

## Some Generalized Fibonomial Sums related with the Gaussian $q$ -Binomial sums

by

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### Abstract

In this paper, we consider some generalized Fibonomial sums formulae and then prove them by using the Cauchy binomial theorem and  $q$ -Zeilberger algorithm in Mathematica session.

**Key Words:** Gaussian binomial coefficient, fibonomial coefficient,  $q$ -binomial theorem, sums.

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### 1 Introduction

For all real  $n$  and integer  $k$  with  $k \geq 0$ , the Gaussian  $q$ -binomial coefficients  $\begin{bmatrix} n \\ k \end{bmatrix}_q$  is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}$$

and as zero otherwise, where

$$(x; q)_n := (1 - x)(1 - xq) \dots (1 - xq^{n-1}).$$

Thus far, the  $q$ -identities have taken the interest of many authors. For a detailed information about the  $q$ -identities, we may refer to [1, 6, 7] and its to the list of references.

We recall some well known identities related to the  $q$ -identities: firstly *Gauss' identity* is as follows :

$$\sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix}_q = \prod_{k=1}^n (1 - q^{2k-1})$$

and the Cauchy binomial theorem is given by

$$\sum_{k=0}^n y^k q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{k=1}^n (1 + yq^k). \quad (1.1)$$

Define the non-degenerate second order linear sequences  $\{U_n\}$  and  $\{V_n\}$  by, for  $n > 1$

$$\begin{aligned} U_n &= pU_{n-1} + U_{n-2}, \quad U_0 = 0, \quad U_1 = 1, \\ V_n &= pV_{n-1} + V_{n-2}, \quad V_0 = 2, \quad V_1 = p. \end{aligned}$$

The Binet formulas of  $\{U_n\}$  and  $\{V_n\}$  are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where  $\alpha, \beta = (p \pm \sqrt{p^2 + 4})/2$ .

For  $n \geq k \geq 1$  and any positive integer  $m$ , define the generalized Fibonomial coefficient by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{U;m} := \frac{U_m U_{2m} \cdots U_{nm}}{(U_m U_{2m} \cdots U_{km})(U_m U_{2m} \cdots U_{m(n-k)})} = \prod_{i=1}^k \frac{U_{m(n-i+1)}}{U_{mi}}$$

with  $\left\{ \begin{matrix} n \\ 0 \end{matrix} \right\}_{U;m} = \left\{ \begin{matrix} n \\ n \end{matrix} \right\}_{U;m} = 1$ .

When  $m = 1$ , we obtain the usual Fibonomial coefficient, denoted by  $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_U$ . For more details about the Fibonomial and generalized Fibonomial coefficients, see [2, 3, 4, 9].

The link between the generalized Fibonomial and Gaussian  $q$ -binomial coefficients is

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{U;m} = \alpha^{mk(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^m} \quad \text{with} \quad q = -\alpha^{-2}.$$

By taking  $q = \beta/\alpha$ , the Binet formulae are reduced to the following forms:

$$U_n = \alpha^{n-1} \frac{1 - q^n}{1 - q} \quad \text{and} \quad V_n = \alpha^n (1 + q^n),$$

where  $\mathbf{i} = \sqrt{-1} = \alpha\sqrt{q}$ . All the identities we will derive hold for general  $q$ , and results about generalized Fibonacci and Lucas numbers come out as corollaries for the special choice of  $q$ .

In the present study, we shall consider new generalized Fibonomial sums formulae and prove them by using the Cauchy binomial theorem and the  $q$ -Zeilberger algorithm (for such computer algorithms, we may refer to [5]).

## 2 Some Fibonomial Sums

In this section, we consider some generalized Fibonomial sums and compute them by means of Gaussian  $q$ -binomial sums formulae.

**Lemma 1.** For  $n \geq 0$ ,

$$\begin{aligned} (i) \quad & \sum_{k=0}^{2n+1} \mathbf{i}^{-k^2} a(k) = \frac{1 - \mathbf{i}}{2} \sum_{k=0}^{2n+1} a(k), \\ (ii) \quad & \sum_{k=0}^{2n} \mathbf{i}^{mk^2 \pm k} b(k) = \mathbf{i}^{mn^2} \sum_{k=0}^{2n} \mathbf{i}^{\pm k} b(k), \\ (iii) \quad & \sum_{k=0}^{2n+1} \mathbf{i}^{k(2n+1-k) \pm k} a(k) = \frac{1 \pm (-1)^n \mathbf{i}}{2} \sum_{k=0}^{2n+1} a(k), \end{aligned}$$

where the sequences  $a(k)$  and  $b(k)$  satisfy  $a(k) = a(2n+1-k)$  and  $b(k) = b(2n-k)$ , respectively.

**Proof:** Consider

$$\begin{aligned} (i) \quad & \sum_{k=0}^{2n+1} \mathbf{i}^{-k^2} a(k) = \frac{1}{2} \sum_{k=0}^{2n+1} (\mathbf{i}^{-k^2} + \mathbf{i}^{-(2n+1-k)^2}) a(k) = \frac{1 - \mathbf{i}}{2} \sum_{k=0}^{2n+1} a(k). \\ (ii) \quad & \text{Let } C_{n;m}(k) := \mathbf{i}^{mn^2 \pm k} - \mathbf{i}^{mk^2 \pm k}. \text{ Then we have } C_{n;m}(k) + C_{n;m}(2n-k) = 0. \\ & \text{Therefore,} \end{aligned}$$

$$\sum_{k=0}^{2n} C_{n;m}(k) b(k) = \frac{1}{2} \sum_{k=0}^{2n} (C_{n;m}(k) + C_{n;m}(2n-k)) b(k) = 0.$$

This result together with some manipulations proves (ii).

$$\begin{aligned} (iii) \quad & \sum_{k=0}^{2n+1} \mathbf{i}^{k(2n+1-k) + k} a(k) \\ &= \frac{1}{2} \sum_{k=0}^{2n+1} \left( (-1)^{k(n+1)} \mathbf{i}^{-k^2} + (-1)^{(2n+1-k)(n+1)} \mathbf{i}^{-(2n+1-k)^2} \right) a(k) \\ &= \frac{1 + (-1)^n \mathbf{i}}{2} \sum_{k=0}^{2n+1} a(k). \end{aligned}$$

By a similar think, it is seen that

$$\sum_{k=0}^{2n+1} \mathbf{i}^{k(2n+1-k) - k} a(k) = \frac{1 - (-1)^n \mathbf{i}}{2} \sum_{k=0}^{2n+1} a(k), \text{ which completes the proof.}$$

□

**Lemma 2.** For positive integers  $n$  and  $m$ ,

$$\begin{aligned} & \sum_{k=0}^n (-1)^{\frac{mk(2n+1-k)}{2}} q^{\frac{mk(k-2n-1)}{2}} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{q^m} \\ &= \begin{cases} (-1)^{\binom{n+1}{2}} q^{-m\binom{n+1}{2}} (-q^{2m}; q^{2m})_n & \text{if } m \text{ is odd,} \\ q^{-m\binom{n+1}{2}} (-q^m; q^m)_n^2 & \text{if } m \text{ is even.} \end{cases} \end{aligned}$$

**Proof:** If  $m$  is odd,

$$\begin{aligned} & 2 \sum_{k=0}^n (-q)^{\frac{mk(k-2n-1)}{2}} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{q^m} \\ &= \sum_{k=0}^{2n+1} (-q^m)^{\frac{k(k-2n-1)}{2}} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{q^m} \\ &= \sum_{k=0}^{2n+1} \mathbf{i}^{k^2} q^{m\binom{k+1}{2}} (\mathbf{i}^{-(2n+1)} q^{-m(n+1)})^k \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{q^m} \end{aligned}$$

which, by (i) in Lemma 1, gives us

$$\begin{aligned} &= \frac{2}{1-\mathbf{i}} \sum_{k=0}^{2n+1} q^{m\binom{k+1}{2}} (\mathbf{i}^{-(2n+1)} q^{-m(n+1)})^k \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{q^m} \\ &= (1+\mathbf{i}) \prod_{k=1}^{2n+1} (1+\mathbf{i}^{-(2n+1)} q^{m(k-n-1)}) \\ &= (1+\mathbf{i})(1+\mathbf{i}^{-(2n+1)}) \prod_{j=1}^n (1+\mathbf{i}^{-(2n+1)} q^{mj})(1+\mathbf{i}^{-(2n+1)} q^{-mj}) \\ &= 2\mathbf{i}^{n^2} \prod_{j=1}^n \mathbf{i}^{-(2n+1)} (q^{mj} + q^{-mj}) \\ &= 2(-q^{-m})^{\binom{n+1}{2}} (-q^{2m}; q^{2m})_n. \end{aligned}$$

Thus the proof is complete for the case  $m$  is odd.

If  $m$  is even,

$$\begin{aligned}
& 2 \sum_{k=0}^n (-q)^{\frac{mk(k-2n-1)}{2}} \left[ \begin{matrix} 2n+1 \\ k \end{matrix} \right]_{q^m} \\
&= \sum_{k=0}^{2n+1} q^{\frac{mk(k-2n-1)}{2}} \left[ \begin{matrix} 2n+1 \\ k \end{matrix} \right]_{q^m} \\
&= \sum_{k=0}^{2n+1} q^{m \binom{k+1}{2}} q^{-mk(n+1)} \left[ \begin{matrix} 2n+1 \\ k \end{matrix} \right]_{q^m} \\
&= \prod_{k=1}^{2n+1} (1 + q^{m(k-n-1)}) = 2 \prod_{j=1}^n (1 + q^{mj})(1 + q^{-mj}) \\
&= 2 \prod_{j=1}^n (2 + q^{mj} + q^{-mj}) = 2 \prod_{j=1}^n q^{-mj} (1 + q^{mj})^2 \\
&= 2q^{-m \binom{n+1}{2}} (-q^m; q^m)_n^2.
\end{aligned}$$

So we have the conclusion.  $\square$

**Theorem 1.** For positive integers  $n$  and  $m$ ,

$$\sum_{k=0}^n \left\{ \begin{matrix} 2n+1 \\ k \end{matrix} \right\}_{U;m} = \begin{cases} \prod_{k=1}^n V_{2mk} & \text{if } m \text{ is odd,} \\ \prod_{k=1}^n V_{mk}^2 & \text{if } m \text{ is even.} \end{cases}$$

**Proof:** The proof could be obtained from Lemma 2 by taking  $q = \beta/\alpha$ .  $\square$

**Lemma 3.** For even  $m$  and any integer  $t$ ,

$$\sum_{k=0}^{2n} (-1)^{\binom{m}{2}+t} k q^{\frac{mk(k-2n)}{2}} \left[ \begin{matrix} 2n \\ k \end{matrix} \right]_{q^m} = \prod_{k=1}^n \left( (-1)^{\frac{m}{2}+t} q^{-\frac{m(2k-1)}{2}} (1 + q^{m(2k-1)}) + 2 \right).$$

**Proof:** Consider the RHS of the claimed identity, we write

$$\begin{aligned}
& \sum_{k=0}^{2n} (-1)^{\left(\frac{m}{2}+t\right)k} q^{\frac{mk(k-2n)}{2}} \left[ \begin{matrix} 2n \\ k \end{matrix} \right]_{q^m} \\
&= \sum_{k=0}^{2n} q^{m\binom{k+1}{2}} \left( (-1)^{\frac{m}{2}+t} q^{-\frac{m(2n+1)}{2}} \right)^k \left[ \begin{matrix} 2n \\ k \end{matrix} \right]_{q^m} \\
&= \prod_{k=1}^{2n} \left( 1 + (-1)^{\frac{m}{2}+t} q^{\frac{m(2k-2n-1)}{2}} \right) \\
&= \prod_{j=1}^n \left( 1 + (-1)^{\frac{m}{2}+t} q^{\frac{m(2j-1)}{2}} \right) \left( 1 + (-1)^{\frac{m}{2}+t} q^{-\frac{m(2j-1)}{2}} \right) \\
&= \prod_{j=1}^n \left( (-1)^{\frac{m}{2}+t} q^{-\frac{m(2j-1)}{2}} (q^{m(2j-1)} + 1) + 2 \right).
\end{aligned}$$

Thus the proof is complete.  $\square$

**Theorem 2.** For even  $m$  and any integer  $t$ ,

$$\sum_{k=0}^{2n} (-1)^{tk} \left\{ \begin{matrix} 2n \\ k \end{matrix} \right\}_{U;m} = \prod_{k=1}^n \left( (-1)^t V_{m(2k-1)} + 2 \right).$$

As a consequence of Theorem 2, we have the following Corollary:

**Corollary 1.** For even  $m$ ,

$$\begin{aligned}
(i) \quad & \sum_{k=0}^n \left\{ \begin{matrix} 2n \\ 2k \end{matrix} \right\}_{U;m} = \frac{1}{2} \left\{ \prod_{k=1}^n (V_{m(2k-1)} + 2) + \prod_{k=1}^n (-V_{m(2k-1)} + 2) \right\}, \\
(ii) \quad & \sum_{k=1}^n \left\{ \begin{matrix} 2n \\ 2k-1 \end{matrix} \right\}_{U;m} = \frac{1}{2} \left\{ \prod_{k=1}^n (V_{m(2k-1)} + 2) - \prod_{k=1}^n (-V_{m(2k-1)} + 2) \right\}.
\end{aligned}$$

**Lemma 4.** For any positive integers  $n$  and  $m$ ,

$$\sum_{k=0}^{2n} \mathbf{i}^{k(2mn-mk\pm 1)} q^{\frac{mk(k-2n)}{2}} \left[ \begin{matrix} 2n \\ k \end{matrix} \right]_{q^m} = \mathbf{i}^{n(mn\pm 1)} q^{-\frac{mn^2}{2}} (-q^m, q^{2m})_n.$$

**Proof:** Consider the RHS of the claimed identity and by using (ii) in Lemma 1,

we get

$$\begin{aligned}
& \sum_{k=0}^{2n} \mathbf{i}^{k(2mn-mk\pm 1)} q^{\frac{mk(k-2n)}{2}} \left[ \begin{matrix} 2n \\ k \end{matrix} \right]_{q^m} \\
&= \sum_{k=0}^{2n} \mathbf{i}^{-mk^2} q^{m \binom{k+1}{2}} \left( \mathbf{i}^{2mn\pm 1} q^{-\frac{m(2n+1)}{2}} \right)^k \left[ \begin{matrix} 2n \\ k \end{matrix} \right]_{q^m} \\
&= \mathbf{i}^{-mn^2} \sum_{k=0}^{2n} q^{m \binom{k+1}{2}} \left( \mathbf{i}^{2mn\pm 1} q^{-\frac{m(2n+1)}{2}} \right)^k \left[ \begin{matrix} 2n \\ k \end{matrix} \right]_{q^m} \\
&= \mathbf{i}^{-mn^2} \prod_{k=1}^{2n} \left( 1 + \mathbf{i}^{2mn\pm 1} q^{\frac{m(2k-2n-1)}{2}} \right) \\
&= \mathbf{i}^{-mn^2} \prod_{j=1}^n \left( 1 + \mathbf{i}^{2mn\pm 1} q^{\frac{m(2j-1)}{2}} \right) \left( 1 + \mathbf{i}^{2mn\pm 1} q^{-\frac{m(2j-1)}{2}} \right) \\
&= \mathbf{i}^{-mn^2} \prod_{j=1}^n \mathbf{i}^{2mn\pm 1} q^{-\frac{m(2j-1)}{2}} (1 + q^{m(2j-1)}) \\
&= \mathbf{i}^{mn^2 \pm n} q^{-\frac{mn^2}{2}} (-q^m; q^{2m})_n.
\end{aligned}$$

So we have the claimed identity.  $\square$

**Theorem 3.** For any positive integers  $n$  and  $m$ ,

$$\sum_{k=0}^{2n} \mathbf{i}^{\pm k} \left\{ \begin{matrix} 2n \\ k \end{matrix} \right\}_{U;m} = \mathbf{i}^{\pm n} \prod_{k=1}^n V_{m(2k-1)}.$$

Using Theorem 3, the following corollary can be obtained:

**Corollary 2.** For any positive integers  $n$  and  $m$ ,

$$\begin{aligned}
(i) \quad & \sum_{k=0}^n (-1)^k \left\{ \begin{matrix} 2n \\ 2k \end{matrix} \right\}_{U;m} = \cos \frac{n\pi}{2} \prod_{k=1}^n V_{m(2k-1)}, \\
(ii) \quad & \sum_{k=0}^n (-1)^k \left\{ \begin{matrix} 2n \\ 2k-1 \end{matrix} \right\}_{U;m} = -\sin \frac{n\pi}{2} \prod_{k=1}^n V_{m(2k-1)}.
\end{aligned}$$

**Lemma 5.** For any positive integers  $n$  and  $m$ ,

$$\begin{aligned}
& \sum_{k=0}^{2n+1} \mathbf{i}^{k(m(k-2n-1)\pm 1)} q^{\frac{mk(k-2n-1)}{2}} \left[ \begin{matrix} 2n+1 \\ k \end{matrix} \right]_{q^m} \\
&= (1 \pm \mathbf{i}) \mathbf{i}^{\pm n} \begin{cases} (-1)^{\binom{n+1}{2}} q^{-m \binom{n+1}{2}} (-q^m; q^m)_n^2 & \text{if } m \text{ is odd,} \\ q^{-m \binom{n+1}{2}} (-q^{2m}; q^{2m})_n & \text{if } m \text{ is even.} \end{cases}
\end{aligned}$$

**Proof:** If  $m$  is odd, then we write

$$\begin{aligned} & \sum_{k=0}^{2n+1} \mathbf{i}^{mk(2n+1-k)\pm k} q^{\frac{mk(k-2n-1)}{2}} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{q^m} \\ = & \sum_{k=0}^{2n+1} \mathbf{i}^{k(2n+1-k)\pm k} q^{\frac{mk(k-2n-1)}{2}} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{q^m} \end{aligned}$$

which, by (iii) in Lemma 1, gives us

$$\begin{aligned} & = \frac{1 \pm (-1)^n \mathbf{i}}{2} \sum_{k=0}^{2n+1} q^{m \binom{k+1}{2}} q^{-m(n+1)k} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{q^m} \\ & = \frac{1 \pm (-1)^n \mathbf{i}}{2} \prod_{k=1}^{2n+1} (1 + q^{m(k-n-1)}) \\ & = \frac{1 \pm (-1)^n \mathbf{i}}{2} 2 \prod_{j=1}^n (1 + q^{mj})(1 + q^{-mj}) \\ & = (1 \pm (-1)^n \mathbf{i}) \prod_{j=1}^n (2 + q^{mj} + q^{-mj}) \\ & = (1 \pm (-1)^n \mathbf{i}) \prod_{j=1}^n q^{-mj} (1 + q^{mj})^2 \\ & = (1 \pm (-1)^n \mathbf{i}) q^{-m \binom{n+1}{2}} (-q^m; q^m)_n^2 \\ & = (1 \pm \mathbf{i}) \mathbf{i}^{\pm n} (-q^{-m})^{\binom{n+1}{2}} (-q^m; q^m)_n^2. \end{aligned}$$

Thus the claim is seen for odd  $m$ .



If  $m$  is even, then we similarly obtain by (1.1),

$$\begin{aligned}
& \sum_{k=0}^{2n+1} \mathbf{i}^{mk(2n+1-k)\pm k} q^{\frac{mk(k-2n-1)}{2}} \left[ \begin{matrix} 2n+1 \\ k \end{matrix} \right]_{q^m} \\
&= \sum_{k=0}^{2n+1} \mathbf{i}^{\pm k} q^{\frac{mk(k-2n-1)}{2}} \left[ \begin{matrix} 2n+1 \\ k \end{matrix} \right]_{q^m} \\
&= \sum_{k=0}^{2n+1} q^{m\binom{k+1}{2}} \left( \mathbf{i}^{\pm 1} q^{-m(n+1)} \right)^k \left[ \begin{matrix} 2n+1 \\ k \end{matrix} \right]_{q^m} \\
&= \prod_{k=1}^{2n+1} (1 + \mathbf{i}^{\pm 1} q^{m(k-n-1)}) = (1 + \mathbf{i}^{\pm 1}) \prod_{j=1}^n (1 + \mathbf{i}^{\pm 1} q^{mj})(1 + \mathbf{i}^{\pm 1} q^{-mj}) \\
&= (1 \pm \mathbf{i}) \prod_{j=1}^n \mathbf{i}^{\pm 1} (q^{mj} + q^{-mj}) = (1 \pm \mathbf{i}) \mathbf{i}^{\pm n} \prod_{j=1}^n q^{-mj} (1 + q^{2mj}) \\
&= (1 \pm \mathbf{i}) \mathbf{i}^{\pm n} q^{-m\binom{n+1}{2}} (-q^{2m}; q^{2m})_n.
\end{aligned}$$

□

**Theorem 4.** For any positive integers  $n$  and  $m$ ,

$$\sum_{k=0}^{2n+1} \mathbf{i}^{\pm k} \left\{ \begin{matrix} 2n+1 \\ k \end{matrix} \right\}_{U;m} = (1 \pm \mathbf{i}) \mathbf{i}^{\pm n} \begin{cases} \prod_{k=1}^n V_{mk}^2 & \text{if } m \text{ is odd,} \\ \prod_{k=1}^n V_{2mk} & \text{if } m \text{ is even.} \end{cases}$$

Using Theorem 4, we have the following Corollary:

**Corollary 3.** For any positive integers  $n$  and  $m$ ,

$$\begin{aligned}
(i) \sum_{k=0}^n (-1)^k \left\{ \begin{matrix} 2n+1 \\ 2k \end{matrix} \right\}_{U;m} &= (-1)^{\binom{n+1}{2}} \begin{cases} \prod_{k=1}^n V_{mk}^2 & \text{if } m \text{ is odd,} \\ \prod_{k=1}^n V_{2mk} & \text{if } m \text{ is even.} \end{cases} \\
(ii) \sum_{k=0}^n (-1)^k \left\{ \begin{matrix} 2n+1 \\ 2k+1 \end{matrix} \right\}_{U;m} &= (-1)^{\binom{n}{2}} \begin{cases} \prod_{k=1}^n V_{mk}^2 & \text{if } m \text{ is odd,} \\ \prod_{k=1}^n V_{2mk} & \text{if } m \text{ is even.} \end{cases}
\end{aligned}$$

**Lemma 6.** For positive integers  $n$  and  $m$ ,

$$\sum_{k=0}^{2n} (-1)^k q^{mk(k-2n)} \left[ \begin{matrix} 2n \\ k \end{matrix} \right]_{q^m}^2 = (-1)^n q^{-mn^2} \frac{(-q^{2m}; q^{2m})_n (q^{2m}, q^{4m})_n}{(q^{2m}; q^{2m})_n}. \quad (2.1)$$

**Proof:** We will show how to evaluate and prove such sums entirely mechanically. We used Zeilberger's own version [8], which is a Mathematica program. After loading the package `qZeil.m` in Mathematica, define

$$\text{SUM}[n] := \sum_{k=0}^{2n} (-1)^k q^{mk(k-2n)} \begin{bmatrix} 2n \\ k \end{bmatrix}_{q^m}^2. \quad (2.2)$$

If we run the RHS of (2.2) in the program, then we obtain following first order recurrence relation:

$$\text{SUM}[n] = -\frac{q^{m-2mn}(1+q^{2mn})(1-q^{2m(2n-1)})}{1-q^{2mn}} \text{SUM}[n-1].$$

Clearly the right hand side of equation (2.1) satisfies the same recurrence relation (and the both sides of (2.1) have the same initial conditions). Also, if we solve this recurrence relation, we obtain

$$\text{SUM}[n] = \frac{(-1)^{n-1} q^{m-mn^2} (-q^{3m}; q^{2m})_{n-1} (q^{3m}; q^{2m})_{n-1} (-q^{4m}; q^{2m})_{n-1}}{(q^{4m}; q^{2m})_{n-1}}. \quad (2.3)$$

Thus, the right hand side of (2.1) and (2.3) are the same; it is no more necessary to distinguish the parity of  $m$ . Consequently, the proof is complete.  $\square$

**Theorem 5.** For positive integers  $n$  and  $m$ ,

$$\sum_{k=0}^{2n} (-1)^{(m+1)k} \left\{ \begin{matrix} 2n \\ k \end{matrix} \right\}_{U;m}^2 = (-1)^{(m+1)n} \prod_{k=1}^n \frac{V_{2mk} U_{2m(2k-1)}}{U_{2mk}}.$$

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