RIORDAN GROUP APPROACHES IN MATRIX FACTORIZATIONS

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Abstract. In this paper, we consider an arbitrary binary polynomial sequence $\{A_n\}$ and then give a lower triangular matrix representation of this sequence. As main result, we obtain a factorization of the infinite generalized Pascal matrix in terms of this new matrix, using a Riordan group approach. Further some interesting results and applications are derived.

1. INTRODUCTION

For $n > 0$, the $n \times n$ Pascal matrix $P_n = [p_{ij}]$ is defined as follows [5]:

$$p_{ij} = \begin{cases} \binom{i-1}{j-1} & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

The authors [1] are the first to give matrix representations of the Pascal triangle. In [12], for a nonzero real x, the Pascal matrices $P_n[x] = [P_n(x; i, j)]$ and $Q_n[x] = [Q_n(x; i, j)]$ are generalized as follows

$$P_n(x; i, j) = \begin{cases} \binom{i-1}{j-1} x^{i-j} & \text{if } i \geq j, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Q_n(x; i, j) = \begin{cases} \binom{i-1}{j-1} x^{i+j-2} & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

Further in [13], the authors generalize the Pascal matrices $P_n[x]$ and $Q_n[x]$ for two nonzero real numbers x and y as follows

$$\varphi[x, y]_{ij} = \begin{cases} \binom{i-1}{j-1} x^{i-j} y^{i+j-2} & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

The Fibonacci and Lucas sequences have been discussed in so many studies. Besides, various generalizations and matrix representations of these sequences have been also introduced and investigated.

For nonnegative integers A and B such that $A^2 + 4B \neq 0$, the generalized Fibonacci and Lucas type sequences $\{U_n(A, B)\}$ and $\{V_n(A, B)\}$ are defined by for $n > 0$

$$\begin{aligned} U_{n+1}(A, B) &= AU_n(A, B) + BU_{n-1}(A, B), \\ V_{n+1}(A, B) &= AV_n(A, B) + BV_{n-1}(A, B) \end{aligned}$$

2000 *Mathematics Subject Classification.* 15A36, 11B37.
Key words and phrases. Factorization, Binary Recurrences, Pascal matrix.

where $U_0(A,B)=0$, $U_1(A,B)=1$ and $V_0(A,B)=2$, $V_1(A,B)=A$, respectively. For example, $U_n(1,1)=F_n$ (nth Fibonacci number) and $V_n(1,1)=L_n$ (nth Lucas number).

For the polynomial versions of generalized Fibonacci and Lucas numbers, we refer to [2]. Even more general cases of these polynomials are considered in [4], where two of us define the polynomial sequence $\{A_n(a,b;p,q)(x)\}$ (briefly $\{A_n(x)\}$) satisfying

$$A_{n+1}(x)=p(x)A_n(x)-q(x)A_{n-1}(x) \tag{1.1}$$

with $A_0(x)=a(x)$, $A_1(x)=b(x)$, where a, b, p,q are polynomials of x with real coefficients. In their study, the authors of [4] show that for $n>0$, any integer k and $n\equiv c \pmod{|k|}$, the sequence $\{A_n\}$ satisfies the following recursion:

$$A_{p(n+1,k,c)}=s_kA_{p(n,k,c)}-z_kA_{p(n-1,k,c)}$$

where $s_{\pm k}=\alpha^k+\beta^k$, $z_k=q^k$ and $p(n,k,c)=nk+c$ (c constant) and $\alpha,\beta=\left(p\pm\sqrt{p^2-4q}\right)/2$.

Further, in [6], the authors define the $n\times n$ Fibonacci matrix $\mathcal{F}_n=[f_{ij}]$ in the form

$$[f_{ij}]=\begin{cases} F_{i-j+1} & \text{if } i-j+1\geq 0,\\ 0 & \text{otherwise,}\end{cases}$$

where F_n is the nth Fibonacci number. This was generalized in [7], where the authors introduce the $n\times n$ generalized Fibonacci matrix $\mathcal{F}[x,y]_n=\left[f[x,y]_{ij}\right]$ as shown

$$f[x,y]_{ij}=\begin{cases} F_{i-j+1}x^{i-j}y^{i+j-2} & \text{if } i\geq j,\\ 0 & \text{otherwise.}\end{cases}$$

Also the authors define the infinite generalized Fibonacci matrix in the form

$$\mathcal{F}[x,y]=\begin{bmatrix} 1 & 0 & 0 & \dots \\ xy & y^2 & 0 & \dots \\ 2x^2y^2 & xy^3 & y^4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{1.2}$$

and the infinite generalized Pell matrix by

$$S[x,y]=\begin{bmatrix} 1 & 0 & 0 & \dots \\ 2xy & y^2 & 0 & \dots \\ 5x^2y^2 & 2xy^3 & y^4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{1.3}$$

Similarly, they define the infinite matrices $L[x,y]=\left[l[x,y]_{ij}\right]$ and $M[x,y]=\left[m[x,y]_{ij}\right]$ as follows:

$$l[x,y]_{ij}=\left(\tbinom{i-1}{j-1}-\tbinom{i-2}{j-1}-\tbinom{i-3}{j-1}\right)x^{i-j}y^{j-i} \tag{1.4}$$

and

$$m[x,y]_{ij}=\left(\tbinom{i-1}{j-1}-2\tbinom{i-2}{j-1}-\tbinom{i-3}{j-1}\right)x^{i-j}y^{j-i}. \tag{1.5}$$

They also show that the matrices $\mathcal{F}[x,y]$, $L[x,y]$, $S[x,y]$ and $M[x,y]$ satisfy $\Phi[x,y]=\mathcal{F}[x,y]*L[x,y]$ and $\Phi[x,y]=S[x,y]*M[x,y]$ where $\Phi[x,y]$ is the infinite generalized Pascal matrix defined by

$$\Phi[x,y]=\begin{bmatrix} 1 & 0 & 0 & \dots \\ xy & y^2 & 0 & \dots \\ x^2y^2 & 2xy^3 & y^4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{1.6}$$

In [14],the authors define an $n\times n$ matrix $R_n=[r_{i,j}]$, where

$$r_{ij}=\binom{i-1}{j-1}-\binom{i-1}{j}-\binom{i-1}{j+1}, \tag{1.7}$$

which they use to show that $P_n=R_n\mathcal{F}_n$ and the following factorization

$$\begin{aligned}\binom{n-1}{r-1} &= F_{n-r+1}+(n-2)F_{n-r}+\tfrac{1}{2}\left(n^2-5n+2\right)F_{n-r-1}\\ &+\sum_{k=r}^{n-3}\binom{n-1}{k-1}\left[2-\tfrac{n}{k}-\tfrac{(n-k)(n-k-1)}{k(k+1)}\right]F_{k-r+1}.\end{aligned}$$

where $\mathcal{F}_n$ and P_n are defined as before.

Stănică [9] looks at a more general case of the results of [6, 14]: he considers the $n\times n$ matrix $\mathcal{U}_n=(u_{ij})$ in terms of the sequence $\{U_n(A,B)\}$, where

$$u_{ij}=\begin{cases} U_{i-j+1} & \text{if } i\geq j,\\ 0 & \text{otherwise.}\end{cases}$$

Then the author give the factorization of *any* matrix in terms of the matrix $\mathcal{U}_n$.

In [8], the Riordan group was defined as follows: Let $R=[r_{ij}]_{i,j\geq 0}$ be an infinite matrix whose entries are complex numbers and $c_i(t)=\sum_{n\geq 0}^{\infty} r_{n,i}t^n$ be the generating function of the ith column of R. If $c_i(t)=g(t)[f(t)]^i$ where

$$g(t)=1+g_1t+g_2t^2+g_3t^3+\cdots,\ \text{and}\ f(t)=t+f_2t^2+f_3t^3+\cdots,$$

then R is a Riordan matrix. When $\Re$ denotes the set of Riordan matrices, the set $\Re$ is a group under matrix multiplication $*$, with the following properties:

(R_1) $(g(t),f(t))*(h(t),l(t))=(g(t)h(f(t)),l(f(t)))$.

(R_2) $I=(1,t)$ is the identity element.

(R_3) The inverse of R is given by $R^{-1}=\left(\frac{1}{g\left(\overline{f}(t)\right)},\overline{f}(t)\right)$, where $\overline{f}(t)$ is the compositional inverse of $f(t)$, i.e., $f\left(\overline{f}(t)\right)=\overline{f}(f(t))=t$.

(R_4) If $(a_0,a_1,a_2,...)^T$ is a column vector with generating function $A(t)$, then multiplying $R=(g(t),f(t))$ on the right by this column vector yields a column vector with generating function $B(t)=g(t)A(f(t))$.

In [6], the authors generalize the infinite Pascal, Fibonacci and Pell matrices and then give factorizations of the infinite generalized Pascal matrix by using Riordan method.

Let $R_n=[r_{i,j}]$ be the $n\times n$ matrix given as before. In [10], using the equations $P_n=R_n\mathcal{F}_n$ and $P_nE_n=R_n\mathcal{F}_nE_n$ for the $n\times n$ Fibonacci matrix $\mathcal{F}_n=[f_{ij}]$, the $n\times n$ Pascal matrix $P_n=[p_{ij}]$ and the $n\times 1$ matrix $E_n=(1,1,...,1)^T$, the authors show that

$$n+1=\sum_{l=1}^{n}\tfrac{(n-1)!}{(l+1)!(n-l)!}\left[l^2+(n+1)l-n^2\right]F_{l+2}$$

where $1\leq i,j\leq n$ and F_n is the nth Fibonacci number.

In this paper, we consider the arbitrary binary polynomial sequence $\{A_n\}$ and then give a lower triangular matrix representation of this sequence. By the definition of Riordan matrices, we obtain a factorization of the infinite generalized Pascal matrix in terms of this new matrix. Further some interesting results and applications are derived.

2. A Factorization of the Generalized Pascal Matrix

For any two nonzero real variables x and y, an infinite matrix $H[x,y]=\left[h[x,y]_{ij}\right]$ is defined as follows:

$$h[x,y]_{ij}=\begin{cases} A_{p(i-j+1,\pm k,c)}x^{i-j}y^{i+j-2} & \text{if } i\geq j, \\ 0 & \text{otherwise,}\end{cases}$$

where $\left\{A_{p(n+1,\pm k,c)}\right\}$ and $p(n+1,\pm k,c)$ are defined as before.

Clearly the matrix $H[x,y]$ is of the form

$$H[x,y]=\begin{bmatrix} A_{p(1,\pm k,c)} & 0 & 0 & \cdots \\ A_{p(2,\pm k,c)}xy & A_{p(1,\pm k,c)}y^2 & 0 & \cdots \\ A_{p(3,\pm k,c)}x^2y^2 & A_{p(2,\pm k,c)}xy^3 & A_{p(1,\pm k,c)}y^4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Now we give the Riordan representation of the infinite matrix $H[x,y]$. Let the Riordan representation of $H[x,y]$ be $(g_H(t),f_H(t))$. Here the generating function of the jth column of $H[x,y]$ is $c_j(t)=g_H(t)[f_H(t)]^j$. Since the first column vector of $H[x,y]$ is $\left(A_{p(1,\pm k,c)},A_{p(2,\pm k,c)}xy,A_{p(3,\pm k,c)}x^2y^2,\ldots\right)^T$, we can write

$$\begin{aligned} g_H(t) &= A_{p(1,\pm k,c)}+A_{p(2,\pm k,c)}xyt+A_{p(3,\pm k,c)}x^2y^2t^2+\ldots \\ -s_{\pm k}xytg_H(t) &= -s_{\pm k}A_{p(1,\pm k,c)}xyt-s_{\pm k}A_{p(2,\pm k,c)}x^2y^2t^2-s_{\pm k}A_{p(3,\pm k,c)}x^3y^3t^3-\ldots \\ z_{\pm k}x^2y^2t^2g_H(t) &= z_{\pm k}A_{p(1,\pm k,c)}x^2y^2t^2+z_{\pm k}A_{p(2,\pm k,c)}x^3y^3t^3+z_{\pm k}A_{p(3,\pm k,c)}x^3y^3t^3+\ldots \end{aligned}$$

By summing the above three equalities side by side, we get

$$g_H(t)=\frac{A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}xyt}{1-s_{\pm k}xyt+z_{\pm k}(xyt)^2}.$$

Since $h[x,y]_{ij}=y^2h[x,y]_{i-1,j-1}$ for $j\geq 2$, we have that $c_j(t)=y^2tc_{j-1}(t)$ and $g_H(t)[f_H(t)]^j=y^2tg_H(t)[f_H(t)]^{j-1}$. Hence we get $f_H(t)=y^2t$. Consequently the Riordan representation of $H[x,y]$ is given by

$$H[x,y]=\left(\frac{A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}xyt}{1-s_{\pm k}xyt+z_{\pm k}(xyt)^2},y^2t\right). \quad (2.1)$$

For two nonzero real numbers x and y, define the infinite matrix $C[x,y]=\left[c[x,y]_{ij}\right]$ with $c[x,y]_{ij}=\Bigg(\frac{1}{A_{p(1,\pm k,c)}}\binom{i-1}{j-1}-\frac{A_{p(2,\pm k,c)}}{A^2_{p(1,\pm k,c)}}\binom{i-2}{j-1}$
$-\ z_{\pm k}\left(\frac{A_{p(0,\pm k,c)}A_{p(2,\pm k,c)}-A^2_{p(1,\pm k,c)}}{A^3_{p(1,\pm k,c)}}\right)\binom{i-3}{j-1}-z_{\pm k}\left(\frac{A_{p(0,\pm k,c)}A_{p(2,\pm k,c)}-A^2_{p(1,\pm k,c)}}{A^3_{p(1,\pm k,c)}}\right)$
$\times\left(\sum_{m=1}^{i-3}\binom{i-m-3}{j-1}\left(\frac{z_{\pm k}A_{p(0,\pm k,c)}}{A_{p(1,\pm k,c)}}\right)^m\right)\Bigg)x^{i-j}y^{j-i}$ if $i\geq j$ and 0 otherwise.

We now give the following theorem.

Theorem 1.
$$\Phi[x,y]=H[x,y]*C[x,y].$$

Proof. Since $C[x,y]$ is a Riordan matrix, we write $C[x,y]=(g_C(t),f_C(t))$. Considering the first column vector of $C[x,y]$, we have

$$\begin{aligned}
&g_C(t)\\
=\;& \frac{1}{A_{p(1,\pm k,c)}}+\left(\frac{1}{A_{p(1,\pm k,c)}}-\frac{A_{p(2,\pm k,c)}}{A^2_{p(1,\pm k,c)}}\right)xy^{-1}t+\\
&\left(\frac{1}{A_{p(1,\pm k,c)}}-\frac{A_{p(2,\pm k,c)}}{A^2_{p(1,\pm k,c)}}-z_{\pm k}\left(\frac{A_{p(0,\pm k,c)}A_{p(2,\pm k,c)}-A^2_{p(1,\pm k,c)}}{A^3_{p(1,\pm k,c)}}\right)\right)\left(xy^{-1}t\right)^2+\\
&\left(\frac{1}{A_{p(1,\pm k,c)}}-\frac{A_{p(2,\pm k,c)}}{A^2_{p(1,\pm k,c)}}-z_{\pm k}\left(\frac{A_{p(0,\pm k,c)}A_{p(2,\pm k,c)}-A^2_{p(1,\pm k,c)}}{A^3_{p(1,\pm k,c)}}\right)\right)\left(xy^{-1}t\right)^3+\cdots\\
=\;& \left(1+xy^{-1}t+\left(xy^{-1}t\right)^2+\cdots\right)\left(\frac{1}{A_{p(1,\pm k,c)}}-\frac{A_{p(2,\pm k,c)}}{A^2_{p(1,\pm k,c)}}xy^{-1}t\right)-\\
&z_{\pm k}\left(1+xy^{-1}t+\left(xy^{-1}t\right)^2+\cdots\right)\left(\frac{A_{p(0,\pm k,c)}A_{p(2,\pm k,c)}-A^2_{p(1,\pm k,c)}}{A^3_{p(1,\pm k,c)}}\right)\left(xy^{-1}t\right)^2\\
&\left(1+\left(\frac{z_{\pm k}A_{p(0,\pm k,c)}}{A_{p(1,\pm k,c)}}\right)xy^{-1}t+\left(\frac{z^2_{\pm k}A^2_{p(0,\pm k,c)}}{A^2_{p(1,\pm k,c)}}\right)\left(xy^{-1}t\right)^2+\cdots\right)\\
=\;& \left(\frac{1}{1-xy^{-1}t}\right)\left(\frac{1-s_{\pm k}xy^{-1}t+z_{\pm k}\left(xy^{-1}t\right)^2}{\left(A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}xy^{-1}t\right)}\right).
\end{aligned}$$

Let the generating function of the jth column of $C[x,y]$ be $c_j(t)=g_C(t)[f_C(t)]^j$. Considering
$$c[x,y]_{ij}=c[x,y]_{i-1,j-1}+xy^{-1}c[x,y]_{i-1,j}$$
for $j\geq 2$, we obtain
$$c_j(t)=tc_{j-1}(t)+xy^{-1}tc_j(t)$$
and
$$g_C(t)[f_C(t)]^j=tg_C(t)[f_C(t)]^{j-1}+xy^{-1}tg_C(t)[f_C(t)]^j.$$
Hence we have $f_C(t)=\frac{t}{1-xy^{-1}t}$. Thus, the Riordan representation of matrix $C[x,y]$ is
$$C[x,y]=\left(\frac{1-s_{\pm k}xy^{-1}t+z_{\pm k}\left(xy^{-1}t\right)^2}{\left(A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}xy^{-1}t\right)\left(1-xy^{-1}t\right)},\frac{t}{1-xy^{-1}t}\right).$$
From [7], we have that $\Phi[x,y]=\left(\frac{1}{1-xyt},\frac{y^2t}{1-xyt}\right)$. Then
$$\begin{aligned}
&H[x,y]*C[x,y]\\
=\;& \left(\frac{A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}xyt}{1-s_{\pm k}xyt+z_{\pm k}(xyt)^2},y^2t\right)*\left(\frac{1-s_{\pm k}xy^{-1}t+z_{\pm k}\left(xy^{-1}t\right)^2}{\left(A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}xy^{-1}t\right)\left(1-xy^{-1}t\right)},\frac{t}{1-xy^{-1}t}\right)\\
=\;& \left(\frac{\left(A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}xyt\right)\left(1-s_{\pm k}xy^{-1}y^2t+z_{\pm k}\left(xy^{-1}y^2t\right)^2\right)}{\left(1-s_{\pm k}xyt+z_{\pm k}(xyt)^2\right)\left(A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}xy^{-1}y^2t\right)\left(1-xy^{-1}y^2t\right)},\frac{y^2t}{1-xy^{-1}y^2t}\right)\\
=\;& \left(\frac{1}{1-xyt},\frac{y^2t}{1-xyt}\right)=\Phi[x,y].
\end{aligned}$$
Thus the proof is complete. □

Now, we consider some special cases.

If we take $A_{p(1,\pm k,c)}=1$, $A_{p(0,\pm k,c)}=0$, $s_{\pm k}=\pm 1$, $z_{\pm k}=-1$, the matrix $H[x,y]$ is reduced to the Fibonacci matrix $\mathcal{F}[x,y]$. In Theorem 1, taking $\mathcal{F}[x,y]$

instead of $H[x,y]$ gives us the matrix $L[x,y]$ such that $\Phi[x,y]=\mathcal{F}[x,y]*L[x,y]$ from [7]. So the matrix $L[x,y]$ is a special case of $C[x,y]$. When $A_{p(1,\pm k,c)}=1$, $A_{p(0,\pm k,c)}=0$, $s_{\pm k}=\pm 2$, $z_{\pm k}=-1$, the matrix $H[x,y]$ is reduced to the Pell matrix $S[x,y]$ defined in [7]. Also taking $S[x,y]$ instead of $H[x,y]$, we get the matrix $M[x,y]$ such that $\Phi[x,y]=S[x,y]*M[x,y]$ given in [7]. The matrix $M[x,y]$ is a special case of the matrix $C[x,y]$.

By taking the finite matrices $C_n[x,y]=[c_n[x,y]]_{ij}$ and $H_n[x,y]=[h_n[x,y]]_{ij}$, we give the following result.

Corollary 1.

$$\sum_{r=1}^{i}\binom{i-1}{r-1}x^{i-r}y^{i+r-2}=\sum_{j=1}^{i}\left(A_{p(i-j+1,\pm k,c)}x^{i-j}y^{i+j-2}\left(\sum_{m=1}^{j}c_{im}\right)\right)$$

where $i,j=1,2,...,n$ and c_{im} is the $(i,m)-$element of $C_n[x,y]$.

Proof. Considering the $n\times n$ Pascal matrix $\Phi_n[x,y]$ and since $\Phi[x,y]=H[x,y]*C[x,y]$ from Theorem 1, we have $\Phi_n[x,y]=H_n[x,y]C_n[x,y]$ and $\Phi_n[x,y]E_n=H_n[x,y]C_n[x,y]E_n$, where $E_n=(1,1,...,1)^T$. Therefore, we obtain the desired result. □

Corollary 2. *For $n>0$ and $i,j=1,2,...,n$,*

$$\binom{n-1}{r-1}=\sum_{j=r}^{n}\left(\frac{A_{p(n-j+1,\pm k,c)}}{A_{p(1,\pm k,c)}}\right)\left(\binom{j-1}{r-1}-\frac{A_{p(2,\pm k,c)}}{A_{p(1,\pm k,c)}}\binom{j-2}{r-1}\right.$$
$$-z_{\pm k}\left(\frac{A_{p(0,\pm k,c)}A_{p(2,\pm k,c)}-A^2_{p(1,\pm k,c)}}{A^2_{p(1,\pm k,c)}}\right)\binom{j-3}{r-1}$$
$$\left.-z_{\pm k}\left(\frac{A_{p(0,\pm k,c)}A_{p(2,\pm k,c)}-A^2_{p(1,\pm k,c)}}{A^2_{p(1,\pm k,c)}}\right)\left(\sum_{m=1}^{n-3}\binom{n-m-3}{r-1}\left(\frac{z_{\pm k}A_{p(0,\pm k,c)}}{A_{p(1,\pm k,c)}}\right)^m\right)\right).$$

Proof. Use $x=y=1$ in the equality $\Phi_n[x,y]=H_n[x,y]*C_n[x,y]$. □

If $r=1$ in the previous corollary, we have

$$\sum_{j=1}^{n}\left(\frac{A_{p(n-j+1,\pm k,c)}}{A_{p(1,\pm k,c)}}\right)\left(1-\frac{A_{p(2,\pm k,c)}}{A_{p(1,\pm k,c)}}-z_{\pm k}\left(\frac{A_{p(0,\pm k,c)}A_{p(2,\pm k,c)}-A^2_{p(1,\pm k,c)}}{A^2_{p(1,\pm k,c)}}\right)\right.$$
$$\left.-z_{\pm k}\left(\frac{A_{p(0,\pm k,c)}A_{p(2,\pm k,c)}-A^2_{p(1,\pm k,c)}}{A^2_{p(1,\pm k,c)}}\right)\left(\sum_{m=1}^{n-3}\left(\frac{z_{\pm k}A_{p(0,\pm k,c)}}{A_{p(1,\pm k,c)}}\right)^m\right)\right)=1$$

For example consider the case $A_{p(1,\pm k,c)}=2$, $A_{p(0,\pm k,c)}=0$. For $k=3$, $p=1$ and $q=-1$, the sequence $\left\{A_{p(n,\pm k,c)}\right\}$ is reduced to the Fibonacci subsequence $\{F_{\pm 3n}\}$. By Corollary 2, we obtain

$$\binom{n-1}{r-1}=\sum_{j=r}^{n}\left(\frac{F_{\pm 3(n-j+1)}}{F_{\pm 3}}\right)\left(\binom{j-1}{r-1}-\frac{F_{\pm 6}}{F_{\pm 3}}\binom{j-2}{r-1}-\binom{j-3}{r-1}\right).$$

Now we give an another factorization of the generalized Pascal matrix with a matrix associated with the sequence $\left\{A_{p(n,\pm k,c)}\right\}$. First, for two nonzero real numbers x and y, we define the infinite matrix $C'[x,y]=\left[c'[x,y]_{ij}\right]$ with $c'[x,y]_{ij}=$

$$\left(\frac{1}{A_{p(1,\pm k,c)}}\binom{i-1}{j-1}-\frac{A_{p(2,\pm k,c)}}{A^2_{p(1,\pm k,c)}}\binom{i-1}{j}-z_{\pm k}\left(\frac{A_{p(0,\pm k,c)}A_{p(2,\pm k,c)}-A^2_{p(1,\pm k,c)}}{A^3_{p(1,\pm k,c)}}\right)\binom{i-1}{j+1}\right.$$

$$- z_{\pm k}\left(\frac{A_{p(0,\pm k,c)}A_{p(2,\pm k,c)}-A^2_{p(1,\pm k,c)}}{A^3_{p(1,\pm k,c)}}\right)\times\left(\sum_{m=1}^{i-3}\binom{i-1}{j+m+1}\left(\frac{z_{\pm k}A_{p(0,\pm k,c)}}{A_{p(1,\pm k,c)}}\right)^m\right)\Bigg) x^{i-j}y^{i+j-2}$$

if $i \geq j$ and 0 otherwise.

Second we define the infinite matrix $H'[x,y] = \left[h'[x,y]_{ij}\right]$ with $h'[x,y]_{ij} = A_{p(i-j+1,\pm k,c)}x^{i-j}y^{j-i}$ if $i \geq j$ and 0 otherwise. Then we can give the following theorem.

Theorem 2.

$$\Phi[x,y] = C'[x,y] * H'[x,y].$$

Proof. From Theorem 1, the Riordan representation of the matrix $C[x,y]$ is known. Thus we get the Riordan representations of $C'[x,y]$ and $H'[x,y]$ as following:

$$C'[x,y] = \left(\frac{1-(2+s_{\pm k})xyt+(1+s_{\pm k}+z_{\pm k})(xyt)^2}{\left(A_{p(1,\pm k,c)}-\left(A_{p(1,\pm k,c)}+z_{\pm k}A_{p(0,\pm k,c)}\right)xyt\right)(1-xyt)^2}, \frac{y^2t}{1-xyt}\right)$$

and

$$H'[x,y] = \left(\frac{A_{p(1,\pm k,c)} - z_{\pm k}A_{p(0,\pm k,c)}xy^{-1}t}{1 - s_{\pm k}xy^{-1}t + z_{\pm k}\left(xy^{-1}t\right)^2}, t\right).$$

From property (R$_1$), we have

$$\begin{aligned}
& C'[x,y] * H'[x,y] \\
= & \left(\frac{1-(2+s_{\pm k})xyt+(1+s_{\pm k}+z_{\pm k})(xyt)^2}{\left(A_{p(1,\pm k,c)}-\left(A_{p(1,\pm k,c)}+z_{\pm k}A_{p(0,\pm k,c)}\right)xyt\right)(1-xyt)^2}, \frac{y^2t}{1-xyt}\right) \\
& * \left(\frac{A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}xy^{-1}t}{1-s_{\pm k}xy^{-1}t+z_{\pm k}(xy^{-1}t)^2}, t\right) \\
= & \left(\frac{\left(1-(2+s_{\pm k})xyt+(1+s_{\pm k}+z_{\pm k})(xyt)^2\right)\left(A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}xy^{-1}\frac{y^2t}{1-xyt}\right)}{\left(A_{p(1,\pm k,c)}-\left(A_{p(1,\pm k,c)}+z_{\pm k}A_{p(0,\pm k,c)}\right)xyt\right)(1-xyt)^2\left(1-s_{\pm k}xy^{-1}\frac{y^2t}{1-xyt}+z_{\pm k}\left(xy^{-1}\frac{y^2t}{1-xyt}\right)^2\right)}, \frac{y^2t}{1-xyt}\right) \\
= & \left(\frac{1}{1-xyt}, \frac{y^2t}{1-xyt}\right) = \Phi[x,y].
\end{aligned}$$

Thus the proof is complete. □

Considering the finite matrices $C'_n[x,y] = [c'_n[x,y]]_{ij}$ and $H'_n[x,y] = [h'_n[x,y]]_{ij}$ we can give the following result.

Corollary 3.

$$\sum_{r=1}^{i}\binom{i-1}{r-1}x^{i-r}y^{i+r-2} = \sum_{j=1}^{i}\left(c'_{ij}\left(\sum_{m=1}^{j}A_{p(m,\pm k,c)}x^{m-1}y^{1-m}\right)\right)$$

where $i,j = 1,2,...,n$ *and* c'_{ij} *is the* $(i,j)-$*element of* $C'_n[x,y]$.

Proof. Since $\Phi[x,y] = C'[x,y]*H'[x,y]$ in Theorem 2, we have $\Phi_n[x,y] = C'_n[x,y]\,H'_n[x,y]$ and $\Phi_n[x,y]\,E_n = C'_n[x,y]\,H'_n[x,y]\,E_n$ where $E_n = (1,1,...,1)^T$. So we obtain the desired result. □

Corollary 4. *For $n>0$ and $i,j=1,2,...,n$,*

$$\begin{aligned}\binom{n-1}{r-1} &= \sum_{j=r}^{n}\left(\left(\binom{n-1}{j-1}-\frac{A_{p(2,\pm k,c)}}{A_{p(1,\pm k,c)}}\binom{n-1}{j}-z_{\pm k}\left(\frac{A_{p(0,\pm k,c)}A_{p(2,\pm k,c)}-A^2_{p(1,\pm k,c)}}{A^2_{p(1,\pm k,c)}}\right)\times\right.\right.\\ &\quad\binom{n-1}{j+1}-z_{\pm k}\left(\frac{A_{p(0,\pm k,c)}A_{p(2,\pm k,c)}-A^2_{p(1,\pm k,c)}}{A^2_{p(1,\pm k,c)}}\right)\times\\ &\quad\left.\left(\sum_{m=1}^{n-3}\binom{n-1}{j+m+1}\left(\frac{z_{\pm k}A_{p(0,\pm k,c)}}{A_{p(1,\pm k,c)}}\right)^m\right)\right)\frac{A_{p(j-r+1,\pm k,c)}}{A_{p(1,\pm k,c)}}.\end{aligned}$$

Proof. By taking $x=y=1$ in $\Phi[x,y]=C'[x,y]*H'[x,y]$, we have the conclusion. □

Particularly, if we take $r=1$ in Corollary 4, we get

$$\begin{aligned}&\sum_{j=1}^{n}\left(\left(\binom{n-1}{j-1}-\frac{A_{p(2,\pm k,c)}}{A_{p(1,\pm k,c)}}\binom{n-1}{j}-z_{\pm k}\left(\frac{A_{p(0,\pm k,c)}A_{p(2,\pm k,c)}-A^2_{p(1,\pm k,c)}}{A^2_{p(1,\pm k,c)}}\right)\times\right.\right.\\ &\binom{n-1}{j+1}-z_{\pm k}\left(\frac{A_{p(0,\pm k,c)}A_{p(2,\pm k,c)}-A^2_{p(1,\pm k,c)}}{A^2_{p(1,\pm k,c)}}\right)\times\\ &\left.\left.\sum_{m=1}^{n-3}\binom{n-1}{j+m+1}\left(\frac{z_{\pm k}A_{p(0,\pm k,c)}}{A_{p(1,\pm k,c)}}\right)^{m-1}\right)\right)\frac{A_{p(j,\pm k,c)}}{A_{p(1,\pm k,c)}}\\ =\ &1.\end{aligned}$$

As an example, consider the case $A_{p(1,\pm k,c)}=\pm 2$, $A_{p(0,\pm k,c)}=0$. When $k=2$, $p=2$ and $q=-1$, the sequence $\{A_{p(n,\pm k,c)}\}$ is reduced to the Pell subsequence $\{P_{\pm 2n}\}$. By Corollary 4, we obtain

$$\begin{aligned}&\binom{n-1}{r-1}\\ =&\sum_{j=r}^{n}\left(\left(\binom{n-1}{j-1}-\frac{A_{p(2,\pm 2,c)}}{A_{p(1,\pm 2,c)}}\binom{n-1}{j}-z_{\pm 2}\left(\frac{A_{p(0,\pm k,c)}A_{p(2,\pm 2,c)}-A^2_{p(1,\pm 2,c)}}{A^2_{p(1,\pm 2,c)}}\right)\right.\right.\\ &\left.\left.\times\binom{n-1}{j+1}-z_{\pm 2}\left(\sum_{m=1}^{n-3}\binom{n-1}{j+m+1}\left(\frac{z_{\pm 2}A_{p(0,\pm 2,c)}}{A_{p(1,\pm 2,c)}}\right)^m\right)\right)\right)\frac{A_{p(j-r+1,\pm 2,c)}}{A_{p(1,\pm 2,c)}}\\ =&\sum_{j=r}^{n}\left(\left(\binom{n-1}{j-1}-\frac{P_{\pm 4}}{P_{\pm 2}}\binom{n-1}{j}+\binom{n-1}{j+1}\right)\right)\frac{P_{\pm 2(j-r+1)}}{P_{\pm 2}}.\end{aligned}$$

From property (R_3), we can find the inverses of $H[x,y]$, $C[x,y]$ and $C'[x,y]$. Using the computation of the inverse of $\Phi[x,y]$ from [7], we can give the next two results.

Lemma 1. *The inverses of matrices $H[x,y]$, $C[x,y]$, $C'[x,y]$ and $H'[x,y]$ are given by*

$$H[x,y]^{-1}=\left(\frac{1-s_{\pm k}xy^{-1}t+z_{\pm k}\left(xy^{-1}t\right)^2}{\left(A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}xy^{-1}t\right)},y^{-2}t\right),$$

$$C[x,y]^{-1}=\left(\frac{A_{p(1,\pm k,c)}+\left(A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}\right)xy^{-1}t}{1+(2-s_{\pm k})xy^{-1}t+(1-s_{\pm k}+z_{\pm k})\left(xy^{-1}t\right)^2},\frac{t}{1+xy^{-1}t}\right),$$

$$C'[x,y]^{-1}=\left(\frac{A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}xy^{-1}t}{\left(1-s_{\pm k}xy^{-1}t+z_{\pm k}\left(xy^{-1}t\right)^{2}\right)\left(1+xy^{-1}t\right)},\frac{t}{y^{2}+xyt}\right)$$

and

$$H'[x,y]^{-1}=\left(\frac{1-s_{\pm k}xy^{-1}t+z_{\pm k}\left(xy^{-1}t\right)^{2}}{\left(A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}xy^{-1}t\right)},t\right).$$

Proof. First, we look at the matrix $H[x,y]$. Since $f_H(t)=y^2t$, we get $\overline{f}_H(t)=y^{-2}t$. Substituting $\overline{f}_H(t)$ in $\left(g_H\left(\overline{f}_H(t)\right)\right)^{-1}$, we obtain

$$\frac{1}{g_H\left(\overline{f}_H(t)\right)}=\frac{1-s_{\pm k}xy^{-1}t+z_{\pm k}\left(xy^{-1}t\right)^{2}}{\left(A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}xy^{-1}t\right)\left(1-xy^{-1}t\right)},$$

and hence, the Riordan representation of $H[x,y]^{-1}$ is

$$H[x,y]^{-1}=\left(\frac{1-s_{\pm k}xy^{-1}t+z_{\pm k}\left(xy^{-1}t\right)^{2}}{\left(A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}xy^{-1}t\right)},y^{-2}t\right).$$

Secondly, since $f_C(t)=\frac{t}{1-xy^{-1}t}$ for the matrix $C[x,y]$, we get $\overline{f}_C(t)=t\left(1+xy^{-1}t\right)^{-1}$ and

$$\frac{1}{g_C\left(\overline{f}_C(t)\right)}=\frac{A_{p(1,\pm k,c)}+\left(A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}\right)xy^{-1}t}{1+(2-s_{\pm k})xy^{-1}t+(1-s_{\pm k}+z_{\pm k})\left(xy^{-1}t\right)^{2}}.$$

Thus, the Riordan representation of $C[x,y]^{-1}$ is

$$C[x,y]^{-1}=\left(\frac{A_{p(1,\pm k,c)}+\left(A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}\right)xy^{-1}t}{1+(2-s_{\pm k})xy^{-1}t+(1-s_{\pm k}+z_{\pm k})\left(xy^{-1}t\right)^{2}},\frac{t}{1+xy^{-1}t}\right).$$

Thirdly, since $f_{C'}(t)=\frac{y^2t}{1-xyt}$ for the matrix $C'[x,y]$, we get $\overline{f}_{C'}(t)=t\left(y^2+xyt\right)^{-1}$ and

$$\frac{1}{g_{C'}\left(\overline{f}_{C'}(t)\right)}=\frac{A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}xy^{-1}t}{\left(1-s_{\pm k}xy^{-1}t+z_{\pm k}\left(xy^{-1}t\right)^{2}\right)\left(1+xy^{-1}t\right)}.$$

Thus the Riordan representation of $C'[x,y]^{-1}$ is

$$C'[x,y]^{-1}=\left(\frac{A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}xy^{-1}t}{\left(1-s_{\pm k}xy^{-1}t+z_{\pm k}\left(xy^{-1}t\right)^{2}\right)\left(1+xy^{-1}t\right)},\frac{t}{y^{2}+xyt}\right).$$

Finally, since $f_{H'}(t)=t$ for the matrix $H'[x,y]$, we get $\overline{f}_{H'}(t)=t$ and

$$\frac{1}{g_{H'}\left(\overline{f}_{H'}(t)\right)}=\frac{1-s_{\pm k}xy^{-1}t+z_{\pm k}\left(xy^{-1}t\right)^{2}}{\left(A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}xy^{-1}t\right)}.$$

Thus the Riordan representation of $C'[x,y]^{-1}$ is

$$H'[x,y]^{-1}=\left(\frac{1-s_{\pm k}xy^{-1}t+z_{\pm k}\left(xy^{-1}t\right)^{2}}{\left(A_{p(1,\pm k,c)}-z_{\pm k}A_{p(0,\pm k,c)}xy^{-1}t\right)},t\right).$$

□

When $A_{p(1,\pm k,c)} = 1$, $A_{p(0,\pm k,c)} = 0$, $s_{\pm k} = \pm 1$, $z_{\pm k} = -1$, the inverses of $H[x,y]$ and $C[x,y]$ are the inverses of the infinite generalized Fibonacci matrix $\mathcal{F}[x,y]$ and the matrix $L[x,y]$, respectively. Also when $A_{p(1,\pm k,c)} = 1$, $A_{p(0,\pm k,c)} = 0$, $s_{\pm k} = \pm 2$, $z_{\pm k} = -1$, the inverses of $H[x,y]$ and $C[x,y]$ are the inverses of the generalized Pell matrix $S[x,y]$ and the matrix $M[x,y]$, respectively.

Corollary 5. *For the generalized Pascal matrix* $\Phi[x,y]$,

$$\Phi[x,y]^{-1} = C[x,y]^{-1} * H[x,y]^{-1} \tag{2.2}$$

and

$$\Phi[x,y]^{-1} = H'[x,y]^{-1} * C'[x,y]^{-1}. \tag{2.3}$$

Proof. From [7], we have the inverse of $\Phi[x,y]$ as

$$\Phi[x,y]^{-1} = \left(\frac{1}{1+xy^{-1}t}, \frac{t}{y^2+xyt}\right). \tag{2.4}$$

From Theorems 1 and 2, we know that $\Phi[x,y] = H[x,y] * C[x,y]$, $\Phi[x,y] = C'[x,y] * H'[x,y]$, respectively. Thus the proof is complete. $\square$

From Lemma 1 and (2.1), we have the following result.

Corollary 6. *For* $n \geq 1$,

(i) $H[x,y]^n = \left(\prod_{m=1}^{n} \frac{A_{p(1,\pm k,c)} - z_{\pm k} A_{p(0,\pm k,c)} xy^{2m-1} t}{1 - s_{\pm k} xy^{2m-1} t + z_{\pm k} (xy^{2m-1} t)^2},\ y^{2n}\ t\right)$,

(ii) $H[x,y]^{-n} = \left(\prod_{m=1}^{n} \frac{1 - s_{\pm k} xy^{-2m+1} t + z_{\pm k} \left(xy^{-2m+1} t\right)^2}{A_{p(1,\pm k,c)} - z_{\pm k} A_{p(0,\pm k,c)} xy^{-2m+1} t},\ y^{-2n}\ t\right)$.

Proof. The result follows from induction, using

$$H[x,y] = \left(\frac{A_{p(1,\pm k,c)} - z_{\pm k} A_{p(0,\pm k,c)} xyt}{1 - s_{\pm k} xyt + z_{\pm k} (xyt)^2}, y^2 t\right)$$

and

$$H[x,y]^{-1} = \left(\frac{1 - s_{\pm k} xy^{-1} t + z_{\pm k} x^2 y^{-2} t^2}{\left(A_{p(1,\pm k,c)} - z_{\pm k} A_{p(0,\pm k,c)} xy^{-1} t\right)}, y^{-2} t\right).$$

$\square$

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