

FORMULAS FOR WEIGHTED BINOMIAL SUMS WITH THE POWERS OF TERMS OF BINARY RECURRENCES

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ABSTRACT. In this paper, we give general formulas for some weighted binomial sums with the powers of terms of certain binary recurrences. As applications of our results, we show that the weighted binomial sums with the generalized Fibonacci and Lucas numbers can be expressed via the second kinds of both Chebyshev polynomials and Stirling numbers.

1. INTRODUCTION

For $n > 1$, define the binary recurrences $\{U_n\}$ and $\{V_n\}$ by

$$U_n = pU_{n-1} - U_{n-2} \quad \text{and} \quad V_n = pV_{n-1} - V_{n-2}$$

where $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = p$, respectively.

The Fibonacci subsequence $\{F_{2n}\}$ and the Pell subsequence $\{P_{2n}\}$ are the special cases of the sequence $\{U_n\}$ for the cases $p = 3$ and $p = 6$, respectively. It is also known that the natural numbers are special cases of the sequence $\{U_n\}$ for $p = 2$.

The Binet formulas of $\{U_n\}$ and $\{V_n\}$ are as follows:

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n, \quad (1.1)$$

where $\alpha, \beta = (p \pm \sqrt{p^2 - 4})/2$.

From [8], we have that for $k \geq 0$ and $n > 1$,

$$\begin{aligned} U_{kn} &= V_k U_{k(n-1)} - U_{k(n-2)}, \\ V_{kn} &= V_k V_{k(n-1)} - V_{k(n-2)}. \end{aligned}$$

Wiemann and Cooper [5] mentioned about some conjectures of Melham for the sum:

$$\sum_{h=1}^n F_{2h}^{2m+1},$$

where F_n stands for the n^{th} Fibonacci number.

Ozeki [2] considered Melham's sum and then he gave an explicit expansion for Melham's sum as a polynomial in F_{2n+1} .

In general, Prodinger [3] derive the general formula for the sum:

$$\sum_{h=0}^n F_{2h+\delta}^{2m+\varepsilon},$$

2000 *Mathematics Subject Classification.* 11B37, 11B39, 05A19.

Key words and phrases. Binary recurrences, weighted binomial sums, Stirling numbers of the second kind, Chebyshev polynomials of the second kind.

where $\varepsilon, \delta \in \{0, 1\}$, as well as the evaluations of the corresponding sums for Lucas numbers.

In [6], we gave alternating Melham's sums for Fibonacci and Lucas numbers of the form $\sum_{h=1}^n (-1)^h F_{2h+\delta}^{2m+\varepsilon}$ and $\sum_{h=1}^n (-1)^h L_{2h+\delta}^{2m+\varepsilon}$, where $\varepsilon, \delta \in \{0, 1\}$.

Recently Khan and Kwong [7] considered the sums

$$\sum_{h=0}^n h^m \binom{n}{h} U_h \quad \text{and} \quad \sum_{h=0}^n (-1)^{n+h} h^m \binom{n}{h} U_h, \quad (1.2)$$

and express them in terms of two associated sequences. Special cases of $m = 2, 3$, lead to interesting binomial and Fibonacci identities.

In this paper, we shall give general formulas for the sums

$$\begin{aligned} & \sum_{h=0}^n \binom{n}{h} h^m U_{ht}^{2m+\varepsilon}, \quad \sum_{h=0}^n \binom{n}{h} h^m V_{ht}^{2m+\varepsilon}, \\ & \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m U_{ht}^{2m+\varepsilon}, \quad \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m V_{ht}^{2m+\varepsilon}, \end{aligned}$$

where t is an positive integer and $\varepsilon \in \{0, 1\}$. In order to do this, firstly we will give general cases of the sums given by (1.2) and we shall derive similar formulas for their Lucas counterparts in the second section. After this, by using these results, we give our main results in the third section.

2. GENERALIZED WEIGHTED BINOMIAL IDENTITIES

In this section, we will give generalizations of the results of [7] by considering the sequence $\{U_{nk}\}$ instead of the sequence $\{U_n\}$. We also give similar formulas for the Lucas sequence $\{V_n\}$. While giving these results, we follow the proof strategy given in [7].

Define the sequences $\{X_{kn}\}$, $\{Y_{kn}\}$, $\{W_{kn}\}$ and $\{Z_{kn}\}$ for $n \geq 2$ as follows:

$$\begin{aligned} X_0 &= 0, \quad X_k = U_k, \quad X_{kn} = (V_k + 2)(X_{k(n-1)} - X_{k(n-2)}), \\ Y_0 &= 0, \quad Y_k = U_k, \quad Y_{kn} = (V_k - 2)(Y_{k(n-1)} + Y_{k(n-2)}), \\ W_0 &= 2, \quad W_k = V_k + 2, \quad W_{kn} = (V_k + 2)(W_{k(n-1)} - W_{k(n-2)}), \\ Z_0 &= 2, \quad Z_k = V_k - 2, \quad Z_{kn} = (V_k - 2)(Z_{k(n-1)} + Z_{k(n-2)}). \end{aligned}$$

The Binet formulas of $\{X_{kn}\}$, $\{Y_{kn}\}$, $\{W_{kn}\}$ and $\{Z_{kn}\}$ are

$$\begin{aligned} X_{kn} &= \frac{(1 + \alpha^k)^n - (1 + \beta^k)^n}{\alpha - \beta}, \quad Y_{kn} = \frac{(\alpha^k - 1)^n - (\beta^k - 1)^n}{\alpha - \beta}, \\ W_{kn} &= (1 + \alpha^k)^n + (1 + \beta^k)^n, \quad \text{and} \quad Z_{kn} = (\alpha^k - 1)^n + (\beta^k - 1)^n, \end{aligned}$$

where $\alpha^k, \beta^k = (V_k \pm \sqrt{V_k^2 - 4})/2$.

Lemma 1. For $n \geq 0$, we have

$$\sum_{h=0}^n \binom{n}{h} U_{hk} = X_{nk}, \quad (2.1)$$

$$\sum_{h=0}^n \binom{n}{h} h U_{hk} = n (X_{kn} - X_{k(n-1)}), \quad (2.2)$$

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} U_{hk} = Y_{kn}, \quad (2.3)$$

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h U_{hk} = n (Y_{kn} + Y_{k(n-1)}). \quad (2.4)$$

Proof. Since

$$\sum_{h=0}^n \binom{n}{h} \alpha^{hk} = (1 + \alpha^k)^n \quad \text{and} \quad \sum_{h=0}^n \binom{n}{h} \beta^{hk} = (1 + \beta^k)^n,$$

the first claim follows from the Binet formula of $\{U_{hk}\}$. Similarly by considering

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} \alpha^{hk} = (\alpha^k - 1)^n, \quad \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} \beta^{hk} = (\beta^k - 1)^n,$$

we have the third claim.

Considering

$$\begin{aligned} \sum_{h=0}^n \binom{n}{h} h \alpha^{hk} &= \frac{\alpha}{k} \cdot \frac{d}{d\alpha} \left[\sum_{h=0}^n \binom{n}{h} \alpha^{hk} \right] = \frac{\alpha}{k} \frac{d}{d\alpha} \left((1 + \alpha^k)^n \right) \\ &= n \alpha^k (1 + \alpha^k)^{n-1} = n \left((1 + \alpha^k)^n - (1 + \alpha^k)^{n-1} \right), \end{aligned} \quad (2.5)$$

and similarly

$$\sum_{h=0}^n \binom{n}{h} h \beta^{hk} = n \left((1 + \beta^k)^n - (1 + \beta^k)^{n-1} \right), \quad (2.6)$$

one can easily obtain the rest of claimed identities. \square

Define the operators D_U and Δ_U on X_{kn} and Y_{kn} for $n \geq 1$, respectively, as follows:

$$\begin{aligned} D_U (X_{kn}) &= n (X_{kn} - X_{k(n-1)}), \\ \Delta_U (Y_{kn}) &= n (Y_{kn} + Y_{k(n-1)}). \end{aligned}$$

Lemma 2. For $n \geq 0$

$$\begin{aligned} \sum_{h=0}^n \binom{n}{h} V_{hk} &= W_{nk}, \\ \sum_{h=0}^n \binom{n}{h} h V_{hk} &= n (W_{kn} - W_{k(n-1)}), \\ \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} V_{hk} &= Z_{kn}, \\ \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h V_{hk} &= n (Z_{kn} + Z_{k(n-1)}). \end{aligned}$$

Proof. The proof is similar to the proof of Lemma 1. \square

Define the operators D_V and Δ_V on W_{kn} and Z_{kn} for $n \geq 1$, respectively, as follows:

$$\begin{aligned} D_V(W_{kn}) &= n (W_{kn} - W_{k(n-1)}), \\ \Delta_V(Z_{kn}) &= n (Z_{kn} + Z_{k(n-1)}). \end{aligned}$$

In [7], the authors stated that if $\sum_{h=0}^n h^m \binom{n}{h} U_h$ is of the form $\sum_{k \geq 0} a_k X_k$, then $\sum_{h=0}^n h^m \binom{n}{h} U_h = D \left(\sum_{k \geq 0} a_k X_k \right)$. Hence the coefficients a_k can be computed iteratively as follows.

For $m \geq 0$, define the polynomials $a_{m,r}(n)$ recursively as follows [7]:

$$a_{m,r}(n) = (n-r) a_{m-1,r}(n) - (n-r+1) a_{m-1,r-1}(n), \quad m \geq 1, \quad (2.7)$$

with the initial value $a_{0,0}(n) = 1$ and the convention that $a_{m,r}(n) = 0$ if $r < 0$ or $r > m$.

Thus, we see that similar direction is also valid for the sum $\sum_{h=0}^n \binom{n}{h} h^m U_{hk}$ and so give the following result.

Theorem 1. For $n \geq 0$

$$\sum_{h=0}^n \binom{n}{h} h^m U_{hk} = \sum_{r=0}^m a_{m,r}(n) X_{k(n-r)}, \quad (2.8)$$

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m U_{hk} = \sum_{r=0}^m (-1)^r a_{m,r}(n) Y_{k(n-r)}, \quad (2.9)$$

$$\sum_{h=0}^n \binom{n}{h} h^m V_{hk} = \sum_{r=0}^m a_{m,r}(n) W_{k(n-r)}, \quad (2.10)$$

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m V_{hk} = \sum_{r=0}^m (-1)^r a_{m,r}(n) Z_{k(n-r)}. \quad (2.11)$$

Proof. It is known that

$$\sum_{h=0}^n \binom{n}{h} h^m U_{hk} = D \left[\sum_{h=0}^n \binom{n}{h} h^{m-1} U_{hk} \right].$$

Thus

$$\begin{aligned}
 \sum_{r=0}^m a_{m,r}(n) X_{k(n-r)} &= D \left[\sum_{r=0}^{m-1} a_{m-1,r}(n) X_{k(n-r)} \right] \\
 &= \sum_{r=0}^{m-1} a_{m-1,r}(n) (n-r) (X_{k(n-r)} - X_{k(n-r-1)}) \\
 &= a_{m-1,0}(n) X_{kn} \\
 &\quad + \sum_{r=1}^{m-1} (n-r) a_{m-1,r}(n) - (n-r+1) a_{m-1,r}(n) X_{k(n-r)} \\
 &\quad - (n-m+1) a_{m-1,m-1}(n) X_{k(n-m)}.
 \end{aligned}$$

Since $a_{m,r}(n) = 0$ if $r < 0$ or $r > m$, we write

$$\sum_{r=0}^m a_{m,r}(n) X_{k(n-r)} = \sum_{r=0}^m (n-r) a_{m-1,r}(n) - (n-r+1) a_{m-1,r}(n) X_{k(n-r)}.$$

The recurrence for $a_{m,r}(n)$ follows directly by comparing coefficients. The rest of claimed identities could be similarly proved. \square

For example, when $p = 2$ in (2.8), we have that $\alpha = \beta = 1$ and so $X_n = n2^{n-1}$, which was also given in [7]. For different examples of our results, suppose that $k = m = 2$, by using the results above, we obtain

$$\sum_{h=0}^n \binom{n}{h} h^2 U_{2h} = \sum_{r=0}^2 a_{2,r}(n) X_{2(n-r)} = 2^{n-2} n^2 (n+3).$$

Now let $p = 3$, $k = 2$ and $m = 1$. Then $U_{2h} = F_{4h}$ and so we have

$$Y_{2n} = \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} F_{4h} = \begin{cases} 5^{(n-1)/2} L_{2n}, & \text{if } n \text{ is odd,} \\ 5^{n/2} F_{2n}, & \text{if } n \text{ is even.} \end{cases} \quad (2.12)$$

By (2.12), we also get

$$\begin{aligned}
 \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h F_{4h} &= \sum_{r=0}^1 (-1)^r a_{1,r}(n) Y_{2(n-r)} \\
 &= a_{1,0}(n) Y_{2n} - a_{1,1}(n) Y_{2(n-1)} \\
 &= \begin{cases} n5^{(n-1)/2} (L_{2n} + F_{2n-2}), & \text{if } n \text{ is odd,} \\ n5^{(n-2)/2} (5F_{2n} + L_{2n-2}), & \text{if } n \text{ is even.} \end{cases}
 \end{aligned}$$

Similar to the above examples, one can obtain various results for each different values of k and p from the results above.

3. THE MAIN RESULTS

In this section, we give general formulas for the sums:

$$\begin{aligned}
 &\sum_{h=0}^n \binom{n}{h} h^m U_{ht}^{2m+\varepsilon}, \quad \sum_{h=0}^n \binom{n}{h} h^m V_{ht}^{2m+\varepsilon}, \\
 &\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m U_{ht}^{2m+\varepsilon} \quad \text{and} \quad \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m V_{ht}^{2m+\varepsilon},
 \end{aligned}$$

where t is an positive integer and $\varepsilon \in \{0, 1\}$.

We shall assume that p in the definition of the sequence $\{U_n\}$ is a positive integer different from 2 throughout in this section.

For the readers convenience and later use in the next result, it would be convenient to recall some facts from [4]: For any real numbers m and n ,

$$(m+n)^t = \begin{cases} \sum_{i=0}^{(t-1)/2} \binom{t}{i} (mn)^i (m^{t-2i} + n^{t-2i}) & \text{if } t \text{ is odd,} \\ \sum_{i=0}^{t/2-1} \binom{t}{i} (mn)^i (m^{t-2i} + n^{t-2i}) + \binom{t}{t/2} (mn)^{t/2} & \text{if } t \text{ is even,} \end{cases} \quad (3.1)$$

and

$$(m-n)^t = \begin{cases} \sum_{i=0}^{(t-1)/2} \binom{t}{i} (mn)^i (-1)^i (m^{t-2i} - n^{t-2i}) & \text{if } t \text{ is odd,} \\ \sum_{i=0}^{t/2-1} \binom{t}{i} (mn)^i (-1)^i (m^{t-2i} + n^{t-2i}) \\ + \binom{t}{t/2} (mn)^{t/2} (-1)^{t/2} & \text{if } t \text{ is even.} \end{cases} \quad (3.2)$$

Theorem 2. *i) For $t, m > 0$,*

$$\begin{aligned} \sum_{h=0}^n \binom{n}{h} h^m U_{ht}^{2m} &= \frac{U_t^{2m}}{(V_t^2 - 4)^m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{r=0}^m a_{m,r}(n) W_{t(2m-2i)(n-r)} \\ &+ \frac{U_t^{2m}}{(V_t^2 - 4)^m} \binom{2m}{m} (-1)^m 2^{n-m} P(n), \end{aligned}$$

where $P(n)$ is a monic polynomial of degree m satisfying $\sum_{h=0}^n \binom{n}{h} h^m = 2^{n-m} P(n)$.

ii) For $t, m > 0$,

$$\sum_{h=0}^n \binom{n}{h} h^m U_{ht}^{2m+1} = \frac{U_t^{2m}}{(V_t^2 - 4)^m} \sum_{i=0}^m (-1)^i \binom{2m}{i} \sum_{r=0}^m a_{m,r}(n) X_{t(2m+1-2i)(n-r)}.$$

Proof. *i)* For $t > 0$, by the Binet formula of $\{U_n\}$, we have

$$\sum_{h=0}^n \binom{n}{h} h^m U_{ht}^{2m} = \sum_{h=0}^n \binom{n}{h} h^m \left(\frac{\alpha^{ht} - \beta^{ht}}{\alpha - \beta} \right)^{2m}.$$

Using (3.2), we write

$$\begin{aligned} &\sum_{h=0}^n \binom{n}{h} h^m U_{ht}^{2m} \\ &= \frac{1}{(\alpha - \beta)^{2m}} \sum_{h=0}^n \binom{n}{h} h^m \left[\sum_{i=0}^{m-1} \binom{2m}{i} (-1)^i (\alpha^{ht(2m-2i)} + \beta^{ht(2m-2i)}) \right. \\ &\quad \left. + \binom{2m}{m} (-1)^m \right] \\ &= \frac{1}{(p^2 - 4)^m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{h=0}^n \binom{n}{h} h^m V_{ht(2m-2i)} \\ &\quad + \binom{2m}{m} (-1)^m \sum_{h=0}^n \binom{n}{h} h^m. \end{aligned}$$

By taking $k = 2t(m - i)$ in (2.10) and $\sum_{h=0}^n \binom{n}{h} h^m = 2^{n-m} P(n)$, where $P(n)$ is a monic polynomial of degree m (for the coefficients of this polynomials, see the sequence *A102573* in the OEIS or see [1], p. 135), we write

$$\begin{aligned} & \sum_{h=0}^n \binom{n}{h} h^m U_{ht}^{2m} \\ &= \frac{U_t^{2m}}{(V_t^2 - 4)^m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{r=0}^m a_{m,r}(n) W_{t(2m-2i)(n-r)} \\ & \quad + \frac{U_t^{2m}}{(V_t^2 - 4)^m} \binom{2m}{m} (-1)^m 2^{n-m} P(n). \end{aligned}$$

ii) The proof is similar to the proof of i). □

For example, when $p = 6$, $m = 3$ and $t = 2$, we derive

$$\begin{aligned} \sum_{h=0}^n \binom{n}{h} h^3 U_{2h}^7 &= \frac{1}{32} \sum_{i=0}^3 (-1)^i \binom{6}{i} \sum_{r=0}^3 a_{3,r}(n) X_{(7-2i)(n-r)} \\ &= \frac{1}{32} \left(\sum_{r=0}^3 a_{3,r}(n) X_{7(n-r)} - 6 \sum_{r=0}^3 a_{3,r}(n) X_{5(n-r)} \right. \\ & \quad \left. + 15 \sum_{r=0}^3 a_{3,r}(n) X_{3(n-r)} - 20 \sum_{r=0}^3 a_{3,r}(n) X_{(n-r)} \right). \end{aligned}$$

For $p = 6$ in the definition of sequence $\{U_n\}$, we have $U_n = \frac{1}{2} P_{2n}$, where P_n is the n^{th} Pell number. Thus

$$X_n = \frac{1}{2} \sum_{h=0}^n \binom{n}{h} P_{2n} = U_n(\sqrt{2}), \tag{3.3}$$

where $U_n(x)$ is the Chebyshev's polynomials of the second kind. We have also another formula for X_n as shown:

$$X_n = \frac{1}{2} \sum_{h=0}^n \binom{n}{h} P_{2n} = W_{n+1}(0, 1; 8, -8),$$

where $W_n(a, b; p, q)$ is the Horadam sequence (see [9]).

Using (3.3), we get

$$\begin{aligned}
\sum_{h=0}^n \binom{n}{h} h^3 U_{2h}^7 &= \frac{1}{32} \left[n^3 U_{7n}(\sqrt{2}) - n(2n^2 + 2n - 1) U_{7(n-1)}(\sqrt{2}) \right. \\
&\quad + 3n(n-1)^2 U_{7(n-2)}(\sqrt{2}) - n(n-1)(n-2) U_{7(n-3)}(\sqrt{2}) \\
&\quad - 6 \left(n^3 U_{5n}(\sqrt{2}) - n(2n^2 + 2n - 1) U_{5(n-1)}(\sqrt{2}) \right. \\
&\quad \left. \left. + 3n(n-1)^2 U_{5(n-2)}(\sqrt{2}) - n(n-1)(n-2) U_{5(n-3)}(\sqrt{2}) \right) \right. \\
&\quad + 15 \left(n^3 U_{3n}(\sqrt{2}) - n(2n^2 + 2n - 1) U_{3(n-1)}(\sqrt{2}) \right. \\
&\quad \left. + 3n(n-1)^2 U_{3(n-2)}(\sqrt{2}) - n(n-1)(n-2) U_{3(n-3)}(\sqrt{2}) \right) \\
&\quad \left. - 20 \left(n^3 U_n(\sqrt{2}) - n(2n^2 + 2n - 1) U_{(n-1)}(\sqrt{2}) \right. \right. \\
&\quad \left. \left. + 3n(n-1)^2 U_{(n-2)}(\sqrt{2}) - n(n-1)(n-2) U_{(n-3)}(\sqrt{2}) \right) \right].
\end{aligned}$$

Theorem 3. i) For $t, m > 0$,

$$\begin{aligned}
\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m U_{ht}^{2m} &= \frac{U_t^{2m}}{(V_t^2 - 4)^m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{r=0}^m (-1)^r a_{m,r}(n) Z_{2t(m-i)(n-r)} \\
&\quad + \frac{U_t^{2m}}{(V_t^2 - 4)^m} \binom{2m}{m} (-1)^m n! S(m, n),
\end{aligned}$$

where $S(m, n)$ is the Stirling numbers of the second kind.

ii) For $t, m > 0$,

$$\begin{aligned}
&\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m U_{ht}^{2m+1} \\
&= \frac{U_t^{2m}}{(V_t^2 - 4)^m} \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \sum_{r=0}^m (-1)^r a_{m,r}(n) Y_{t(2m+1-2i)(n-r)}.
\end{aligned}$$

Proof. i) For $t > 0$, consider

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m U_{ht}^{2m} = \sum_{h=0}^n \binom{n}{h} (-1)^h h^m \left(\frac{\alpha^{ht} - \beta^{ht}}{\alpha - \beta} \right)^{2m}.$$

By (3.2), we write

$$\begin{aligned}
 & \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m U_{ht}^{2m} \\
 = & \frac{1}{(\alpha - \beta)^{2m}} \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m \left[\sum_{i=0}^{m-1} \binom{2m}{i} (-1)^i (\alpha^{ht(2m-2i)} + \beta^{ht(2m-2i)}) \right. \\
 & \left. + \binom{2m}{m} (-1)^m \right] \\
 = & \frac{1}{(\alpha - \beta)^{2m}} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m V_{ht(2m-2i)} \\
 & + \frac{1}{(\alpha - \beta)^{2m}} \binom{2m}{m} (-1)^m \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m.
 \end{aligned}$$

By taking $k = 2t(m - i)$ in (2.11) and since $\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m = n!S(m, n)$, where $S(m, n)$ is the Stirling numbers of the second kind, the claim is obtained.
 ii) The proof is similar to the proof of i). \square

For example, when $p = 3$, we get $U_n = F_{2n}$. For $m = t = 2$, we have

$$\begin{aligned}
 & \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^2 U_{2h}^5 \\
 = & \frac{1}{5^2} \sum_{i=0}^2 (-1)^i \binom{5}{i} \sum_{r=0}^2 (-1)^r a_{2,r}(n) Y_{2(5-2i)(n-r)} \\
 = & \frac{1}{5^2} \left(\sum_{r=0}^2 (-1)^r a_{2,r}(n) Y_{10(n-r)} - 5 \sum_{r=0}^2 (-1)^r a_{2,r}(n) Y_{6(n-r)} \right. \\
 & \left. + 10 \sum_{r=0}^2 (-1)^r a_{2,r}(n) Y_{2(n-r)} \right).
 \end{aligned}$$

Since

$$Y_n = \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} F_{2h} = F_n,$$

we get

$$\begin{aligned}
 \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^2 U_{2h}^5 &= \frac{1}{5^2} [nF_{10n} + n(2n-1)F_{10(n-1)} + n(n-1)F_{10(n-2)} \\
 & - 5(nF_{6n} + n(2n-1)F_{6(n-1)} + n(n-1)F_{6(n-2)}) \\
 & + 10(nF_{2n} + n(2n-1)F_{2(n-1)} + n(n-1)F_{2(n-2)})].
 \end{aligned}$$

Theorem 4. i) For $t, m > 0$,

$$\sum_{h=0}^n \binom{n}{h} h^m V_{ht}^{2m} = \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{r=0}^m a_{m,r}(n) W_{t(2m-2i)(n-r)} + \binom{2m}{m} 2^{n-m} P(n),$$

where the polynomial $P(n)$ is defined as before.

ii) For $t, m > 0$,

$$\sum_{h=0}^n \binom{n}{h} h^m V_{ht}^{2m+1} = \sum_{i=0}^m \binom{2m+1}{i} \sum_{r=0}^m a_{m,r}(n) W_{t(2m+1-2i)(n-r)}.$$

Proof. i) For $t > 0$, by the Binet formula of $\{V_n\}$, we write

$$\sum_{h=0}^n \binom{n}{k} h^m V_{ht}^{2m} = \sum_{h=0}^n \binom{n}{h} h^m (\alpha^{ht} + \beta^{ht})^{2m},$$

which, by (3.1) and since $\alpha\beta = 1$, satisfies

$$\begin{aligned} \sum_{h=0}^n \binom{n}{h} h^m V_{ht}^{2m} &= \sum_{h=0}^n \binom{n}{h} h^m \left[\sum_{i=0}^{m-1} \binom{2m}{i} (\alpha^{ht(2m-2i)} + \beta^{ht(2m-2i)}) \right. \\ &\quad \left. + \binom{2m}{m} (\alpha\beta)^{thm} \right] \\ &= \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{h=0}^n \binom{n}{h} h^m V_{ht(2m-2i)} + \binom{2m}{m} \sum_{h=0}^n \binom{n}{h} h^m. \end{aligned}$$

By taking $k = 2t(m-i)$ in (2.10) and since $\sum_{h=0}^n \binom{n}{h} h^m = 2^{n-m} P(n)$, where $P(n)$ is defined as before, we get

$$\sum_{h=0}^n \binom{n}{h} h^m V_{ht}^{2m} = \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{r=0}^m a_{m,r}(n) W_{t(2m-2i)(n-r)} + \binom{2m}{m} 2^{n-m} P(n).$$

ii) The proof is similar to the proof of i). \square

Theorem 5. i) For $t, m > 0$,

$$\begin{aligned} \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m V_{ht}^{2m} &= \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{r=0}^m (-1)^r a_{m,r}(n) Z_{t(2m-2i)(n-r)} \\ &\quad + \binom{2m}{m} n! S(m, n), \end{aligned}$$

where $S(m, n)$ is the Stirling numbers of the second kind.

ii) For $t, m > 0$,

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m V_{ht}^{2m+1} = \sum_{i=0}^m \binom{2m+1}{i} \sum_{r=0}^m (-1)^r a_{m,r}(n) Z_{t(2m+1-2i)(n-r)}.$$

Proof. i) From the Binet formula of $\{V_n\}$, we write

$$\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m V_{ht}^{2m} = \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m (\alpha^{ht} + \beta^{ht})^{2m},$$

which, by (3.1) and since $\alpha\beta = 1$, is equivalent to

$$\begin{aligned} & \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m V_{ht}^{2m} \\ = & \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m \left(\sum_{i=0}^{m-1} \binom{2m}{i} (\alpha^{ht(2m-2i)} + \beta^{ht(2m-2i)}) \right) \\ & + \binom{2m}{m} (\alpha\beta)^{tm} \\ = & \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m V_{ht(2m-2i)} \\ & + \binom{2m}{m} \sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m. \end{aligned}$$

By taking $k = 2t(m - i)$ in (2.11) and since $\sum_{h=0}^n \binom{n}{h} (-1)^{n+h} h^m = n!S(m, n)$, where $S(m, n)$ is defined as before, the claim is obtained.

ii) The proof is similar to the proof of *i)*. □

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