

ALTERNATING SUMS OF THE POWERS OF FIBONACCI AND LUCAS NUMBERS

EMRAH KILIÇ¹, NEŞE ÖMÜR², AND YÜCEL TÜRKER ULUTAŞ³

ABSTRACT. We shall consider alternating Melham's sums for Fibonacci and Lucas numbers of the form $\sum_{k=1}^n (-1)^k F_{2k+\delta}^{2m+\varepsilon}$ and $\sum_{k=1}^n (-1)^k L_{2k+\delta}^{2m+\varepsilon}$, where $\varepsilon, \delta \in \{0, 1\}$.

1. INTRODUCTION

The Fibonacci numbers F_n and Lucas numbers L_n are defined by the following recursions: for $n > 0$,

$$F_{n+1} = F_n + F_{n-1} \quad \text{and} \quad L_{n+1} = L_n + L_{n-1},$$

where $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$, respectively.

If the roots of the characteristic equation $x^2 - x - 1 = 0$ are α and β , then the Binet formulas for them are given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n.$$

Wiemann and Cooper [6] mentioned about some conjectures of Melham for the sum:

$$\sum_{k=1}^n F_{2k}^{2m+1}.$$

Ozeki [2] considered Melham's sum and then he gave an explicit expansion for Melham's sum as a polynomial in F_{2n+1} .

In a general perspective, Prodinger [3] derive the general formula for the sum:

$$\sum_{k=0}^n F_{2k+\delta}^{2m+\varepsilon},$$

where $\varepsilon, \delta \in \{0, 1\}$, as well as the evaluations of the corresponding sums for Lucas numbers.

In this paper, we consider alternating analogous of Melham's sums. We derive explicit formulas for the sums:

$$\sum_{k=1}^n (-1)^k F_{2k+\delta}^{2m+\varepsilon} \quad \text{and} \quad \sum_{k=1}^n (-1)^k L_{2k+\delta}^{2m+\varepsilon},$$

where $\varepsilon, \delta \in \{0, 1\}$.

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2. ALTERNATING MELHAM'S SUMS FOR FIBONACCI NUMBERS

In this section we will start with some lemmas and then we shall derive our results about the alternating Melham's sums.

Lemma 1. *For positive integers n, m and t such that $m \geq t$,*

$$\begin{aligned} i) \quad & (-1)^{t+1} F_{(2m-2t+1)n} + F_{(2m-2t+1)(n+1)} \\ &= \sum_{j=0}^{2m-2t} (-1)^{j(t-2)} F_{(2m-2t+1)n+j+(1-3(-1)^t)/2}, \end{aligned}$$

and

$$ii) \quad F_{2(m-t)(n+1)} - (-1)^t F_{2(m-t)n} = \begin{cases} F_{2(m-t)n+m-t} L_{m-t} & \text{if } m \text{ is odd,} \\ L_{2(m-t)n+m-t} F_{m-t} & \text{if } m \text{ is even.} \end{cases}$$

Proof. *i)* We can write

$$\begin{aligned} & (-1)^{t+1} F_{(2m-2t+1)n} + F_{(2m-2t+1)(n+1)} \\ = & (-1)^{t+1} \left(\frac{\alpha^{(2m-2t+1)n} - \beta^{(2m-2t+1)n}}{\alpha - \beta} \right) \\ & + \left(\frac{\alpha^{(2m-2t+1)(n+1)} - \beta^{(2m-2t+1)(n+1)}}{\alpha - \beta} \right) \\ = & \frac{\alpha^{(2m-2t+1)n} \left((-1)^{t+1} + \alpha^{2m-2t+1} \right) - \beta^{(2m-2t+1)n} \left((-1)^{t+1} + \beta^{2m-2t+1} \right)}{\alpha - \beta} \\ = & \begin{cases} \sum_{j=0}^{2m-2t} (-1)^j (F_{(2m-2t+1)n+j} + F_{(2m-2t+1)n+j+1}) & \text{if } t \text{ is odd,} \\ - \sum_{j=0}^{2m-2t} F_{(2m-2t+1)n+j} - F_{(2m-2t+1)n+j+1} & \text{if } t \text{ is even,} \end{cases} \\ = & \begin{cases} \sum_{j=0}^{2m-2t} (-1)^j F_{(2m-2t+1)n+j+2} & \text{if } t \text{ is odd,} \\ \sum_{j=0}^{2m-2t} F_{(2m-2t+1)n+j-1} & \text{if } t \text{ is even,} \end{cases} \\ = & \sum_{j=0}^{2m-2t} (-1)^{j(t-2)} F_{(2m-2t+1)n+j+(1-3(-1)^t)/2}. \end{aligned}$$

ii) The proof is similarly seen from the Binet formulas of $\{F_n\}$ and $\{L_n\}$. \square

From [4], we have the following result for the Gibonacci sequence $\{G_n\}$, defined by for $n > 0$

$$G_{n+1} = G_n + G_{n-1},$$

with arbitrary initials G_0 and G_1 .

Lemma 2. *Let $a, p \neq 0$, q be arbitrary integers. Then for $n > 0$,*

$$\sum_{i=a}^n G_{pi+q} = \frac{G_{p(n+1)+q} + (-1)^{p+1} G_{pn+q} + (-1)^p G_{p(a-1)+q} - G_{pa+q}}{L_p - 1 - (-1)^p},$$

and

$$\begin{aligned} & \sum_{i=a}^n (-1)^i G_{pi+q} \\ &= \frac{(-1)^n G_{p(n+1)+q} + (-1)^{p+n} G_{pn+q} + (-1)^a G_{pa+q} + (-1)^{a+p} G_{p(a-1)+q}}{1 + (-1)^p + L_p}. \end{aligned}$$

As a consequence of Lemma 2, for further use we give the following result:

Corollary 1. *For any integer r and positive even integer t ,*

$$\sum_{k=1}^n (-1)^{kr} L_{kt} = \frac{(-1)^{nr} F_{t(n+1)} + (-1)^{nr-r} F_{tn} - F_t}{F_t}$$

and

$$\sum_{k=1}^n (-1)^{kr} F_{k(t+1)} = \frac{(-1)^{(n+1)r} F_{(t+1)n} + (-1)^{nr} F_{(t+1)(n+1)} - F_{t+1}}{L_{t+1}}.$$

Proof. Clearly

$$\sum_{k=1}^n (-1)^{kr} L_{kt} = (-1)^r L_t + L_{2t} + \dots + (-1)^{nr} L_{nt}.$$

For the first claim, we consider two cases: first case n is an odd integer. Here

$$\sum_{k=1}^n (-1)^{kr} L_{kt} = (-1)^r \sum_{j=1}^{(n+1)/2} L_{(2j-1)t} + \sum_{j=1}^{(n-1)/2} L_{2jt}. \quad (2.1)$$

If we take $a = 1$, $p = 2t$, $q = -t$ and $n \rightarrow \frac{n+1}{2}$ in Lemma 2, then we get

$$\sum_{j=1}^{(n+1)/2} L_{(2j-1)t} = \frac{L_{t(n+2)} - L_{tn} + L_{-t} - L_t}{L_{2t} - 2}.$$

The following identities are well known [1, 5]:

$$L_{c+t} - L_{c-t} = 5F_c F_t \quad (2.2)$$

for even t , and

$$L_{2c} - (-1)^c 2 = 5F_c^2 \text{ and } L_{-c} = (-1)^c L_c \quad (2.3)$$

for any integer c . Thus we have

$$\sum_{j=1}^{(n+1)/2} L_{(2j-1)t} = \frac{5F_{t(n+1)}F_t}{5F_t^2} = \frac{F_{t(n+1)}}{F_t}. \quad (2.4)$$

Meanwhile, if we take $a = 1$, $p = 2t$, $q = 0$ and $n \rightarrow \frac{n-1}{2}$ in Lemma 2, then we get

$$\sum_{j=1}^{(n-1)/2} L_{2jt} = \frac{L_{t(n+1)} - L_{t(n-1)} + L_0 - L_{2t}}{L_{2t} - 2}.$$

Since t is even, by (2.2) and (2.3), we rewrite the last equation as

$$\sum_{j=1}^{(n-1)/2} L_{2jt} = \frac{5F_{tn}F_t}{5F_t^2} - 1 = \frac{F_{tn}}{F_t} - 1. \quad (2.5)$$

If we put (2.4) and (2.5) in (2.1), then we obtain

$$\begin{aligned} \sum_{k=1}^n (-1)^{kr} L_{kt} &= (-1)^r \left(\frac{F_{t(n+1)}}{F_t} \right) + \left(\frac{F_{tn}}{F_t} - 1 \right) \\ &= \frac{(-1)^r F_{t(n+1)} + F_{tn} - F_t}{F_t}. \end{aligned} \quad (2.6)$$

As a second case, let n be an even integer, thus

$$\sum_{k=1}^n (-1)^{kr} L_{kt} = (-1)^r \sum_{j=1}^{n/2} L_{(2j-1)t} + \sum_{j=1}^{n/2} L_{2jt}. \quad (2.7)$$

By taking $a = 1$, $p = 2t$, $q = -t$ and $n \rightarrow \frac{n}{2}$ and $a = 1$, $p = 2t$, $q = 0$ and $n \rightarrow \frac{n}{2}$ in Lemma 2, respectively, we obtain the following result by (2.2) and (2.3), get for even t ,

$$\sum_{j=1}^{n/2} L_{(2j-1)t} = \frac{F_{tn}}{F_t}, \quad (2.8)$$

$$\sum_{j=1}^{n/2} L_{2jt} = \frac{F_{t(n+1)}}{F_t} - 1. \quad (2.9)$$

If we put (2.8) and (2.9) in (2.7), we get

$$\begin{aligned} \sum_{k=1}^n (-1)^{kr} L_{kt} &= (-1)^r \left(\frac{F_{tn}}{F_t} \right) + \left(\frac{F_{t(n+1)}}{F_t} - 1 \right) \\ &= \frac{(-1)^r F_{tn} + F_{t(n+1)} - F_t}{F_t}. \end{aligned} \quad (2.10)$$

Combining (2.6) and (2.10), we get the combined result:

$$\sum_{k=1}^n (-1)^{kr} L_{kt} = \frac{(-1)^{nr} F_{t(n+1)} + (-1)^{nr-r} F_{tn} - F_t}{F_t},$$

as claimed.

Finally by taking $a = 1$, $p = t + 1$, $q = 0$ in Lemma 2, the second claim is obtained similar to the first claim. \square

Theorem 1. *i) For positive odd m ,*

$$\begin{aligned} \sum_{k=1}^n (-1)^k F_k^{2m} &= \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^{i(n+1)+n} \binom{2m}{i} \frac{F_{(m-i)(2n+1)}}{F_{m-i}} \\ &\quad - \frac{1}{5^m} \binom{2m-1}{m} - \frac{1}{5^m} \binom{2m}{m} n, \end{aligned}$$

and, for positive even m ,

$$\begin{aligned} & \sum_{k=1}^n (-1)^k F_k^{2m} \\ = & \begin{cases} \frac{1}{5^m} \left(\sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{(m-i)(2n+1)}}{L_{m-i}} + \binom{2m-1}{m} \right) & \text{if } n \text{ is even,} \\ -\frac{1}{5^m} \left(\sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{(m-i)(2n+1)}}{L_{m-i}} + \binom{2m-1}{m-1} \right) & \text{if } n \text{ is odd,} \end{cases} \end{aligned}$$

and

$$\begin{aligned} ii) \sum_{k=1}^n (-1)^k F_k^{2m+1} &= \frac{1}{5^m} \sum_{i=0}^m \binom{2m+1}{i} \frac{(-1)^{i(n+1)+n}}{L_{2m-2i+1}} \\ & \quad \times \sum_{j=0}^{2(m-i)} (-1)^{j(i-2)} F_{n(2m-2i+1)+j+\frac{1-3(-1)^i}{2}} \\ & \quad - \frac{1}{5^m} \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \frac{F_{2m-2i+1}}{L_{2m-2i+1}}. \end{aligned}$$

Proof. i) For odd m , consider

$$\begin{aligned} & \sum_{k=1}^n (-1)^k F_k^{2m} \\ = & \sum_{k=1}^n (-1)^k \left(\frac{\alpha^k - \beta^k}{\alpha - \beta} \right)^{2m} \\ = & \sum_{k=1}^n (-1)^k \frac{1}{(\alpha - \beta)^{2m}} \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} \alpha^{ki} \beta^{k(2m-i)} \\ = & \sum_{k=1}^n \frac{(-1)^k}{5^m} \left(\sum_{i=0}^m (-1)^i \binom{2m}{i} \left(\alpha^{k(2m-i)} \beta^{ki} + \alpha^{ki} \beta^{k(2m-i)} \right) \right. \\ & \quad \left. - (-1)^m \binom{2m}{m} (\alpha\beta)^{km} \right) \\ = & \frac{1}{5^m} \left(\sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{k=1}^n (-1)^{k(i+1)} L_{2k(m-i)} + \binom{2m}{m} \sum_{k=1}^n (-1)^{m(k+1)+k} \right). \end{aligned}$$

By taking $i + 1 = r$ and $2(m - i) = t$ in Corollary 1, we write

$$\begin{aligned}
& \sum_{k=1}^n (-1)^k F_k^{2m} \\
&= \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \\
&\quad \times \left(\frac{(-1)^{n(i+1)} F_{2(m-i)(n+1)} + (-1)^{(n+1)(i+1)} F_{2(m-i)n} - F_{2(m-i)}}{F_{2(m-i)}} \right) \\
&\quad + \frac{1}{5^m} \binom{2m}{m} \sum_{k=1}^n (-1)^{m(k+1)+k} \\
&= \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^{i(n+1)+n} \binom{2m}{i} \left(\frac{F_{2(m-i)(n+1)} - (-1)^i F_{2(m-i)n}}{F_{2(m-i)}} \right) \\
&\quad - \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} + \frac{1}{5^m} \binom{2m}{m} \sum_{k=1}^n (-1)^{m(k+1)+k}.
\end{aligned}$$

Using (ii) in Lemma 1, we have the claimed result. For even m , the desired result is also obtained.

ii) Consider

$$\begin{aligned}
& \sum_{k=1}^n (-1)^k F_k^{2m+1} \\
&= \sum_{k=1}^n (-1)^k \left(\frac{\alpha^k - \beta^k}{\alpha - \beta} \right)^{2m+1} \\
&= \sum_{k=1}^n \frac{(-1)^k}{(\alpha - \beta)^{2m+1}} \sum_{i=0}^{2m+1} \binom{2m+1}{i} (-1)^{i+1} \alpha^{ki} \beta^{k(2m+1-i)} \\
&= \sum_{k=1}^n \frac{(-1)^k}{(\alpha - \beta)^{2m+1}} \sum_{i=0}^m \binom{2m+1}{i} (-1)^i \left(\alpha^{k(2m+1-i)} \beta^{ki} - \alpha^{ki} \beta^{k(2m+1-i)} \right) \\
&= \frac{1}{5^m} \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \sum_{k=1}^n (-1)^{k(i+1)} F_{k(2m-2i+1)}.
\end{aligned}$$

By taking $r = i + 1$ and $t = 2(m - i)$ in Corollary 1, we write

$$\begin{aligned}
& \sum_{k=1}^n (-1)^k F_k^{2m+1} \\
&= \frac{1}{5^m} \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \times \\
&\quad \frac{(-1)^{(n+1)(i+1)} F_{(2m-2i+1)n} + (-1)^{n(i+1)} F_{(2m-2i+1)(n+1)} - F_{2m-2i+1}}{L_{2m-2i+1}} \\
&= \frac{1}{5^m} \sum_{i=0}^m (-1)^{i+n(i+1)} \binom{2m+1}{i} \frac{(-1)^{i+1} F_{(2m-2i+1)n} + F_{(2m-2i+1)(n+1)}}{L_{2m-2i+1}} \\
&\quad - \frac{1}{5^m} \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \frac{F_{2m-2i+1}}{L_{2m-2i+1}}.
\end{aligned}$$

By Lemma 1, we obtain the claimed result. \square

For further use, we give a consequence of Lemma 2:

Corollary 2. *For even positive integer t and $n > 0$,*

$$\sum_{k=1}^n (-1)^k L_{2tk} = \frac{(-1)^n L_{t(2n+1)}}{L_t} - 1.$$

Proof. Substituting $a = 1$, $p = 2t$ and $q = 0$ in Lemma 2, we get

$$\begin{aligned} \sum_{k=1}^n (-1)^k L_{2tk} &= \frac{(-1)^n L_{2t(n+1)} + (-1)^n L_{2tn} - L_{2t} - 2}{2 + L_{2t}} \\ &= \frac{(-1)^n (L_{2t(n+1)} + L_{2tn}) - (L_{2t} + 2)}{2 + L_{2t}}. \end{aligned}$$

For even t , from [1, 5], we have that

$$L_{c+t} + L_{c-t} = L_c L_t \quad (2.11)$$

and for any c ,

$$L_{2c} + (-1)^c 2 = L_c^2. \quad (2.12)$$

Thus we obtain

$$\sum_{k=1}^n (-1)^k L_{2tk} = (-1)^n \frac{L_{(2n+1)t}}{L_t} - 1,$$

as claimed. \square

Theorem 2. *For $m > 0$,*

$$\begin{aligned} & i) \sum_{k=1}^n (-1)^k F_{2k}^{2m} \\ &= \begin{cases} \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} + \frac{(-1)^m (2m-1)}{5^m \binom{m-1}{m-1}} & \text{if } n \text{ is even,} \\ \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^{i+1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - \frac{(-1)^m (2m-1)}{5^m \binom{m-1}{m-1}} & \text{if } n \text{ is odd,} \end{cases} \end{aligned}$$

and

$$ii) \sum_{k=1}^n (-1)^k F_{2k}^{2m+1} = \frac{1}{5^{m+1}} \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \frac{(-1)^n L_{(2m-2i+1)(2n+1)} - L_{2m-2i+1}}{F_{2m-2i+1}}.$$

Proof. *i)* We write

$$\begin{aligned}
& \sum_{k=1}^n (-1)^k F_{2k}^{2m} = \sum_{k=1}^n (-1)^k \left(\frac{\alpha^{2k} - \beta^{2k}}{\alpha - \beta} \right)^{2m} \\
&= \sum_{k=1}^n (-1)^k \frac{1}{(\alpha - \beta)^{2m}} \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} \alpha^{2ki} \beta^{2k(2m-i)} \\
&= \sum_{k=1}^n \frac{(-1)^k}{5^m} \left(\sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \left(\alpha^{2k(2m-i)} \beta^{2ki} + \alpha^{2ki} \beta^{2k(2m-i)} \right) \right. \\
&\quad \left. + (-1)^m \binom{2m}{m} (\alpha\beta)^{2km} \right) \\
&= \sum_{k=1}^n \frac{(-1)^k}{5^m} \left(\sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} L_{4k(m-i)} + (-1)^m \binom{2m}{m} \right) \\
&= \frac{1}{5^m} \left(\sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{k=1}^n (-1)^k L_{4k(m-i)} + (-1)^m \binom{2m}{m} \sum_{k=1}^n (-1)^k \right).
\end{aligned}$$

If we take $t = 2(m - i)$ in Corollary 2, we write

$$\begin{aligned}
& \sum_{k=1}^n (-1)^k F_{2k}^{2m} \\
&= \frac{1}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} (-1)^i \left(\frac{(-1)^n L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 1 \right) \\
&\quad + \frac{1}{5^m} \binom{2m}{m} (-1)^m \sum_{k=1}^n (-1)^k \\
&= \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^{n+i} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \\
&\quad + \frac{1}{5^m} \binom{2m}{m} (-1)^m \sum_{k=1}^n (-1)^k \\
&= \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^{n+i} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} \\
&\quad + \frac{1}{5^m} (-1)^m \left(\binom{2m-1}{m-1} + \binom{2m}{m} \left(\frac{(-1)^n - 1}{2} \right) \right).
\end{aligned}$$

According to choosing of n as an odd and even number, we have the conclusion.

ii) The proof is obtained similar to the proof of (*i*). \square

For further use, we give the following result:

Corollary 3. For positive even integer t and $n > 0$,

$$\sum_{k=1}^n (-1)^k L_{(2k+1)t} = \frac{(-1)^n L_{2t(n+1)} - L_{2t}}{L_t}.$$

Proof. Substituting $a = 1$, $p = 2t$ and $q = t$ in Lemma 2, we get

$$\begin{aligned}
\sum_{k=1}^n (-1)^k L_{(2k+1)t} &= \frac{(-1)^n L_{(2n+3)t} + (-1)^n L_{(2n+1)t} - L_{3t} - L_t}{2 + L_{2t}} \\
&= \frac{(-1)^n (L_{(2n+3)t} + L_{(2n+1)t}) - (L_{3t} + L_t)}{2 + L_{2t}}.
\end{aligned}$$

By (2.11) and (2.12), we rewrite the last equation as for even t ,

$$\begin{aligned} \sum_{k=1}^n (-1)^k L_{(2k+1)t} &= (-1)^n \frac{L_t L_{2t(n+1)}}{L_t^2} - \frac{L_{3t} + L_t}{L_t^2} \\ &= (-1)^n \frac{L_{2t(n+1)}}{L_t} - \frac{L_{2t}}{L_t}. \end{aligned}$$

Thus we have the conclusion. \square

Theorem 3. For $m > 0$,

$$\begin{aligned} &i) \sum_{k=1}^n (-1)^k F_{2k+1}^{2m} \\ &= \begin{cases} \frac{-1}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{4(m-i)(n+1)} + L_{4(m-i)}}{L_{2(m-i)}} - \frac{1}{5^m} \binom{2m}{m} & \text{if } n \text{ is odd} \\ \frac{1}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{4(m-i)(n+1)} - L_{4(m-i)}}{L_{2(m-i)}} & \text{if } n \text{ is even,} \end{cases} \end{aligned}$$

and

$$ii) \sum_{k=1}^n (-1)^k F_{2k+1}^{2m+1} = \frac{1}{5^{m+1}} \sum_{i=0}^m \binom{2m+1}{i} \frac{(-1)^n L_{(4m-4i+2)(n+1)} - L_{4m-4i+2}}{F_{2m-2i+1}}.$$

Proof. i) Consider

$$\begin{aligned} &\sum_{k=1}^n (-1)^k F_{2k+1}^{2m} \\ &= \sum_{k=1}^n (-1)^k \left(\frac{\alpha^{2k+1} - \beta^{2k+1}}{\alpha - \beta} \right)^{2m} \\ &= \sum_{k=1}^n \frac{(-1)^k}{(\alpha - \beta)^{2m}} \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} \alpha^{(2k+1)i} \beta^{(2k+1)(2m-i)} \\ &= \sum_{k=1}^n \frac{(-1)^k}{5^m} \left(\sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \left(\alpha^{(2k+1)(2m-i)} \beta^{(2k+1)i} + \alpha^{(2k+1)i} \beta^{(2k+1)(2m-i)} \right) \right. \\ &\quad \left. + \binom{2m}{m} (\alpha\beta)^{2km} \right) \\ &= \sum_{k=1}^n \frac{(-1)^k}{5^m} \left(\sum_{i=0}^{m-1} \binom{2m}{i} L_{2(2k+1)(m-i)} + \binom{2m}{m} \right) \\ &= \frac{1}{5^m} \left(\sum_{i=0}^{m-1} \binom{2m}{i} \sum_{k=1}^n (-1)^k L_{2(2k+1)(m-i)} + \binom{2m}{m} \sum_{k=1}^n (-1)^k \right) \end{aligned}$$

If we take $t = 2(m - i)$ in Corollary 3, we write

$$\begin{aligned}
& \sum_{k=1}^n (-1)^k F_{2k+1}^{2m} \\
&= \frac{1}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{(-1)^n L_{4(m-i)(n+1)} - L_{4(m-i)}}{L_{2(m-i)}} + \frac{1}{5^m} \binom{2m}{m} \sum_{k=1}^n (-1)^k \\
&= \frac{(-1)^n}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{4(m-i)(n+1)}}{L_{2(m-i)}} \\
&\quad - \frac{1}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{4(m-i)}}{L_{2(m-i)}} + \frac{1}{5^m} \binom{2m}{m} \left(\frac{(-1)^{n-1}}{2} \right).
\end{aligned}$$

According to choosing of n as an odd or even number, we have the conclusion.

ii) The proof of *(ii)* is obtained similar to the proof of *(i)*. \square

3. ALTERNATING MELHAM'S SUM FOR LUCAS NUMBERS

Theorem 4. *i)* For positive odd m ,

$$\begin{aligned}
\sum_{k=1}^n (-1)^k L_k^{2m} &= \sum_{i=0}^{m-1} (-1)^{(i+1)n} \binom{2m}{i} \frac{F_{(m-i)(2n+1)}}{F_{m-i}} - 2^{2m-1} \\
&\quad + \binom{2m-1}{m} + \binom{2m}{m} n,
\end{aligned}$$

and, for even m ,

$$\begin{aligned}
& \sum_{k=1}^n (-1)^k L_k^{2m} \\
&= \begin{cases} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{(m-i)(2n+1)}}{L_{m-i}} - 2^{2m-1} + \binom{2m-1}{m} & \text{if } n \text{ is even,} \\ \sum_{i=0}^{m-1} (-1)^{i+1} \binom{2m}{i} \frac{L_{(m-i)(2n+1)}}{L_{m-i}} - 2^{2m-1} \\ \quad + \binom{2m-1}{m} - \binom{2m}{m} & \text{if } n \text{ is odd,} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
ii) \sum_{k=1}^n (-1)^k L_k^{2m+1} &= \sum_{i=0}^m \binom{2m+1}{i} \frac{(-1)^{(i+1)n+1}}{L_{2m-2i+1}} \\
&\quad \times \sum_{j=0}^{2(m-i)} (-1)^{j+1} L_{n(2m-2i+1)+j+\frac{1-3(-1)^i}{2}} \\
&\quad + 2 \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \frac{1}{L_{2m-2i+1}} - 2^{2m}.
\end{aligned}$$

Proof. *i)* By the Binet formula of $\{L_n\}$, we have

$$\begin{aligned} \sum_{k=1}^n (-1)^k L_k^{2m} &= \sum_{k=1}^n (-1)^k (\alpha^k + \beta^k)^{2m} \\ &= \sum_{k=1}^n (-1)^k \left(\sum_{i=0}^{m-1} \binom{2m}{i} (\alpha^{k(2m-i)} \beta^{ki} + \alpha^{ki} \beta^{k(2m-i)}) + \binom{2m}{m} (\alpha\beta)^{km} \right). \end{aligned}$$

Since $\alpha\beta = -1$, we get

$$\sum_{k=1}^n (-1)^k L_k^{2m} = \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{k=1}^n (-1)^{k(i+1)} L_{2k(m-i)} + \binom{2m}{m} \sum_{k=1}^n (-1)^{k(m+1)}.$$

From Corollary 1, we write

$$\begin{aligned} &\sum_{k=1}^n (-1)^k L_k^{2m} \\ &= \sum_{i=0}^{m-1} \binom{2m}{i} \left(\frac{(-1)^{(i+1)n} F_{2(m-i)(n+1)} + (-1)^{(n+1)(i+1)} F_{2(m-i)n} - F_{2(m-i)}}{F_{2(m-i)}} \right) \\ &\quad + \binom{2m}{m} \sum_{k=1}^n (-1)^{k(m+1)} \\ &= \sum_{i=0}^{m-1} (-1)^{(i+1)n} \binom{2m}{i} \frac{F_{2(m-i)(n+1)} + (-1)^{i+1} F_{2(m-i)n}}{F_{2(m-i)}} \\ &\quad - \sum_{i=0}^{m-1} \binom{2m}{i} + \binom{2m}{m} \sum_{k=1}^n (-1)^{k(m+1)} \\ &= \begin{cases} \sum_{i=0}^{m-1} (-1)^{(i+1)n} \binom{2m}{i} \frac{F_{(m-i)(2n+1)}}{F_{m-i}} - 2^{2m-1} & \text{if } m \text{ is odd,} \\ \quad + \binom{2m-1}{m} + \binom{2m}{m} n & \\ \sum_{i=0}^{m-1} (-1)^{(i+1)n} \binom{2m}{i} \frac{L_{(m-i)(2n+1)}}{L_{m-i}} - 2^{2m-1} & \text{if } m \text{ is even.} \\ \quad + \binom{2m-1}{m} + \binom{2m}{m} \left(\frac{(-1)^n - 1}{2} \right) & \end{cases} \end{aligned}$$

According to choosing of n , the claim is obtained.

ii) The proof is similar to the proof of *i)*. □

Theorem 5. *i)* For $m > 0$ and even $n > 0$,

$$\sum_{k=1}^n (-1)^k L_{2k}^{2m} = \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 2^{2m-1} + \binom{2m-1}{m},$$

and, for odd $n > 0$,

$$\sum_{k=1}^n (-1)^k L_{2k}^{2m} = - \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 2^{2m-1} - \binom{2m-1}{m}.$$

ii) For positive integers m and n ,

$$\sum_{k=1}^n (-1)^k L_{2k}^{2m+1} = (-1)^n \sum_{i=0}^m \binom{2m+1}{i} \frac{F_{(2m-2i+1)(2n+1)}}{F_{2m-2i+1}} - 2^{2m}.$$

Proof. *i)* We write

$$\begin{aligned}
& \sum_{k=1}^n (-1)^k L_{2k}^{2m} \\
&= \sum_{k=1}^n (-1)^k (\alpha^{2k} + \beta^{2k})^{2m} \\
&= \sum_{k=1}^n (-1)^k \left(\sum_{i=0}^{m-1} \binom{2m}{i} (\alpha^{2k(2m-i)} \beta^{2ki} + \alpha^{2ki} \beta^{2k(2m-i)}) + \binom{2m}{m} (\alpha\beta)^{2km} \right) \\
&= \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{k=1}^n (-1)^k L_{4k(m-i)} + \binom{2m}{m} \sum_{k=1}^n (-1)^k.
\end{aligned}$$

If we take $t = 2(m - i)$ in Corollary 2, we write

$$\begin{aligned}
& \sum_{k=1}^n (-1)^k L_{2k}^{2m} \\
&= \sum_{i=0}^{m-1} \binom{2m}{i} \left(\frac{(-1)^n L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 1 \right) + \binom{2m}{m} \sum_{k=1}^n (-1)^k \\
&= (-1)^n \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - \sum_{i=0}^{m-1} \binom{2m}{i} + \binom{2m}{m} \sum_{k=1}^n (-1)^k \\
&= (-1)^n \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 2^{2m-1} + \binom{2m-1}{m-1} + \binom{2m}{m} \left(\frac{(-1)^{n-1}}{2} \right).
\end{aligned}$$

According to choosing of n , the proof is complete by the fact that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

ii) The proof of *(ii)* is similar to the proof of *(i)*. \square

Theorem 6. For $m > 0$,

$$\begin{aligned}
& i) \sum_{k=1}^n (-1)^k L_{2k+1}^{2m} \\
&= \begin{cases} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)(n+1)} - L_{4(m-i)}}{L_{2(m-i)}} & \text{if } n \text{ is even,} \\ (-1)^{m+1} \binom{2m}{m} - \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)(n+1)} + L_{4(m-i)}}{L_{2(m-i)}} & \text{if } n \text{ is odd,} \end{cases}
\end{aligned}$$

and

$$ii) \sum_{k=1}^n (-1)^k L_{2k+1}^{2m+1} = \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \frac{(-1)^n F_{2(2m-2i+1)(n+1)} - F_{2(2m-2i+1)}}{F_{2m-2i+1}}.$$

Proof. *i)* Using the Binet formula of $\{L_n\}$, we have

$$\begin{aligned}
 & \sum_{k=1}^n (-1)^k L_{2k+1}^{2m} \\
 = & \sum_{k=1}^n (-1)^k \left(\alpha^{2k+1} + \beta^{2k+1} \right)^{2m} \\
 = & \sum_{k=1}^n (-1)^k \left(\sum_{i=0}^{m-1} \binom{2m}{i} \left(\alpha^{(2k+1)(2m-i)} \beta^{(2k+1)i} + \alpha^{(2k+1)i} \beta^{(2k+1)(2m-i)} \right) \right. \\
 & \left. + \binom{2m}{m} (\alpha\beta)^{(2k+1)m} \right) \\
 = & \sum_{k=1}^n (-1)^k \left(\sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \left(\alpha^{(2k+1)(2m-2i)} + \beta^{(2k+1)(2m-2i)} \right) \right. \\
 & \left. + \binom{2m}{m} (-1)^m \right) \\
 = & \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{k=1}^n (-1)^k L_{2(2k+1)(m-i)} + \binom{2m}{m} (-1)^m \sum_{k=1}^n (-1)^k.
 \end{aligned}$$

If we take $t = 2(m - i)$ in Corollary 3, we get

$$\begin{aligned}
 & \sum_{k=1}^n (-1)^k L_{2k+1}^{2m} \\
 = & \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \left(\frac{(-1)^n L_{4(m-i)(n+1)} - L_{4(m-i)}}{L_{2(m-i)}} \right) + \binom{2m}{m} (-1)^m \sum_{k=1}^n (-1)^k \\
 = & \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{(-1)^n L_{4(m-i)(n+1)}}{L_{2(m-i)}} - \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)}}{L_{2(m-i)}} \\
 & + \binom{2m}{m} (-1)^m \sum_{k=1}^n (-1)^k \\
 = & (-1)^n \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)(n+1)}}{L_{2(m-i)}} - \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)}}{L_{2(m-i)}} \\
 & + \binom{2m}{m} (-1)^m \left(\frac{(-1)^n - 1}{2} \right).
 \end{aligned}$$

According to choosing of n , the claimed result is seen.

ii) The proof is similar to the proof of *i)*. □

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¹TOBB ECONOMICS AND TECHNOLOGY UNIVERSITY MATHEMATICS DEPARTMENT 06560 ANKARA
TURKEY
E-mail address: `ekilic@etu.edu.tr`¹

^{2,3}KOCAELI UNIVERSITY MATHEMATICS DEPARTMENT 41380 IZMIT TURKEY
E-mail address: `neseomur@kocaeli.edu.tr`², `turkery@kocaeli.edu.tr`³