the above equation gives the generating function

$$\Re_{k,t,r}(x) = \left(\det(\delta_{k,t,r})\right) / \left(\det(\Delta_{k,r})\right),$$

as claimed. □

3. Applications

Now we state some applications of our main result by the following tables:

k	t	r	The generating function $\Re_{k,t,r}(x;0,1,1,1)$
1	1	2	$\frac{1}{1-3x+x^2}$
2	1	2	$\frac{1+x}{(1-x)(1-7x+x^2)}$
3	1	2	$\frac{1+6x+x^2}{1-21x+56x^2-21x^3+x^4}$
4	1	2	$\frac{16+1712x+1712x^2+17x^3}{(1-x)(1-34x+x^2)(1-1154x+x^2)}$

Table 1: The generating function for the powers of Fibonacci Numbers

k	t	r	The generating function $\Re_{k,t,r}(x;2,1,1,1)$
1	1	2	$\frac{3-2x}{1-3x+x^2}$
2	1	2	$\frac{9-23x+4x^2}{(1-x)(1-7x+x^2)}$
3	1	2	$\frac{27-224x+141x^2-8x^3}{1-21x+56x^2-21x^3+x^4}$
4	1	2	$\frac{81-2054x+452913226x^2-78298x^3-2864x^4}{(1-x)(1-7x+x^2)(1-47x+x^2)}$

Table 2: The generating function for the powers of Lucas Number

k	t	r	The generating function $\Re_{k,t,r}(x;0,1,2,1)$
1	1	2	$\frac{2}{x^2-6x+1}$
2	1	2	$\frac{4+4x}{(1-x)(1-34x+x^2)}$
3	1	2	$\frac{8\left(1+12x+x^2\right)}{1-204x+1190x^2-204x^3+x^4}$
4	1	2	$\frac{16(x+1)\left(1+106x+x^2\right)}{(1-x)(1-34x+x^2)(1-1154x+x^2)}$

Table 3: The generating function for the powers of Pell Numbers

and

k	t	r	The generating function $\Re_{k,t,r}(x;1,2t,2t,-1)$
1	1	2	$\frac{-1+4t^2-x}{1+(2-4t^2)x+x^2}$
2	1	2	$\frac{\left(16t^4-8t^2+1\right)+\left(16t^2-16t^4-2\right)x+x^2}{(1-x)(1+(-2+12t^2)x+x^2)}$
3	1	2	$\frac{12t^2-48t^4+64t^6+27-\left(4-24t^2+288t^4-256t^6+576t^8\right)x+\left(40t^2-336t^4+64t^6-3\right)x^2-x^3}{1+(-64t^6+96t^4-40t^2+4)x+(256t^8-512t^6+336t^4-80t^2+6)x^2+(-64t^6+96t^4-40t^2+4)x^3+x^4}$

Table 4: The generating function for the powers of Chebyshev polynomials of the second kind

Fibonacci numbers. If $a=0$ and $p=q=b=1$, then Theorem 1 for $k=1,2,3,4$ yields Table 1.

Lucas numbers. If $a=2$ and $p=q=b=1$, then Theorem 1 for $k=1,2,3,4$ yields Table 2.

Pell numbers. If $a=0$ and $p=2, q=b=1$, then Theorem 1 for $k=1,2,3,4$ yields Table 3.

Chebyshev polynomials of the second kind. If $a=1$, $b=p=2t$ and $q=-1$, then Theorem 1 for $k=1,2,3$ yields Table 4.

Applying Theorem 1 for $k=1,2,3$, then we give the following corollary.

Corollary 1. *Let* $k=1,2,3$. *Then the generating function* $\Re_{k,t,r}(x;a,b,p,q)$ *is given by* $\hat{A}_{k,t,r}(x)/\hat{E}_{k,t,r}(x)$, *where*

$$\begin{aligned}
\hat{A}_{1,1,2}(x) &= aq+bp-aq^2x, \\
\hat{A}_{2,1,2}(x) &= a^2q^2+b^2p^2+2abpq+q^2\left(-2a^2q^2+b^2p^2-2abp^3-2a^2p^2q-2abpq\right)x \\
&\quad +a^2q^6x^2, \\
\hat{A}_{3,1,2}(x) &= b^3p^3+3ab^2p^2q+3a^2bpq^2+a^3q^3-(3a^3p^4q^4+7a^3p^2q^5 \\
&\quad +3a^3q^6+6a^2bp^5q^3+15a^2bp^3q^4+6a^2bpq^5+6ab^2p^4q^3 \\
&\quad -2b^3p^5q^2-4b^3p^3q^3+3ab^2p^6q^2)x+(3a^3p^4q^7+7a^3p^2q^8+3a^3q^9 \\
&\quad +3a^2bp^5q^6+6a^2bp^3q^7+3a^2bpq^8-3ab^2p^4q^6 \\
&\quad -3ab^2p^2q^7+b^3p^3q^6)x^2-a^3q^{12}x^3
\end{aligned}$$

and

$$\begin{aligned}
\hat{E}_{1,1,2}(x) &= 1-\left(p^2+2q\right)x+q^2x^2, \\
\hat{E}_{2,1,2}(x) &= \left(q^2x-1\right)\left(-1+\left(p^4+4p^2q+2q^2\right)x-q^4x^2\right), \\
\hat{E}_{3,1,2}(x) &= \left(-1+q^2\left(12q+p^2\right)x-q^6x^2\right) \\
&\quad \times\left(-1+\left(2q+p^2\right)\left(4p^2q+p^4+q^2x-q^6x^2\right)\right).
\end{aligned}$$

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