

## THE GENERALIZED BINET FORMULA, REPRESENTATION AND SUMS OF THE GENERALIZED ORDER- $k$ PELL NUMBERS

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**Abstract.** In this paper we give a new generalization of the Pell numbers in matrix representation. Also we extend the matrix representation and we show that the sums of the generalized order- $k$  Pell numbers could be derived directly using this representation. Further we present some identities, the generalized Binet formula and combinatorial representation of the generalized order- $k$  Pell numbers.

### 1. INTRODUCTION

It is well-known that the Pell sequence  $\{P_n\}$  is defined recursively by the equation, for  $n \geq 1$

$$(1.1) \quad P_{n+1} = 2P_n + P_{n-1}$$

in which  $P_0 = 0$ ,  $P_1 = 1$ .

In [3], Horadam showed that some properties involving Pell numbers. Also in [2], Ercolano gave the matrix method for generating the Pell sequence as follows:

$$(1.2) \quad M = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

and by taking successive positive powers of the matrix  $M$  one can easily verify that

$$M^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}.$$

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The Pell sequence is a special case of a sequence which is defined recursively as a linear combination of the preceding  $k$  terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \dots + c_{k-1} a_{n+k-1}$$

where  $c_0, c_1, \dots, c_{k-1}$  are real constants. In [4], Kalman derived a number of closed-form formulas for the generalized sequence by companion matrix method as follows:

$$(1.3) \quad A_k = [a_{ij}]_{k \times k} = \begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_{k-2} & c_{k-1} \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

Then by an inductive argument he obtained that

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

Further in [7], we defined the generalized order- $k$  Lucas sequence in matrix representation with employing the matrix methods of Kalman.

Also in [5], we gave the generalized Binet formula, combinatorial representation and some relations involving the generalized order- $k$  Fibonacci and Lucas numbers.

Now we give a new generalization of the Pell numbers in matrix representation and extend the matrix representation so we give sums of the generalized Pell numbers could be derived directly using this representation.

## 2. THE MAIN RESULTS

Define  $k$  sequences of the generalized order- $k$  Pell numbers as shown:

$$(2.1) \quad P_n^i = 2P_{n-1}^i + P_{n-2}^i + \dots + P_{n-k}^i$$

for  $n > 0$  and  $1 \leq i \leq k$ , with initial conditions

$$P_n^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1 - k \leq n \leq 0,$$

where  $P_n^i$  is the  $n$ th term of the  $i$ th sequence. When  $k = 2$ , the generalized order- $k$  Pell sequence,  $\{P_n^k\}$ , is reduced to the usual Pell sequence,  $\{P_n\}$ .

When  $i = k$  in (2.1), we call  $P_n^k$  the generalized  $k$ -Pell number.

By (2.1), we can write

$$(2.2) \quad \begin{bmatrix} P_{n+1}^i \\ P_n^i \\ P_{n-1}^i \\ \vdots \\ P_{n-k+2}^i \end{bmatrix} = \begin{bmatrix} 2 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \begin{bmatrix} P_n^i \\ P_{n-1}^i \\ P_n^i \\ \vdots \\ P_{n-k+1}^i \end{bmatrix}$$

for the generalized order- $k$  Pell sequences. Letting

$$(2.3) \quad R = [r_{ij}]_{k \times k} = \begin{bmatrix} 2 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

The matrix  $R$  is said to be generalized order- $k$  Pell matrix.

To deal with the  $k$  sequences of the generalized order- $k$  Pell sequences simultaneously, we define a  $k \times k$  matrix  $E_n$  as follows:

$$(2.4) \quad E_n = [e_{ij}]_{k \times k} = \begin{bmatrix} P_n^1 & P_n^2 & \dots & P_n^k \\ P_{n-1}^1 & P_{n-1}^2 & \dots & P_{n-1}^k \\ \vdots & \vdots & & \vdots \\ P_{n-k+1}^1 & P_{n-k+1}^2 & \dots & P_{n-k+1}^k \end{bmatrix}.$$

Generalizing Eq. (2.2), we derive

$$(2.5) \quad E_{n+1} = R \cdot E_n.$$

**Lemma 1.** *Let  $E_n$  and  $R$  be as in (2.4) and (2.3), respectively. Then, for all integers  $n \geq 0$*

$$E_{n+1} = R^{n+1}.$$

*Proof.* By (2.4), we have  $E_{n+1} = R \cdot E_n$ . Then, by an inductive argument, we may rewrite it as

$$(2.6) \quad E_{n+1} = R^n \cdot E_1.$$

Since by definition of the generalized order- $k$  Pell number,  $E_1 = R$ ; therefore

$$E_{n+1} = R^{n+1}.$$

So the proof is complete. ■

**Theorem 1.** *Let  $E_n$  be as in (2.4). Then*

$$\det E_n = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ (-1)^n & \text{if } k \text{ is even.} \end{cases}$$

*Proof.* From Lemma 1, we have  $E_{n+1} = R^{n+1}$ . Then

$$\det E_{n+1} = \det (R^{n+1}) = (\det R)^{n+1}$$

where  $\det R = (-1)^{k+1}$ . Thus

$$\det E_{n+1} = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ (-1)^{n+1} & \text{if } k \text{ is even.} \end{cases}$$

So the proof is complete. ■

Now we give some relations involving the generalized order- $k$  Pell numbers.

**Theorem 2.** *Let  $P_n^i$  be the  $n$ th generalized order- $k$  Pell number, for  $1 \leq i \leq k$ . Then, for all positive integers  $n$  and  $m$*

$$P_{n+m}^i = \sum_{j=1}^k P_m^j P_{n-j+1}^i.$$

*Proof.* From Lemma 1, we know that  $E_n = R^n$ ; we may rewrite it as

$$(2.7) \quad E_{n+1} = E_n E_1 = E_1 E_n.$$

In other words,  $E_1$  is commutative under matrix multiplication. Hence, more generalizing Eq. (2.7), we can write

$$(2.8) \quad E_{n+m} = E_n E_m = E_m E_n.$$

Consequently, an element of  $E_{n+m}$  is the product of a row  $E_n$  and a column of  $E_m$ ; that is

$$P_{n+m}^i = \sum_{j=1}^k P_m^j P_{n-j+1}^i.$$

Thus the proof is complete. ■

For example, if we take  $k = 2$  in Theorem 2, we have

$$\begin{aligned} P_{n+m}^2 &= \sum_{j=1}^2 P_m^j P_{n-j+1}^2 \\ &= P_m^1 P_n^2 + P_m^2 P_{n-1}^2 \end{aligned}$$

and, since  $P_n^1 = P_{n+1}^2$  for all  $n \in \mathbb{Z}^+$  and  $k = 2$ , we obtain

$$P_{n+m}^2 = P_{m+1}^2 P_n^2 + P_m^2 P_{n-1}^2$$

where  $P_n^2$  is the usual Pell number. Indeed, we generalize the following relation involving the usual Pell numbers:

$$P_{n+m} = P_{m+1} P_n + P_m P_{n-1}.$$

**Lemma 2.** *Let  $P_n^i$  be the  $n$ th generalized order- $k$  Pell number. Then*

$$\begin{aligned} (2.9) \quad P_{n+1}^i &= P_n^1 + P_n^{i+1}, \quad \text{for } 2 \leq i \leq k-1, \\ P_{n+1}^1 &= 2P_n^1 + P_n^2, \\ P_{n+1}^k &= P_n^1. \end{aligned}$$

*Proof.* From Eq. (2.7), we have  $E_{n+1} = E_n E_1$ . Since using a property of matrix multiplication, the proof is readily seen. ■

### 3. SUMS OF THE PELL NUMBERS

Now we extend the matrix representation and show that the sums of the generalized Pell numbers.

To calculate the sums  $S_n$ ,  $n \geq 0$ , of the generalized order- $k$  Pell numbers, defined by

$$S_n = \sum_{i=0}^n P_i^1.$$

Let  $T$  be a  $(k + 1) \times (k + 1)$  square matrix, such that

$$(3.1) \quad T = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & & & & \\ 0 & R & & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}$$

where  $R$  is the  $k \times k$  matrix as in (2.3).

**Theorem 3.** Let  $S_n$ ,  $n \geq 0$ , denote the sums of the generalized Pell numbers. Then  $S_n$  is  $(2, 1)$  entry of the matrix  $T^{n+1}$  in which  $T$  is the  $(k+1) \times (k+1)$  matrix as in (3.1).

*Proof.* Let  $C_n$  be a  $(k+1) \times (k+1)$  square matrix, such that

$$C_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ S_{n-1} & & & & \\ S_{n-2} & & E_n & & \\ \vdots & & & & \\ S_{n-k} & & & & \end{bmatrix}$$

where  $E_n$  is the  $k \times k$  matrix as in (2.4). Then, by Eq. (2.9) and

$$(3.2) \quad S_{n+1} = P_{n+1}^1 + S_n,$$

we derive a recurrence equation

$$(3.3) \quad C_{n+1} = C_n \cdot T.$$

Inductively, we also have

$$(3.4) \quad C_{n+1} = C_1 \cdot T^n.$$

Since  $S_{-i} = 0$ ,  $1 \leq i \leq k$ , we thus infer  $C_1 = T$ , and in general,  $C_n = T^n$ . Since  $S_n = (C_{n+1})_{2,1}$  and  $C_n = T^n$ , the proof is readily seen. ■

From Eqs. (3.3) and (3.4), we reach the following equation:

$$(3.5) \quad C_{n+1} = C_n C_1 = C_1 C_n$$

which shows that  $C_1$  is commutative as well under matrix multiplication. By an application of Eq. (3.5), the sums of the generalized order- $k$  Pell numbers satisfy the following recurrence relation:

$$S_n = 1 + 2S_{n-1} + \sum_{i=2}^k S_{n-i}.$$

Substituting  $S_n = P_n^1 + S_{n-1}$ , an instance of Eq. (3.2), into Eq. (3.4), we express  $P_n^1$  in terms of the sums of the generalized order- $k$  Pell numbers:

$$(3.6) \quad P_n^1 = 1 + \sum_{i=1}^k S_{n-i}.$$

When  $k = 2$ , this equation is reduced to

$$P_n^1 = 1 + S_{n-1} + S_{n-2}.$$

So we derive the well-known result [3]:

$$\sum_{i=1}^n P_i = \frac{P_{n+1} + P_n - 1}{2}$$

where  $P_n$  is the  $n$ th term of the usual Pell sequence.

#### 4. GENERALIZED BINET FORMULA

In [6], Levesque gave a Binet formula for the Fibonacci sequence. In this section, we derive a generalized Binet formula for the generalized order- $k$  Pell sequence by using the determinant.

**Lemma 3.** *The equation  $x^{k+1} - 3x^k + x^{k-1} + 1 = 0$  does not have multiple roots for  $k \geq 2$ .*

*Proof.* Let  $f(x) = x^k - 2x^{k-1} - x^{k-2} - \dots - x - 1$  and let  $h(x) = (x - 1)f(x)$ . Then  $h(x) = x^{k+1} - 3x^k + x^{k-1} + 1$ . So 1 is a root but not a multiple root of  $h(x)$ , since  $k \geq 2$  and  $f(1) \neq 1$ . Suppose that  $\alpha$  is a multiple root of  $h(x)$ . Note that  $\alpha \neq 0$  and  $\alpha \neq 1$ . Since  $\alpha$  is a multiple root,  $h(\alpha) = \alpha^{k+1} - 3\alpha^k + \alpha^{k-1} + 1 = 0$  and

$$\begin{aligned} h'(\alpha) &= (k + 1)\alpha^k - 3k\alpha^{k-1} + (k - 1)\alpha^{k-2} \\ &= \alpha^{k-2}((k + 1)\alpha^2 - 3k\alpha + k - 1) = 0. \end{aligned}$$

Thus  $\alpha_{1,2} = \frac{3k \mp \sqrt{5k^2 + 4}}{2(k+1)}$  and hence, for  $\alpha_1$

$$\begin{aligned} 0 &= \alpha_1^{k-1}(-\alpha_1^2 + 3\alpha_1 - 1) - 1 \\ (4.1) \quad &= \left(\frac{3k + \sqrt{5k^2 + 4}}{2(k+1)}\right)^{k-1} \left(\frac{5k - 4 + 3\sqrt{5k^2 + 4}}{2(k+1)^2}\right) - 1. \end{aligned}$$

We let  $a_k = \left(\left(\frac{3k + \sqrt{5k^2 + 4}}{2(k+1)}\right)^{k-1} \left(\frac{5k - 4 + 3\sqrt{5k^2 + 4}}{2(k+1)^2}\right)\right)$ . Then we write Eq. (4.1) as follows:

$$0 = a_k - 1.$$

Since  $a_k < a_{k+1}$  and  $a_2 = 2, 0887$  for  $k \geq 2$ ,  $a_k \neq 1$ , a contradiction. Similarly, hence, for  $\alpha_2$

$$\begin{aligned}
 0 &= \alpha_2^{k-1} (-\alpha_2^2 + 3\alpha_2 - 1) - 1 \\
 (4.2) \quad &= \left( \frac{3k - \sqrt{5k^2 + 4}}{2(k+1)} \right)^{k-1} \left( \frac{5k - 4 - 3\sqrt{5k^2 + 4}}{2(k+1)^2} \right) - 1.
 \end{aligned}$$

We let  $b_k = \left( \left( \frac{3k - \sqrt{5k^2 + 4}}{2(k+1)} \right)^{k-1} \left( \frac{5k - 4 - 3\sqrt{5k^2 + 4}}{2(k+1)^2} \right) \right)$ . Then we write Eq. (4.2) as follows:

$$0 = b_k - 1.$$

Since  $b_k > b_{k+1}$  and  $b_2 = -8,88662 \times 10^{-2}$  for  $k \geq 2$ ,  $b_k \neq 1$ , a contradiction. Therefore, the equation  $h(x) = 0$  does not have multiple roots. ■

Consequently, from Lemma 3, it is seen that the equation  $x^k - 2x^{k-1} - x^{k-2} - \dots - x - 1 = 0$  does not have multiple roots for  $k \geq 2$ .

Let  $f(\lambda)$  be the characteristic polynomial of the generalized order- $k$  Pell matrix  $R$ . Then  $f(\lambda) = \lambda^k - 2\lambda^{k-1} - \lambda^{k-2} - \dots - \lambda - 1$ , which is a well-known fact. Let  $\lambda_1, \lambda_2, \dots, \lambda_k$  be the eigenvalues of  $R$ . Then, by Lemma 3,  $\lambda_1, \lambda_2, \dots, \lambda_k$  are distinct. Let  $V$  be a  $k \times k$  Vandermonde matrix as follows:

$$V = \begin{bmatrix} \lambda_1^{k-1} & \lambda_2^{k-1} & \dots & \lambda_k^{k-1} \\ \lambda_1^{k-2} & \lambda_2^{k-2} & \dots & \lambda_k^{k-2} \\ \vdots & \vdots & & \vdots \\ \lambda_1 & \lambda_2 & \dots & \lambda_k \\ 1 & 1 & \dots & 1 \end{bmatrix}.$$

Let  $w_k^i$  be a  $k \times 1$  matrix as follows:

$$w_k^i = \begin{bmatrix} \lambda_1^{n+k-i} \\ \lambda_2^{n+k-i} \\ \vdots \\ \lambda_k^{n+k-i} \end{bmatrix}$$

and  $V_j^{(i)}$  be a  $k \times k$  matrix obtained from  $V$  by replacing the  $j$ th column of  $V$  by  $w_k^i$ . Then we obtain the generalized Binet formula for the generalized order- $k$  Pell numbers with the following theorem.

**Theorem 4.** Let  $P_n^i$  be the  $n$ th term of  $i$ th Pell sequence, for  $1 \leq i \leq k$ . Then

$$P_{n-i+1}^j = \frac{\det(V_j^{(i)})}{\det(V)}.$$



*Proof.* Since the eigenvalues of  $R$  are distinct,  $R$  is diagonalizable. It is readily seen that  $RV = VD$ , where  $D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_k)$ . Since  $V$  is invertible

$V^{-1}RV = D$ . Hence,  $R$  is similar to  $D$ . So we obtain  $R^n V = VD^n$ . Then we have the following linear system of equations:

$$\begin{aligned} e_{i1}\lambda_1^{k-1} + e_{i2}\lambda_1^{k-2} + \dots + e_{ik} &= \lambda_1^{n+k-i} \\ e_{i1}\lambda_2^{k-1} + e_{i2}\lambda_2^{k-2} + \dots + e_{ik} &= \lambda_2^{n+k-i} \\ &\vdots \\ e_{i1}\lambda_k^{k-1} + e_{i2}\lambda_k^{k-2} + \dots + e_{ik} &= \lambda_k^{n+k-i} \end{aligned}$$

And, for each  $j = 1, 2, \dots, k$ , we obtain

$$e_{ij} = \frac{\det(V_j^{(i)})}{\det(V)}$$

where  $e_{ij}$  is the  $(i, j)$ th elements of the matrix  $E_n$ , i.e.,  $e_{ij} = P_{n-i+1}^j$ .

So the proof is complete. ■

**Corollary 1.** Let  $P_n^k$  be the  $n$ th generalized  $k$ -Pell number. Then

$$P_n^k = \frac{\det(V_k^{(1)})}{\det(V)}.$$

*Proof.* Since  $e_{ij}$  is the  $(i, j)$ th elements of the matrix  $E_n$ , i.e.,  $e_{ij} = P_{n-i+1}^j$ . If we take  $i = 1$  and  $j = k$ , then  $e_{1,k} = P_n^k$ . Then by using Theorem 4, the proof is immediately seen. ■

### 5. COMBINATORIAL REPRESENTATION

In this section we give a combinatorial representation of the generalized order- $k$  Pell numbers. In [1], the authors obtained an explicit formula for the elements in the  $n$ th power of the companion matrix and gave some interesting applications. The matrix  $A_k$  be as in (1.3), then we find the following Theorem in [1].

**Theorem 5.** The  $(i, j)$  entry  $a_{ij}^{(n)}(c_1, c_2, \dots, c_k)$  in the matrix  $A_k^n(c_1, c_2, \dots, c_k)$  is given by the following formula:

$$(5.1) \quad a_{ij}^{(n)}(c_1, c_2, \dots, c_k) = \sum_{(t_1, t_2, \dots, t_k)} \frac{t_j + t_{j+1} + \dots + t_k}{t_1 + t_2 + \dots + t_k} \times \binom{t_1 + t_2 + \dots + t_k}{t_1, t_2, \dots, t_k} c_1^{t_1} \dots c_k^{t_k}$$

where the summation is over nonnegative integers satisfying  $t_1 + 2t_2 + \dots + kt_k = n - i + j$ , and the coefficients in (5.1) is defined to be 1 if  $n = i - j$ .

Then we have the following Corollary.

**Corollary 2.** Let  $P_n^i$  be the generalized order- $k$  Pell number, for  $1 \leq i \leq k$ . Then

$$P_n^i = \sum_{(r_1, r_2, \dots, r_k)} \frac{r_k}{r_1 + r_2 + \dots + r_k} \times \binom{r_1 + r_2 + \dots + r_k}{r_1, r_2, \dots, r_k} 2^{r_1}$$

where the summation is over nonnegative integers satisfying  $r_1 + 2r_2 + \dots + kr_k = n - i + k$ .

*Proof.* In Theorem 5, if  $j = k$  and  $c_1 = 2$ , then the proof is immediately seen from (2.4). ■

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