

A GENERALIZATION OF A CONJECTURE OF MELHAM

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ABSTRACT. A generalization of one of Melham's conjectures is presented. After writing it in terms of Gaussian q -binomial coefficients, a solution is found using the elementary technique of partial fraction decomposition.

1. INTRODUCTION

The *Fibonomial coefficient* is, for $n \geq m \geq 1$, defined by

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\} := \frac{F_1 F_2 \dots F_n}{(F_1 F_2 \dots F_{n-m})(F_1 F_2 \dots F_m)}$$

with $\left\{ \begin{matrix} n \\ n \end{matrix} \right\} = \left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} = 1$ where F_n is the n th Fibonacci number .

For a detailed discussion of the Fibonomial coefficients, we refer to the list of references in [2].

The Gaussian q -binomial coefficient $\left[\begin{matrix} n \\ m \end{matrix} \right]_q$ is defined, for all real n and integers m with $m \geq 0$, by

$$\left[\begin{matrix} n \\ m \end{matrix} \right]_q := \frac{(q; q)_n}{(q; q)_m (q; q)_{n-m}}$$

and as zero otherwise, where

$$(a; q)_n = (1 - a)(1 - aq) \dots (1 - aq^{n-1}).$$

Thus, $\left[\begin{matrix} n \\ m \end{matrix} \right]_q$ is a rational function of the parameter q . For more details, see [1].

Let

$$\alpha = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1 - \sqrt{5}}{2}.$$

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Then the well known Binet forms give

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n,$$

and thus

$$F_n = \alpha^{n-1} \frac{1 - q^n}{1 - q}, \quad L_n = \alpha^n (1 + q^n),$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\mathbf{i} = \alpha\sqrt{q}$, where F_n is the n th Fibonacci number and L_n is the n th Lucas number. All the identities we are going to derive hold for general q , and results about Fibonacci and Lucas numbers come out as corollaries for the special choice of q .

The link between the Fibonomial and Gaussian q -binomial coefficients is

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \alpha^{m(n-m)} \left[\begin{matrix} n \\ m \end{matrix} \right]_q \quad \text{with } q = -\alpha^{-2}.$$

Melham [3] derived some families of identities between sums of powers of the Fibonacci and Lucas numbers. He also conjectured a complex identity by using a "falling" Fibonacci factorial $(F_n)_{(m)}$, which begins at F_n for $n \neq 0$, and is the product of m Fibonacci numbers excluding F_0 . His conjecture is in two parts: Let $m, n, k \in \mathbb{Z}$ with $m \geq 1$. Then:

(a)

$$\begin{aligned} \sum_{j=0}^{m-1} \frac{F_{n+k+m-j}^{m+1}}{(F_{m-j-1})_{(m-1)} F_{(m+1)k+m-j}} + (-1)^{\frac{m(m-1)}{2}} \frac{F_{n-mk}^{m+1}}{\prod_{j=1}^m F_{(m+1)k+j}} \\ = F_{(m+1)(n+\frac{m}{2})}. \end{aligned}$$

(b) The Lucas counterpart of (a),

$$\begin{aligned} \sum_{j=0}^{m-1} \frac{L_{n+k+m-j}^{m+1}}{(F_{m-j-1})_{(m-1)} F_{(m+1)k+m-j}} + (-1)^{\frac{m(m-1)}{2}} \frac{L_{n-mk}^{m+1}}{\prod_{j=1}^m F_{(m+1)k+j}} \\ = \begin{cases} 5^{\frac{m+1}{2}} F_{(m+1)(n+\frac{m}{2})} & \text{if } m \text{ is odd,} \\ 5^{\frac{m}{2}} L_{(m+1)(n+\frac{m}{2})} & \text{if } m \text{ is even.} \end{cases} \end{aligned}$$

In [2], we rearranged the conjectures (a) and (b) by using Fibonomials and then gave a solution of the conjecture by translating it into a q -expression: we were left with the evaluation of a certain sum. This was achieved using contour integration.

The present paper is organized as follows: (i) We generalize the conjecture of Melham by using indices in arithmetic progression for both, the Fibonacci and the Lucas instance. (ii) Then we give a solution for this

general formula by a partial fraction decomposition method that is even simpler than the contour integration given in [2] (although it is essentially equivalent).

2. A GENERALIZATION OF MELHAM'S CONJECTURE

In this section, we give a generalization of the Fibonomial coefficients in order to state a general version of the conjecture of Melham.

Definition 1. For all integers n, m, r with $m, r \geq 1$, the r -Fibonomial is defined as

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_r := \frac{F_n F_{n-r} \cdots F_{n-r(m-1)}}{F_r F_{2r} \cdots F_{mr}}$$

with $\left\{ \begin{matrix} n \\ 0 \end{matrix} \right\}_r = 1$ where F_n is the n th Fibonacci number.

Now we give a generalization of the conjecture of Melham in terms of r -Fibonomials:

Theorem 1. For any integers m, n and k :

(i) If r odd, then

$$\begin{aligned} & \sum_{j=0}^{m-1} (-1)^{\frac{j(j-1)}{2}} \left\{ \begin{matrix} (m+1)k + mr \\ j \end{matrix} \right\}_r \left\{ \begin{matrix} (m+1)k + r(m-j-1) \\ m-j-1 \end{matrix} \right\}_r \\ & \times F_{n+k+r(m-j)}^{m+1} + (-1)^{\frac{m(m-1)}{2}} F_{n-mk}^{m+1} = \left(\prod_{j=1}^m F_{(m+1)k+rj} \right) F_{(m+1)(n+\frac{rm}{2})}. \end{aligned}$$

(ii) If r even, then

$$\begin{aligned} & \sum_{j=0}^{m-1} (-1)^j \left\{ \begin{matrix} (m+1)k + mr \\ j \end{matrix} \right\}_r \left\{ \begin{matrix} (m+1)k + r(m-j-1) \\ m-j-1 \end{matrix} \right\}_r \\ & \times F_{n+k+r(m-j)}^{m+1} + (-1)^m F_{n-mk}^{m+1} = \left(\prod_{j=1}^m F_{(m+1)k+rj} \right) F_{(m+1)(n+\frac{rm}{2})}. \end{aligned}$$

(iii) *The Lucas counterpart of (i),*

$$\begin{aligned} & \sum_{j=0}^{m-1} (-1)^{\frac{j(j-1)}{2}} \left\{ \begin{matrix} (m+1)k + mr \\ j \end{matrix} \right\}_r \left\{ \begin{matrix} (m+1)k + r(m-j-1) \\ m-j-1 \end{matrix} \right\}_r \times \\ & L_{n+k+r(m-j)}^{m+1} + (-1)^{\frac{m(m-1)}{2}} L_{n-mk}^{m+1} \\ & = \begin{cases} 5^{\frac{m+1}{2}} \left(\prod_{j=1}^m F_{(m+1)k+rj} \right) F_{(m+1)(n+\frac{r}{2}m)} & \text{if } m \text{ is odd,} \\ 5^{\frac{m}{2}} \left(\prod_{j=1}^m F_{(m+1)k+rj} \right) L_{(m+1)(n+\frac{r}{2}m)} & \text{if } m \text{ is even.} \end{cases} \end{aligned}$$

(iv) *The Lucas counterpart of (ii),*

$$\begin{aligned} & \sum_{j=0}^{m-1} (-1)^j \left\{ \begin{matrix} (m+1)k + mr \\ j \end{matrix} \right\}_r \left\{ \begin{matrix} (m+1)k + r(m-j-1) \\ m-j-1 \end{matrix} \right\}_r \\ & \times L_{n+k+r(m-j)}^{m+1} + (-1)^m L_{n-mk}^{m+1} \\ & = \begin{cases} 5^{\frac{m+1}{2}} \left(\prod_{j=1}^m F_{(m+1)k+rj} \right) F_{(m+1)(n+\frac{r}{2}m)} & \text{if } m \text{ is odd,} \\ 5^{\frac{m}{2}} \left(\prod_{j=1}^m F_{(m+1)k+rj} \right) L_{(m+1)(n+\frac{r}{2}m)} & \text{if } m \text{ is even.} \end{cases} \end{aligned}$$

Proof. We want to point out that rewriting (i)–(iv) in terms of q -binomials produces the same expression for (i) and (ii) and also for (iii) and (iv). We can combine all the cases (i)–(iv) in one formula as follows:

$$\begin{aligned} & (1 - q^{(m+1)k+rm}) \frac{(q^{(m+1)k+r}; q^r)_{m-1}}{(q^r; q^r)_{m-1}} \times \\ & \sum_{j=0}^{m-1} (-1)^j q^{\frac{rj(j+1)}{2}} \begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^r} \frac{(1 + (-1)^h q^{n+k+r(m-j)})^{m+1}}{1 - q^{(m+1)k+r(m-j)}} \\ & = \left(1 - (-1)^{(h-1)(m+1)} q^{\frac{(m+1)(2n+rm)}{2}} \right) (q^{(m+1)k+r}; q^r)_m \\ & \quad - (-1)^m q^{\frac{m(m+1)(2k+r)}{2}} (1 + (-1)^h q^{n-mk})^{m+1}. \quad (2.1) \end{aligned}$$

Here, $h = 1$ in (2.1) gives us the q -notation of the cases (i), (ii) and similarly $h = 0$ in (2.1) gives the cases (iii) and (iv). For the proof, let us consider

$$X := \frac{1}{(1-z)(1-zq^r)\dots(1-zq^{r(m-1)})} \frac{(1 + (-1)^h z q^{n+k+rm})^{m+1}}{z(1 - zq^{(m+1)k+rm})}.$$

Performing *partial fraction decomposition*, we get:

$$X = \sum_{j=0}^{m-1} \frac{(-1)^j q^{r \binom{j+1}{2}}}{(q^r; q^r)_j (q^r; q^r)_{m-1-j}} \frac{(1 + (-1)^h q^{n+k+rm-rj})^{m+1}}{1 - q^{(m+1)k+rm-rj}} \\ \times \frac{1}{q^{-rj}(1 - zq^{rj})} + \frac{C}{1 - zq^{(m+1)k+rm}} + \frac{1}{z}$$

with

$$C = \frac{(1 + (-1)^h zq^{n+k+rm})^{m+1}}{z(1-z)(1-zq^r) \dots (1-zq^{r(m-1)})} \Big|_{z=q^{-(m+1)k-rm}}.$$

We note that the degree of numerator and denominator is $m+1$. As $z \rightarrow \infty$,

$$X \sim \frac{B}{z} + O(z^{-2})$$

with

$$B = \frac{(-1)^m ((-1)^h q^{n+k+rm})^{m+1}}{q^{r \binom{m}{2}} - q^{(m+1)k+rm}} = (-1)^{(h+1)(m+1)} q^{n(m+1)+r \binom{m+1}{2}}.$$

Furthermore,

$$zX = \sum_{j=0}^{m-1} \frac{(-1)^j q^{r \binom{j+1}{2}}}{(q^r; q^r)_j (q^r; q^r)_{m-1-j}} \frac{(1 + (-1)^h q^{n+k+rm-rj})^{m+1}}{1 - q^{(m+1)k+rm-rj}} \times \\ \frac{z}{q^{-rj}(1 - zq^{rj})} + \frac{Cz}{1 - zq^{(m+1)k+rm}} + 1,$$

and letting z approach ∞ :

$$B = \sum_{j=0}^{m-1} \frac{(-1)^{j-1} q^{r \binom{j+1}{2}}}{(q^r; q^r)_j (q^r; q^r)_{m-1-j}} \frac{(1 + (-1)^h q^{n+k+rm-rj})^{m+1}}{1 - q^{(m+1)k+rm-rj}} \\ - \frac{C}{q^{(m+1)k+rm}} + 1.$$

Thus, we get:

$$(-1)^{(h-1)(m+1)} q^{n(m+1)+r \binom{m+1}{2}} (q^r; q^r)_{m-1} \\ = \sum_{j=0}^{m-1} (-1)^{j-1} q^{r \binom{j+1}{2}} \left[\begin{matrix} m-1 \\ j \end{matrix} \right]_{q^r} \frac{(1 + (-1)^h q^{n+k+r(m-j)})^{m+1}}{1 - q^{(m+1)k+r(m-j)}} \\ - C \cdot \frac{(q^r; q^r)_{m-1}}{q^{(m+1)k+rm}} + (q^r; q^r)_{m-1},$$

or

$$\begin{aligned}
& \sum_{j=0}^{m-1} (-1)^j q^{r \binom{j+1}{2}} \begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^r} \frac{(1 + (-1)^h q^{n+k+r(m-j)})^{m+1}}{1 - q^{(m+1)k+r(m-j)}} \\
&= -C \cdot \frac{(q^r; q^r)_{m-1}}{q^{(m+1)k+rm}} + (q^r; q^r)_{m-1} \\
&\quad - (-1)^{(h-1)(m+1)} q^{n(m+1)+r \binom{m+1}{2}} (q^r; q^r)_{m-1}. \quad (2.2)
\end{aligned}$$

Now we work out the constant C :

$$\begin{aligned}
C &= \frac{1}{q^{-(m+1)k-rm}} \times \\
&\quad \frac{(1 + (-1)^h q^{n-mk})^{m+1}}{(1 - q^{-(m+1)k-rm})(1 - q^{-(m+1)k-rm} q^r) \dots (1 - q^{-(m+1)k-rm} q^{r(m-1)})} \\
&= \frac{(-1)^m (1 + (-1)^h q^{n-mk})^{m+1} q^{((m+1)k+rm)m-r \binom{m}{2}}}{q^{-(m+1)k-rm} (1 - q^{(m+1)k+rm}) (1 - q^{(m+1)k+rm-r}) \dots (1 - q^{(m+1)k+r})} \\
&= \frac{(-1)^m (1 + (-1)^h q^{n-mk})^{m+1} q^{((m+1)k+rm)(m+1)-r \binom{m}{2}}}{(1 - q^{(m+1)k+rm}) (1 - q^{(m+1)k+rm-r}) \dots (1 - q^{(m+1)k+r})} \\
&= \frac{(-1)^m (1 + (-1)^h q^{n-mk})^{m+1} q^{(m+1)^2 k+rm(m+1)-r \binom{m}{2}}}{(q^{(m+1)k+r}; q^r)_m}. \quad (2.3)
\end{aligned}$$

Using (2.2) and (2.3), we obtain

$$\begin{aligned}
& \sum_{j=0}^{m-1} (-1)^j q^{r \binom{j+1}{2}} \begin{bmatrix} m-1 \\ j \end{bmatrix}_{q^r} \frac{(1 + (-1)^h q^{n+k+r(m-j)})^{m+1}}{1 - q^{(m+1)k+r(m-j)}} \\
&= - \frac{(-1)^m (1 + (-1)^h q^{n-mk})^{m+1} q^{(m+1)mk+r \binom{m+1}{2}}}{(q^{(m+1)k+r}; q^r)_m} \cdot (q^r; q^r)_{m-1} \\
&\quad + (q^r; q^r)_{m-1} - (-1)^{(h-1)(m+1)} q^{n(m+1)+r \binom{m+1}{2}} (q^r; q^r)_{m-1}.
\end{aligned}$$

Now we show that this is equivalent to (2.1). We write (2.1), but replace the sum by the formula that we just obtained and get

$$\begin{aligned}
& (1 - q^{(m+1)k+rm}) \frac{(q^{(m+1)k+r}; q^r)_{m-1}}{(q^r; q^r)_{m-1}} \times \\
& \quad \left[- \frac{(-1)^m (1 + (-1)^h q^{n-mk})^{m+1} q^{(m+1)mk+r \binom{m+1}{2}}}{(q^{(m+1)k+r}; q^r)_m} \cdot (q^r; q^r)_{m-1} \right. \\
& \quad \left. + (q^r; q^r)_{m-1} - (-1)^{(h-1)(m+1)} q^{n(m+1)+r \binom{m+1}{2}} (q^r; q^r)_{m-1} \right] \\
&= \left(1 - (-1)^{(h-1)(m+1)} q^{\frac{(m+1)(2n+rm)}{2}} \right) (q^{(m+1)k+r}; q^r)_m \\
&\quad - (-1)^m q^{\frac{m(m+1)(2k+r)}{2}} (1 + (-1)^h q^{n-mk})^{m+1}.
\end{aligned}$$

Our goal is achieved once we see that this is an identity. For that, we will gradually simplify it until a trivial identity remains.

We need to prove that

$$\begin{aligned}
& (1 - q^{(m+1)k+rm})(q^{(m+1)k+r}; q^r)_{m-1} \times \\
& \left[\frac{(-1)^m (1 + (-1)^h q^{n-mk})^{m+1} q^{(m+1)mk+r \binom{m+1}{2}}}{(q^{(m+1)k+r}; q^r)_m} \right. \\
& \left. + 1 - (-1)^{(h-1)(m+1)} q^{n(m+1)+r \binom{m+1}{2}} \right] \\
& = \left(1 - (-1)^{(h-1)(m+1)} q^{\frac{(m+1)(2n+rm)}{2}} \right) (q^{(m+1)k+r}; q^r)_m \\
& - (-1)^m q^{\frac{m(m+1)(2k+r)}{2}} (1 + (-1)^h q^{n-mk})^{m+1}.
\end{aligned}$$

Combining the first two factors,

$$\begin{aligned}
& (q^{(m+1)k+r}; q^r)_m \times \\
& \left[\frac{(-1)^m (1 + (-1)^h q^{n-mk})^{m+1} q^{(m+1)mk+r \binom{m+1}{2}}}{(q^{(m+1)k+r}; q^r)_m} \right. \\
& \left. + 1 - (-1)^{(h-1)(m+1)} q^{n(m+1)+r \binom{m+1}{2}} \right] \\
& = \left(1 - (-1)^{(h-1)(m+1)} q^{\frac{(m+1)(2n+rm)}{2}} \right) (q^{(m+1)k+r}; q^r)_m \\
& - (-1)^m q^{\frac{m(m+1)(2k+r)}{2}} (1 + (-1)^h q^{n-mk})^{m+1},
\end{aligned}$$

which we further rewrite:

$$\begin{aligned}
& (-1)^{m-1} (1 + (-1)^h q^{n-mk})^{m+1} q^{(m+1)mk+r \binom{m+1}{2}} + (q^{(m+1)k+r}; q^r)_m \\
& - (-1)^{(h-1)(m+1)} q^{n(m+1)+r \binom{m+1}{2}} (q^{(m+1)k+r}; q^r)_m \\
& = \left(1 - (-1)^{(h-1)(m+1)} q^{\frac{(m+1)(2n+rm)}{2}} \right) (q^{(m+1)k+r}; q^r)_m \\
& + (-1)^{m-1} q^{\frac{m(m+1)(2k+r)}{2}} (1 + (-1)^h q^{n-mk})^{m+1},
\end{aligned}$$

or

$$\begin{aligned}
& (q^{(m+1)k+r}; q^r)_m - (-1)^{(h+1)(m+1)} q^{n(m+1)+r \binom{m+1}{2}} (q^{(m+1)k+r}; q^r)_m \\
& = \left(1 - (-1)^{(h-1)(m+1)} q^{\frac{(m+1)(2n+rm)}{2}} \right) (q^{(m+1)k+r}; q^r)_m,
\end{aligned}$$

or

$$\begin{aligned}
& - (-1)^{(h-1)(m+1)} q^{n(m+1)+r \binom{m+1}{2}} (q^{(m+1)k+r}; q^r)_m \\
& = - (-1)^{(h-1)(m+1)} q^{\frac{(m+1)(2n+rm)}{2}} (q^{(m+1)k+r}; q^r)_m.
\end{aligned}$$

Finally, we see that this is correct, and so we have proved the identities (i)–(iv). \square

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