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Research Article

Some Finite Sums Involving Generalized Fibonacci and Lucas Numbers

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By considering Melham's sums (Melham, 2004), we compute various more general nonalternating sums, alternating sums, and sums that alternate according to $(-1)^{\binom{n+1}{2}}$ involving the generalized Fibonacci and Lucas numbers.

1. Introduction

Let a, b, and p be assumed to be arbitrary nonzero complex numbers with $p(p^2+2)(p^2+4) \neq 0$. Define second-order linear recursion $\{W_n\}$ by

$$W_n = pW_{n-1} + W_{n-2,} \tag{1.1}$$

with $W_0 = a, W_1 = b$ for all integers n. Since $\Delta = p^2 + 4 \neq 0$, the roots α and β of $x^2 - px - 1 = 0$ are distinct.

Also define the sequence $\{X_n\}$ via the terms of sequence $\{W_n\}$ as $X_n = W_{n+1} + W_{n-1}$.

The Binet formulas for the sequences $\{W_n\}$ and $\{X_n\}$ are

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \qquad X_n = A\alpha^n + B\beta^n, \tag{1.2}$$

where $A = b - a\beta$ and $B = b - a\alpha$.

For $a = 0, b = 1$, we denote $W_n = U_n$ and so $X_n = V_n$, respectively. When $p = 1, U_n = F_n$ (nth Fibonacci number) and $V_n = L_n$ (nth Lucas number).

Inspired by the well-known identity

$$\sum_{n=1}^{j} F_n^2 = F_j F_{j+1}, \tag{1.3}$$

Clary and Hemenway [1] obtained factored closed-form expressions for all sums of the form $\sum_{n=1}^{j} F_{mn}^3$, where m is an integer. Motivated by the results in [1], Melham [2] computed all sums of the form $\sum_{n=1}^{j} (-1)^n F_{mn}^4$ and $\sum_{n=1}^{j} (-1)^n L_{mn}^4$. In [3], Melham computed various nonalternating sums, alternating sums, and sums that alternate according to $(-1)^{\binom{n+1}{2}}$ for sequences $\{W_n\}$ and $\{X_n\}$. The author gathers his sums in three sets. Here we recall one example from each set for the reader's convenience:

$$\begin{gathered}
\sum_{n=i}^{j} W_n = \begin{cases} \dfrac{1}{p} V_{(j-i+1)/2}\left(W_{(j+i+1)/2} + W_{(j+i-1)/2}\right) & \text{if } j - i \equiv 1 \pmod 4, \\ \dfrac{1}{p} U_{(j-i+1)/2}\left(X_{(j+i+1)/2} + X_{(j+i-1)/2}\right) & \text{if } j - i \equiv 3 \pmod 4, \end{cases} \\
\sum_{n=4i}^{4j+3} (-1)^{\binom{n+1}{2}} W_{2n} = \frac{p}{\Delta - 2} U_{4j-4i+4} X_{4j+4i+3}, \\
\sum_{n=4i+2}^{4j+3} (-1)^{\binom{n+1}{2}} U_n X_n = \frac{p}{\Delta - 2} V_{4j-4i+5} W_{4j+4i+3} + 2W_0.
\end{gathered} \tag{1.4}$$

We refer to [4] for general expansion formulas for sums of powers of Fibonacci and Lucas numbers, as considered by Melham, as well as some extensions such that

$$\sum_{k=0}^{n} F_{2k+\delta}^{2m+\epsilon}, \qquad \sum_{k=0}^{n} L_{2k+\delta}^{2m+\epsilon}, \tag{1.5}$$

where $\delta, \epsilon \in \{0, 1\}$.

For alternating analogues of the results given by Prodinger, that is,

$$\sum_{k=0}^{n} (-1)^k F_{2k+\delta}^{2m+\epsilon}, \qquad \sum_{k=0}^{n} (-1)^k L_{2k+\delta}^{2m+\epsilon}, \tag{1.6}$$

we refer to [5].

Hendel [6] gave the *factorization theorem* which exhibits factorizations of sums of the form $\sum_{j=i}^{n+i-1} F_{aj-b}$. The author also introduced a unified proof method based on formulae for the factorizations of $F_{q-d} + F_{q+d}$.

In [7], Curtin et al. derived formulae for the shifted summations

$$\sum_{j=0}^{d-1} F_{n+j}F_{m+j}, \qquad \sum_{j=0}^{d-1} L_{n+j}L_{m+j}, \qquad \sum_{j=0}^{d-1} F_{n+j}L_{m+j}, \tag{1.7}$$

and the shifted convolutions

$$\sum_{j=0}^{d-1} F_{n+j}F_{d-m-j}, \qquad \sum_{j=0}^{d-1} L_{n+j}L_{d-m-j}, \qquad \sum_{j=0}^{d-1} F_{n+j}L_{d-m-j} \tag{1.8}$$

for positive integers d and arbitrary integers n and m.

In this paper, our main purpose is to consider Melham's sums involving double products of terms of $\{W_n\}$,$\{X_n\}$,$\{U_n\}$, and $\{V_n\}$ given in [3] and then compute several more general nonalternating sums, alternating sums, and sums that alternate according to $(-1)^{\binom{n+1}{2}}$.

2. Certain Finite Sums of Double Products of Terms

In this section, we will investigate certain sums consisting of products of at most two terms of $\{W_n\}$: nonalternating sums, alternating sums and sums that alternate according to $(-1)^{\binom{n+1}{2}}$. From the Binet forms of $\{W_n\}$ and $\{X_n\}$, we give the following lemma for further use without proof.

Lemma 2.1. *Let a, b, and p be as in Section 1, and let $r = aW_2 - bW_1$. Then for all integers k,*

$$\begin{aligned}
b^2U_{2k} + 2abU_{2k-1} + a^2U_{2k-2} &= W_kX_k,\\
b^2U_{2k+1} + 2abU_{2k} + a^2U_{2k-1} &= W_{k+1}X_k + (-1)^k r,\\
b^2V_{2k} + 2abV_{2k-1} + a^2V_{2k-2} &= X_k^2 + (-1)^k 2r,\\
b^2V_{2k+1} + 2abV_{2k} + a^2V_{2k-1} &= X_kX_{k+1} + (-1)^k pr.
\end{aligned} \tag{2.1}$$

Theorem 2.2. *Fix integers c, d, and m.*

(i) *If m is even, then for all integers $j > i$,*

$$\sum_{n=i}^{j} U_{mn+c}W_{mn+d} = \frac{U_{m(j-i+1)}X_{m(j+i)+c+d}}{\Delta U_m} - \frac{(-1)^c(j-i+1)X_{d-c}}{\Delta}. \tag{2.2}$$

(i) *If m is odd, then for all integers $j > i$,*

$$\sum_{n=i}^{j} U_{mn+c}W_{mn+d} = \begin{cases} \dfrac{U_{m(j-i+1)}W_{m(j+i)+c+d}}{V_m} & \text{if } j-i \equiv 1 \pmod 2, \\ \dfrac{V_{m(j-i+1)}X_{m(j+i)+c+d}}{\Delta V_m} - \dfrac{(-1)^{c+j}X_{d-c}}{\Delta} & \text{if } j-i \equiv 0 \pmod 2. \end{cases} \tag{2.3}$$

Proof. Using the Binet formulas, we compute

$$\begin{aligned} \sum_{n=i}^{j} U_{mn+c}W_{mn+d} &= \sum_{n=i}^{j}\left(\frac{\alpha^{mn+c}-\beta^{mn+c}}{\alpha-\beta}\right)\left(\frac{A\alpha^{mn+d}-B\beta^{mn+d}}{\alpha-\beta}\right) \\ &= \frac{1}{(\alpha-\beta)^2}\sum_{n=i}^{j}\left(A\alpha^{2mn+c+d}+B\beta^{2mn+c+d}\right) - \frac{(-1)^{mn+c}}{(\alpha-\beta)^2}\left(A\alpha^{d-c}+B\beta^{d-c}\right) \\ &= \frac{1}{\Delta}\sum_{n=i}^{j}X_{2mn+c+d} - \frac{(-1)^c}{\Delta}X_{d-c}\sum_{n=i}^{j}(-1)^{mn}. \end{aligned} \tag{2.4}$$

Since $X_n = W_{n-1} + W_{n+1}$, we can obtain that for even m

$$\sum_{n=i}^{j} X_{2mn+t} = \frac{U_{m(j-i+1)}X_{m(j+i)+t}}{U_m}. \tag{2.5}$$

The result follows. □

For example, when $i = 2, \quad m = 3, \; a = 0, \; b = c = p = 1$, and $d = 5$, we obtain

$$\sum_{n=2}^{j} F_{3n+1}F_{3n+5} = \frac{F_{3(j-1)}F_{3j+4}}{4}. \tag{2.6}$$

Theorem 2.3. *Fix integers c, d, and m. Let $S = \sum_{n=i}^{j}(-1)^n U_{mn+c}W_{mn+d}$.*

(1) *If m is odd, then S equals*

$$S = \frac{(-1)^j U_{m(j-i+1)}X_{m(j+i)+c+d}}{\Delta U_m} - \frac{(-1)^c(j-i+1)X_{d-c}}{\Delta}. \tag{2.7}$$

(2) *If m is odd and the parities of i and j are the same, then S equals*

$$\frac{(-1)^j V_{m(j-i+1)}X_{m(j+i)+c+d}}{\Delta V_m} - \frac{\left((-1)^j+(-1)^i\right)(-1)^c X_{d-c}}{2\Delta}. \tag{2.8}$$

(3) *If m is odd and the parities of i and j are the different, then S equals*

$$\frac{(-1)^j U_{m(j-i+1)} W_{m(j+i)+c+d}}{V_m} - \frac{\left((-1)^j + (-1)^i\right)(-1)^c X_{d-c}}{2}. \tag{2.9}$$

Proof. Consider

$$\begin{aligned}\sum_{n=i}^{j}(-1)^n U_{mn+c} W_{mn+d} &= \sum_{n=i}^{j}\left(\frac{\alpha^{mn+c}-\beta^{mn+c}}{\alpha-\beta}\right)\left(\frac{A\alpha^{mn+d}-B\beta^{mn+d}}{\alpha-\beta}\right) \\ &= \frac{1}{(\alpha-\beta)^2}\sum_{n=i}^{j}(-1)^n\left(A\alpha^{2mn+c+d}+B\beta^{2mn+c+d}\right)-\frac{(-1)^{mn+c}}{(\alpha-\beta)^2}\left(A\alpha^{d-c}+B\beta^{d-c}\right) \\ &= \frac{1}{\Delta}\sum_{n=i}^{j}(-1)^n X_{2mn+c+d} - \frac{1}{\Delta}X_{d-c}\sum_{n=i}^{j}(-1)^{(m+1)n+c}.\end{aligned} \tag{2.10}$$

Since $X_n = W_{n-1} + W_{n+1}$, for odd m, we find

$$\sum_{n=i}^{j}(-1)^n X_{2mn+c} = \frac{(-1)^j U_{m(j-i+1)} X_{m(i+j)+c}}{U_m}. \tag{2.11}$$

The result is now obtained by considering the values of $\sum_{n=i}^{j}(-1)^{(m+1)n+c}$. □

Theorem 2.4. *Fix integers c, d, and m. For all integers $j > i$,*

$$\sum_{n=4i+1}^{4j}(-1)^{\binom{n+1}{2}} U_{mn+c} W_{mn+d} = \frac{U_{4m(j-i)}}{V_{2m}}\begin{cases} V_m W_{s+m} & \text{if } m \text{ is even}, \\ U_m X_{s+m} & \text{if } m \text{ is odd}, \end{cases} \tag{2.12}$$

$$\sum_{n=4i}^{4j+3}(-1)^{\binom{n+1}{2}} U_{mn+c} W_{mn+d} = \frac{U_{4m(j-i+1)}}{V_{2m}}\begin{cases} U_m X_{s+3m} & \text{if } m \text{ is even}, \\ V_m W_{s+3m} & \text{if } m \text{ is odd}, \end{cases}$$

$$\sum_{n=4i+3}^{4j}(-1)^{\binom{n+1}{2}}U_{mn+c}W_{mn+d}$$
$$=\begin{cases}\dfrac{V_mV_{2m(2(j-i)-1)}X_{s+3m}}{\Delta V_{2m}}-\dfrac{2(-1)^cX_{d-c}}{\Delta} & \text{if } m \text{ is even,}\\ \dfrac{U_mV_{2m(2(j-i)-1)}W_{s+3m}}{V_{2m}} & \text{if } m \text{ is odd,}\end{cases}$$
$$\sum_{n=4i+2}^{4j+3}(-1)^{\binom{n+1}{2}}U_{mn+c}W_{mn+d}$$
$$=\begin{cases}\dfrac{U_mV_{2m(2(j-i)+1)}W_{s+5m}}{V_{2m}} & \text{if } m \text{ is even,}\\ \dfrac{V_mV_{m(4(j-i+1)-2)}X_{s+5m}}{\Delta V_{2m}}+\dfrac{2(-1)^cX_{d-c}}{\Delta} & \text{if } m \text{ is odd,}\end{cases} \tag{2.13}$$

where $s = m(4(j+i)) + c + d$.

Proof. Consider

$$\begin{aligned}\sum_{n=4i+1}^{4j}&(-1)^{\binom{n+1}{2}}U_{mn+c}W_{mn+d}\\ &=\sum_{n=4i+1}^{4j}(-1)^{\binom{n+1}{2}}\left(\frac{\alpha^{mn+c}-\beta^{mn+c}}{\alpha-\beta}\right)\left(\frac{A\alpha^{mn+d}-B\beta^{mn+d}}{\alpha-\beta}\right)\\ &=\frac{1}{(\alpha-\beta)^2}\sum_{n=4i+1}^{4j}(-1)^{\binom{n+1}{2}}\left(A\alpha^{2mn+c+d}+B\beta^{2mn+c+d}\right)-\frac{(-1)^{mn+c}}{(\alpha-\beta)^2}\left(A\alpha^{d-c}+B\beta^{d-c}\right)\\ &=\frac{1}{\Delta}\sum_{n=4i+1}^{4j}(-1)^{\binom{n+1}{2}}X_{2mn+c+d}-\frac{1}{\Delta}X_{d-c}(-1)^c\sum_{n=4i+1}^{4j}(-1)^{\binom{n+1}{2}}.\end{aligned} \tag{2.14}$$

Here we have that $\sum_{n=4i+1}^{4j}(-1)^{\binom{n+1}{2}} = 0$ and, by $X_n = W_{n-1} + W_{n+1}$,

$$\sum_{n=4i+1}^{4j}(-1)^{\binom{n+1}{2}}X_{2mn+c+d}=\frac{\Delta V_mU_{4m(j-i)}W_{m(4(j+i)+1)+c+d}}{V_{2m}}, \tag{2.15}$$

for even m. Now formula (2.12) follows. The remaining formulas are proven in a similar manner. □

Notice that in (2.12)-(2.13), one limit of summation is even while the other is odd. Accordingly we have observed that each of (2.12)-(2.13) has a dual sum that is obtained with the use of the rule below. We highlight this rule since it also applies to get certain groups of sums in Section 2.

From [3], we recall the rule for the formation of the dual sum.

(1) Replace the even limit by the even limit corresponding to the other residue class modulo 4 and the odd limit by the odd limit corresponding to the other residue class modulo 4.

(2) Calculate the subscripts on the right in accordance with the paragraph following (2.13).

(3) Multiply the right side by −1.

For example, for odd integer m, the dual of (2.13) is

$$\sum_{n=4i}^{4j+1}(-1)^{\binom{n+1}{2}}U_{mn+c}W_{mn+d} = -\frac{1}{\Delta}\left(\frac{V_m V_{2m(2(j-i)+1)}X_{s+m}}{V_{2m}} + 2(-1)^c X_{d-c}\right), \tag{2.16}$$

where s is defined as before.

Theorem 2.5. *Fix integers c, d, and m.*

(i) *If c and d have the same parities, then*

$$\begin{aligned}
&\sum_{n=4i+1}^{4j}(-1)^{\binom{n+1}{2}}W_{(2m+1)n+c}W_{(2m+1)n+d} \\
&\quad= \frac{U_{2m+1}U_{4(2m+1)(j-i)}}{V_{2(2m+1)}} \times \left(X_{2(2m+1)(j+i)+t}X_{2(2m+1)(j+i)+t+1} + pr(-1)^t\right), \\
&\sum_{n=4i+1}^{4j}(-1)^{\binom{n+1}{2}}W_{2mn+c}W_{2mn+d} \\
&\quad= \frac{V_{2m}U_{8m(j-i)}}{V_{4m}} \times \left(W_{m(4j+4i)+t}X_{m(4j+4i)+t}\right), \\
&\sum_{n=4i+3}^{4j}(-1)^{\binom{n+1}{2}}W_{(2m+1)n+c}W_{(2m+1)n+d} \\
&\quad= \frac{U_{2m+1}V_{2(2m+1)(2(j-i)-1)}}{V_{2(2m+1)}} \times \left(W_{(2m+1)(2(j+i)+1)+t+1}X_{(2m+1)(2(j+i)+1)+t} - r(-1)^t\right),
\end{aligned}$$

$$\sum_{n=4i+3}^{4j}(-1)^{\binom{n+1}{2}}W_{2mn+c}W_{2mn+d}$$

$$=\frac{V_{2m}V_{4m(2(j-i)-1)}}{\Delta V_{4m}}\times\left(X^2_{m(4(j+i)+2)+t}+2r(-1)^t\right)+\frac{2r(-1)^cV_{d-c}}{\Delta},$$

$$\sum_{n=4i}^{4j+3}(-1)^{\binom{n+1}{2}}W_{(2m+1)n+c}W_{(2m+1)n+d}$$

$$=\frac{V_{2m+1}U_{4(2m+1)(j-i+1)}}{V_{2(2m+1)}}\times\left(W_{(2m+1)(2(j+i)+1)+t+1}X_{(2m+1)(2(j+i)+1)+t}-r(-1)^t\right),$$

$$\sum_{n=4i}^{4j+3}(-1)^{\binom{n+1}{2}}W_{2mn+c}W_{2mn+d}$$

$$=\frac{U_{2m}U_{8m(j-i+1)}}{V_{4m}}\times\left(X^2_{m(4(j+i)+2)+t}+2r(-1)^t\right),$$

$$\sum_{n=4i+2}^{4j+3}(-1)^{\binom{n+1}{2}}W_{(2m+1)n+c}W_{(2m+1)n+d}$$

$$=\frac{V_{2m+1}V_{2(2m+1)(2(j-i)+1)}}{\Delta V_{2(2m+1)}}\times\left(X_{2(2m+1)(j+i+1)+t+1}X_{2(2m+1)(j+i+1)+t}+pr(-1)^t\right)-\frac{2r(-1)^cV_{d-c}}{\Delta},$$

$$\sum_{n=4i+2}^{4j+3}(-1)^{\binom{n+1}{2}}W_{2mn+c}W_{2mn+d}$$

$$=\frac{1}{V_{4m}}\left(U_{2m}V_{4m(2(j-i)+1)}W_{4m(j+i+1)+t}X_{4m(j+i+1)+t}\right),\tag{2.17}$$

where $t=(c+d)/2+m$.

(ii) *If c and d have different parities, then*

$$\sum_{n=4i+1}^{4j}(-1)^{\binom{n+1}{2}}W_{(2m+1)n+c}W_{(2m+1)n+d}$$

$$=\frac{U_{2m+1}U_{4(2m+1)(j-i)}}{V_{2(2m+1)}}\left(X^2_{2(2m+1)(j+i)+v}+2r(-1)^v\right),$$

$$\sum_{n=4i+1}^{4j}(-1)^{\binom{n+1}{2}}W_{2mn+c}W_{2mn+d}$$
$$=\frac{V_{2m}U_{8m(j-i)}}{V_{4m}}\left(X_{4m(j+i)+v-1}W_{4m(j+i)+v}-r(-1)^{v}\right),$$
$$\sum_{n=4i+3}^{4j}(-1)^{\binom{n+1}{2}}W_{(2m+1)n+c}W_{(2m+1)n+d}$$
$$=\frac{1}{V_{2(2m+1)}}\times U_{2m+1}V_{2(2m+1)(2(j-i)-1)}W_{(2m+1)(2(j+i)+1)+v}X_{(2m+1)(2(j+i)+1)+v},$$
$$\sum_{n=4i+3}^{4j}(-1)^{\binom{n+1}{2}}W_{2mn+c}W_{2mn+d}$$
$$=\frac{V_{2m}V_{4m(2(j-i)-1)}}{\Delta V_{4m}}\times\left(X_{m(4(j+i)+2)+v}X_{m(4(j+i)+2)+v-1}+pr(-1)^{v}\right)+\frac{2r(-1)^{c}V_{d-c}}{\Delta},$$
$$\sum_{n=4i}^{4j+3}(-1)^{\binom{n+1}{2}}W_{(2m+1)n+c}W_{(2m+1)n+d}$$
$$=\frac{V_{2m+1}U_{4(2m+1)(j-i+1)}}{V_{2(2m+1)}}\times X_{(2m+1)(2(j+i)+1)+v}W_{(2m+1)(2(j+i)+1)+v},$$
$$\sum_{n=4i}^{4j+3}(-1)^{\binom{n+1}{2}}W_{2mn+c}W_{2mn+d}$$
$$=\frac{U_{2m}U_{8m(j-i+1)}}{V_{4m}}\times\left(X_{m(4(j+i)+2)+v}X_{m(4(j+i)+2)+v-1}-pr(-1)^{v}\right),$$
$$\sum_{n=4i+2}^{4j+3}(-1)^{\binom{n+1}{2}}W_{(2m+1)n+c}W_{(2m+1)n+d}$$
$$=-\frac{2r(-1)^{c}V_{d-c}}{\Delta}+\frac{V_{2m+1}V_{2(2m+1)(2(j-i)+1)}}{\Delta V_{2(2m+1)}}\left(X^{2}_{2(2m+1)(j+i+1)+v}+2r(-1)^{v}\right),$$
$$\sum_{n=4i+2}^{4j+3}(-1)^{\binom{n+1}{2}}W_{2mn+c}W_{2mn+d}$$
$$=\frac{U_{2m}V_{4m(2(j-i)+1)}}{V_{4m}}\times\left(W_{4m(j+i+1)+v}X_{4m(j+i+1)+v-1}-r(-1)^{v}\right),\tag{2.18}$$

where r is defined as before and $v=(c+d+1)/2+m$.

Proof. Suppose that c and d have the same parities. Consider

$$\begin{aligned}
&\sum_{n=4i+2}^{4j+3}(-1)^{\binom{n+1}{2}}W_{2mn+c}W_{2mn+d}\\
&\quad=\frac{1}{\Delta}\sum_{n=4i+2}^{4j+3}(-1)^{\binom{n+1}{2}}\left(A^2\alpha^{4mn+c+d}+B^2\beta^{4mn+c+d}-AB\alpha^{2mn+c}\beta^{2mn+d}-AB\beta^{2mn+c}\alpha^{2mn+d}\right)\\
&\quad=\frac{1}{\Delta}\left(b^2\sum_{n=4i+2}^{4j+3}(-1)^{\binom{n+1}{2}}V_{4mn+c+d}+2ab\sum_{n=4i+2}^{4j+3}(-1)^{\binom{n+1}{2}}V_{4mn+c+d-1}\right.\\
&\qquad\qquad\left.+a^2\sum_{n=4i+2}^{4j+3}(-1)^{\binom{n+1}{2}}V_{4mn+c+d-2}\right)\\
&\qquad+\frac{1}{\Delta}(-1)^c rV_{d-c}\sum_{n=4i+2}^{4j+3}(-1)^{\binom{n+1}{2}}.
\end{aligned}\tag{2.19}$$

From the definition of $\{V_n\}$, we obtain

$$\sum_{n=4i+2}^{4j+3}(-1)^{\binom{n+1}{2}}V_{4mn+c}=\frac{\Delta U_{2m}V_{4m(2(j-i)+1)}U_{2m(4(j+i+1)+1)+c}}{V_{4m}}.\tag{2.20}$$

Since

$$\sum_{n=4i+2}^{4j+3}(-1)^{\binom{n+1}{2}}=0,\tag{2.21}$$

we get

$$\begin{aligned}
&\sum_{n=4i+2}^{4j+3}(-1)^{\binom{n+1}{2}}W_{2mn+c}W_{2mn+d}\\
&\quad=\frac{U_{2m}V_{4m(2(j-i)+1)}}{V_{4m}}\left(b^2U_{2m(4(j+i+1)+1)+c+d}+2abU_{2m(4(j+i+1)+1)+c+d-1}+a^2U_{2m(4(j+i+1)+1)+c+d-2}\right)
\end{aligned}\tag{2.22}$$

Taking $2k=2m(4(j+i+1)+1)+c+d$ in Lemma 2.1, we write

$$\begin{aligned}
&\sum_{n=4i+2}^{4j+3}(-1)^{\binom{n+1}{2}}W_{2mn+c}W_{2mn+d}\\
&\quad=\frac{U_{2m}V_{4m(2(j-i)+1)}W_{m(4(j+i+1)+1)+(d+c)/2}X_{m(4(j+i+1)+1)+(d+c)/2}}{\Delta V_{4m}}.
\end{aligned}\tag{2.23}$$

Thus the result follows. Similar arguments yield the remaining formulas, where we must consider the parities of c, d. □

For example, the dual of (2.17) is given by if c and d have the same parities,

$$\sum_{n=4i}^{4j+1}(-1)^{\binom{n+1}{2}} W_{2mn+c}W_{2mn+d} = -\frac{U_{2m}V_{4m(2(j-i)+1)}W_{4m(j+i)+t}X_{4m(j+i)+t}}{V_{4m}}, \tag{2.24}$$

and the dual of (2.18) is given by if c and d have different parities,

$$\begin{aligned}\sum_{n=4i}^{4j+1}(-1)^{\binom{n+1}{2}}W_{2mn+c}W_{2mn+d} \\ = -\frac{U_{2m}V_{4m(2(j-i)+1)}}{V_{4m}} \times \left(W_{4m(j+i)+v}X_{4m(j+i)+v-1} - r(-1)^{v}\right),\end{aligned} \tag{2.25}$$

where t and v are defined as before.

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