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On the computing of the generalized order-k Pell numbers in log time

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Abstract

In this paper, we consider the generalized order-k Pell numbers and present an algorithm for computing the sums of the generalized order-k Pell numbers, as well as the Pell numbers themselves. The theoretical basis of using a matrix method for deriving the algorithm is also discussed.

Keywords: Generalized order-k Pell number; Sum; Matrix method

1. Introduction

The well-known Fibonacci sequence $\{F_n\}$ is defined by the recurrence relation, for $n > 1$

$$F_{n+1} = F_n + F_{n-1},$$

where $F_0 = 0$, $F_1 = 1$.

The Pell sequence $\{P_n\}$ is defined as, for $n > 1$

$$P_{n+1} = 2P_n + P_{n-1},$$

where $P_0 = 0$, $P_1 = 1$.

In [1], Miles defines the order-k Fibonacci numbers which are defined by the following recurrence relation:

$$f_n = \sum_{i=1}^{k} f_{n-i} \quad \text{for } n > k$$

with initial conditions

$$\begin{cases} f_j = 0, \\ f_k = 1, \end{cases} \quad j = 1, 2 \ldots, k-1,$$

where f_n is the nth order-k Fibonacci number.

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Many algorithms for computing Fibonacci numbers have been studied [2–11].

Gries and Levin [4], Pettorossi [5], and, Wilson and Shortt [12] present O(log$n$) algorithms for computing the order-k Fibonacci numbers, f_n. However, in computing the sums of order-k Fibonacci numbers, $\sum_{i=1}^{n} f_i$, such efficient algorithms might become irrelevant, as each Fibonacci number needs to be generated in order to carry out the summation.

In [13], the author defines k sequences of the generalized order-k Fibonacci numbers as shown: for $n > 0$ and $1 \leqslant i \leqslant k$

$$g_n^i = \sum_{j=1}^{k} g_{n-j}^i$$

with initial conditions

$$g_n^i = \begin{cases} 1 & \text{if } i = 1 - n, \\ 0 & \text{otherwise}, \end{cases} \quad \text{for } 1 - k \leqslant n \leqslant 0,$$

where g_n^i is the nth term of the ith sequence.

Also in [6,7], the author gives a theoretical basis and efficient algorithm for computing the sums of the generalized order-k Fibonacci numbers, g_n^i.

2. Generalized order-k Pell numbers

In this section, we consider the generalization of the usual Pell numbers. The Fibonacci and Pell sequence are the special cases of a sequence which is defined recursively as a linear combination of the preceding k terms

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [14], Kalman derives a number of closed-form formulas for the generalized sequence by companion matrix method.

In [13], the author uses the matrix method, chooses convenient initial conditions and then defines the generalized order-k Fibonacci sequence, $\{g_n^i\}$.

The authors define k sequences of the generalized order-k Pell numbers as follows [15]: for $n > 0$ and $1 \leqslant i \leqslant k$

$$P_n^i = 2P_{n-1}^i + P_{n-2}^i + \cdots + P_{n-k}^i \tag{1}$$

with initial conditions, for $1 - k \leqslant n \leqslant 0$

$$P_n^i = \begin{cases} 1 & \text{if } i = 1 - n, \\ 0 & \text{otherwise}, \end{cases}$$

where P_n^i is the nth term of the ith generalized order-k Pell sequence.

When $k = 2$, the generalized order-k Pell sequence is reduced to the usual Pell sequence $\{P_n\}$.

We immediately note that in an earlier work [16], the author gives an order-k generalization of the Pell numbers. However, this sequence is not convenient for finding a generating matrix, for computing and using.

The following theoretical basis can be found in [15]: letting

$$R = (r_{ij}) = \begin{bmatrix} 2 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{k \times k} \tag{2}$$

and

$$E_n = (e_{ij}) = \begin{bmatrix} P_n^1 & P_n^2 & \cdots & P_n^k \\ P_{n-1}^1 & P_{n-1}^2 & \cdots & P_{n-1}^k \\ \vdots & \vdots & & \vdots \\ P_{n-k+1}^1 & P_{n-k+1}^2 & \cdots & P_{n-k+1}^k \end{bmatrix}_{k\times k}, \tag{3}$$

the authors show that

$$E_n = R^n. \tag{4}$$

Furthermore, the author derives that

$$E_{n+1} = E_1 E_n = E_n E_1, \tag{5}$$

and more generally

$$E_{n+m} = E_n E_m = E_m E_n.$$

Thus we can say that E_1 is commutative under matrix multiplication. From (4) and (5) we can easily obtain that

$$P_n^1 = P_{n+1}^k. \tag{6}$$

Therefore, the sums of the generalized order-k Pell numbers, S_n, are defined as, for $n \geqslant 0$

$$S_n = \sum_{j=0}^{n} P_j^1. \tag{7}$$

To compute the sums of the generalized order-k Pell numbers using a matrix method without generating all P_j^1, for $0 \leqslant j \leqslant n-1$, the sums should appear as matrix elements.

Let T be a $(k+1)\times(k+1)$ matrix as

$$T = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ 0 & & R & \\ \vdots & & & \\ 0 & & & \end{bmatrix}, \tag{8}$$

and let C_n be a $(k+1)\times(k+1)$ matrix as

$$C_n = \begin{bmatrix} 1 & 0 & & 0 \\ S_{n-1} & & & \\ S_{n-2} & & E_n & \\ \vdots & & & \\ S_{n-k} & & & \end{bmatrix}, \tag{9}$$

where R and E_n are given by (2) and (3), respectively, then we have that

$$C_n = T^n. \tag{10}$$

Combining Eqs. (10) and (7), we obtain that

$$S_{n+1} = S_n + P_n^1. \tag{11}$$

More generally Eq. (10), we may write as

$$C_{n+m} = C_n C_m,$$

and when $n = m$,

$$C_{2n} = C_n C_n. \tag{12}$$

Eq. (12) provides us with a mean doubling of the exponents of C. It is obvious that C_n can be computed in O(log$n$) units of time.

3. Algorithm

The total of the Pell numbers that are generalized at the kth level may be estimated through Eq. (11). However, this estimation based on the multiplication of the matrices is not economical for the large values of k and n. As a result, in order to obtain the matrix C_n yielding total of the Pell numbers that are generalized at the kth level, we suggest an algorithm that is solely based on the total of the series rather than the one based on the multiplication of the matrices. The following observations have played the key role in forming such an algorithm: firstly, whatever values n and k take, the first line of the resulting matrix is fixed as follows:

$$C_i(1,j) = \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{if } j > i, \end{cases} \quad \text{for all } i, j,$$

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-------------------------------
clc;clear;S=0;I=0;
% create initial the last row of
% result matrix
for j=1:k
    I(j,j)=1;
end
for l=1:n-k+1
    for i=1:k
          Temp(i)=2*I(1,i);
    for j=2:k
          Temp(i)=Temp(i)+I(j,i);
    end
    end
    S=S+I(1,1);
    for i=k:-1:2
    for j=1:k
        I(i,j)=I(i-1,j);
    end
    end
    for j=1:k
          I(1,j)=Temp(j);
    end
    Temp=0;
end
R=[S,I(1,1:k)];
% fill the result matrix
for j=1:k+1
C(k+1,j)=R(j);
end
C(1,1)=1;
for i=k:-1:2
C(i,k+1)=C(i+1,2);
C(i,1)=C(i+1,1)+C(i+1,2);
C(i,2)=C(i+1,3)+2*C(i+1,2);
    for j=3:k
       C(i,j)=C(i+1,j+1)+C(i+1,2);
    end
end
-------------------------------
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Fig. 1. Algorithm regarding the calculation of the matrix C_n, yielding the total of the Pell numbers generalized at the kth level.

hence, there is no need for any processes for the first line. Secondly, for $i = k$, the elements in the $(i-k+1)$th line of the matrix E_i can be calculated by using the identity matrix I of $k \times k$ size. That is, for $i = k$ and $j = 1, 2, \ldots, k$

$$E_i(i-k+1, j) = 2I(1, j) + \sum_{r=2}^{k} I(r, j). \tag{13}$$

Following this step, when $(i > k)$, the operation to do is to erase the kth line of I and for $j = 2, 3, \ldots, k-1$ to move the jth line into $(j+1)$th line and the $(i-k+1)$th line of E_{i-1} into the first line of I. This operation is repeated by using Eq. (13) until $i = n$. Finally, after obtaining the $(n-k+1)$th line of the matrix E_n, it is easy to calculate the other elements of C_n by using Eqs. (6) and (11). Fig. 1 illustrates the MATLAB code concerning the algorithm.

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