SOME SUBSEQUENCES OF THE GENERALIZED FIBONACCI AND LUCAS SEQUENCES

EMRAH KILIC

Abstract. We derive first-order nonlinear homogeneous recurrence relations for certain subsequences of generalized Fibonacci and Lucas sequences. We also present a polynomial representation for the terms of Lucas subsequence.

1. INTRODUCTION

Let p and q be nonzero integers such that $\Delta = p^2 - 4q \neq 0$. The generalized Fibonacci sequence $\{U_n(p,q)\}$, or briefly $\{U_n\}$, and Lucas sequence $\{V_n(p,q)\}$, or briefly $\{V_n\}$, are defined by for $n > 1$

$$U_n = pU_{n-1} - qU_{n-2} \text{ and } V_n = pV_{n-1} - qV_{n-2},$$

where $U_0 = 0, U_1 = 1$ and $V_0 = 2, V_1 = p$, respectively.

When $p = -q = 1$, $U_n = F_n$ (nth Fibonacci number) and $V_n = L_n$ (nth Lucas number).

The Binet forms of $\{U_n\}$ and $\{V_n\}$ are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n,$$

where α and β are the roots of $x^2 - px + q = 0$.

In [5], the solution of the following first order cubic recursion was asked

$$a_{n+1} = 5a_n^3 - 3a_n, \ a_0 = 1. \tag{1.1}$$

Then the solution was given as $a_n = F_{3^n}$ in [7]. After this, similarly the solution of recurrence

$$P_{n+1} = 25P_n^5 - 25P_n^3 + 5P_n, \ P_0 = 1 \tag{1.2}$$

was also asked. Then the solution was given as $P_n = F_{5^n}$.

2000 *Mathematics Subject Classification.* 11B37, 11B39.

Key words and phrases. Generalized Fibonacci and Lucas sequences, subsequences, recurrence relation.

As an addendum to the solution of the problem given in [7], Klamkin asked the solutions of recurrences:

$$\begin{aligned} A_{n+1} &= A_n^2 - 2,\ A_1 = 3, \\ B_{n+1} &= B_n^4 - 4B_n^2 + 2,\ B_1 = 7, \\ C_{n+1} &= C_n^6 - 6C_n^4 + 9C_n^2 - 2,\ C_1 = 18. \end{aligned}$$

Then the solutions of them were given as $A_n = L_{2^n},\ B_n = L_{4^n}$ and $C_n = L_{6^n}$.

In [1], the author give a recurrence relation for the Fibonacci subsequence $\{F_{k^n}\}$ for positive odd k, which generalize (1.1) and (1.2). In [2], some generalizations of the results of [1] were obtained for the sequences $\{U_n(p,-1)\}$ and $\{V_n(p,-1)\}$.

Meanwhile Prodinger [3] proved a general expansion formula for a sum of powers of Fibonacci numbers, as considered by Melham, as well as some extensions.

In this paper, we find first-order nonlinear recurrence relation for the subsequence $\{U_{k^n}\}$ of generalized Fibonacci sequence $\{U_n\}$ for odd k, and first-order nonlinear recurrence relation for the subsequence $\{V_{k^n}\}$ of generalized Lucas sequence $\{V_n\}$ for both odd and even k. We also give a polynomial representation for the generalized Lucas number V_{k^n} in terms of generalized Fibonacci numbers U_{k^n} of degree k for even k.

2. Recurrence Relations

We find first-order nonlinear recursions for the sequences $\{U_{k^n}\}$ and $\{V_{k^n}\}$ for certain k's. We need the following result for further use.

Lemma 1. *For $n, t \geq 0$,*

i) $U_{(2t+1)n} = U_n \sum_{k=0}^{t} \frac{2t+1}{t+k+1} \binom{t+k+1}{2k+1} \Delta^k q^{n(t-k)} U_n^{2k},$

ii) $V_{2tn} = \sum_{k=0}^{t} \frac{2t}{t+k} \binom{t+k}{2k} \Delta^k q^{n(t-k)} U_n^{2k},$

iii) $V_{(2t+1)n} = V_n \sum_{k=0}^{t} (-1)^{t+k} \frac{2t+1}{t+k+1} \binom{t+k+1}{2k+1} q^{n(t-k)} V_n^{2k}$

iv) $V_{2tn} = \sum_{i=0}^{t} (-1)^{t-i} \frac{2t}{t+i} \binom{t+i}{2i} q^{(t-i)n} V_n^{2i}.$

Proof. In order to prove the claimed identities, it is sufficient to use the following well-known formulas (for more details, see [6]):

$$X^m + Y^m = \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \frac{m}{m-k} \binom{m-k}{k} (XY)^k (X+Y)^{m-2k}, \ m \geq 1, \tag{2.1}$$

and

$$\frac{X^m - Y^m}{X - Y} = \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} (-1)^k \binom{m-k-1}{k} (XY)^k (X+Y)^{m-2k-1}, \ m \geq 1.$$

For example, the claim (iii) follows from by taking $X = \alpha^n$, $Y = \beta^n$ and $m = 2t$ in (2.1). The other claims are similarly obtained. $\square$

For odd k, we give a first-order nonlinear recurrence relation for the sequence $\{U_{k^n}\}$:

Theorem 1. *For odd $k > 0$ and $n \geq 0$,*

$$U_{k^{n+1}} = \triangle^{\frac{k-1}{2}} U_{k^n}^k + \sum_{i=0}^{(k-3)/2} \frac{2k}{k+2i+1} \binom{(k+1)/2+i}{2i+1} \triangle^i q^{k^n \left(\frac{k-1}{2}-i\right)} U_{k^n}^{2i+1}.$$

Proof. From the Binet formula of $\{U_n\}$ and by the binomial theorem, we obtain

$$\begin{aligned} U_{k^n}^k &= \left(\frac{\alpha^{k^n} - \beta^{k^n}}{\alpha - \beta} \right)^k = \frac{1}{\triangle^{k/2}} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \beta^{jk^n} \alpha^{(k-j)k^n} \\ &= \frac{1}{\triangle^{(k-1)/2}} \left(U_{k^{n+1}} + \sum_{j=1}^{(k-1)/2} \binom{k}{j} \left(-q^{k^n}\right)^j U_{(k-2j)k^n} \right), \end{aligned} \tag{2.2}$$

where Δ is defined as before. By (2.2), we obtain for odd k,

$$U_{k^{n+1}} = \triangle^{(k-1)/2} U_{k^n}^k - \sum_{j=1}^{(k-1)/2} \binom{k}{j} q^{k^n j} (-1)^j U_{(k-2j)k^n}. \tag{2.3}$$

By *(i)* in Lemma 1 and (2.3), we conclude

$$\begin{aligned} U_{k^{n+1}} &= \triangle^{(k-1)/2} U_{k^n}^k \\ &\quad - \sum_{j=1}^{\frac{k-1}{2}} \sum_{i=0}^{\frac{k-1}{2}-j} (-1)^j \binom{k}{j} \binom{(k+1)/2+i-j}{2i+1} \frac{k-2j}{\frac{k+1}{2}+i-j} \triangle^i q^{k^n\left(\frac{k-1}{2}-i\right)} U_{k^n}^{2i+1} \end{aligned}$$

which, after reversing the summation order, can be rewritten as

$$U_{k^{n+1}} = \triangle^{(k-1)/2} U_{k^n}^k - \sum_{i=0}^{(k-1)/2} \triangle^i q^{k^n\left(\frac{k-1}{2}-i\right)} A_{i,k} U_{k^n}^{2i+1}, \tag{2.4}$$

where

$$A_{i,k} = \sum_{j=1}^{(k-1)/2-i} (-1)^j \binom{k}{j} \binom{(k+1)/2+i-j}{2i+1} \frac{k-2j}{(k+1)/2+i-j}.$$

Since $A_{(k-1)/2,k}=0$, the equality (2.4) becomes

$$U_{k^{n+1}}=\triangle^{(k-1)/2}U_{k^n}^k-\sum_{i=0}^{(k-3)/2}\triangle^i A_{i,k}q^{k^n\left(\frac{k-1}{2}-i\right)}U_{k^n}^{2i+1}.$$

From (pp. 58, [4]), it is known that for $1\leq m\leq(k-3)/2$

$$\sum_{j=1}^{m}(-1)^j\,\tfrac{k-2j}{k-m-j}\tbinom{k}{j}\tbinom{k-m-j}{m-j}=-\tfrac{k}{k-m}\tbinom{k-m}{m}. \qquad (2.5)$$

In order to obtain $A_{i,k}$ as $\frac{2k}{k+2i+1}\binom{(k+1)/2+i}{2i+1}$, it is sufficient to replace m by $(k-1)/2-i$ in (2.5). Thus we obtain the claimed result. □

For example, when $k=7$, we have that

$$U_{7^{n+1}}=\Delta^3U_{7^n}^7+7\Delta^2q^{7^n}U_{7^n}^5+14\Delta q^{7^n2}U_{7^n}^3+7q^{7^n3}U_{7^n}.$$

We now give a nonlinear first order recurrence relation for the sequence $\{V_{k^n}\}$ for odd k.

Theorem 2. *For $n>0$ and odd $k>1$,*

$$V_{k^{n+1}}=V_{k^n}^k-\sum_{i=0}^{\frac{k-3}{2}}\binom{(k-1)/2+i}{2i+1}\frac{2k}{2i-k+1}(-1)^{i+\frac{k-1}{2}}q^{k^n\left(\frac{k-1}{2}-i\right)}V_{k^n}^{2i+1}.$$

Proof. It is easy to see that

$$V_{k^n}^k=\sum_{j=0}^{k}\tbinom{k}{j}\beta^{jk^n}\alpha^{(k-j)k^n}=V_{k^{n+1}}+\sum_{j=1}^{(k-1)/2}\tbinom{k}{j}q^{jk^n}V_{(k-2j)k^n}.$$

By *(iii)* in Lemma 1, we write

$$\begin{aligned}V_{k^{n+1}} &= V_{k^n}^k-\sum_{j=1}^{\frac{k-1}{2}}\sum_{i=0}^{\frac{k-1}{2}-j}\tbinom{k}{j}\tbinom{(k+1)/2+i-j}{2i+1}(-1)^{\frac{k-1}{2}-j+i}\\ &\quad\times q^{k^n\left(\frac{k-1}{2}-i\right)}\tfrac{k-2j}{(k+1)/2+i-j}V_{k^n}^{2i+1}\end{aligned}$$

which, by reversing the summation order, becomes

$$\begin{aligned}&= V_{k^n}^k-\sum_{i=0}^{\frac{k-3}{2}}\sum_{j=1}^{\frac{k-1}{2}-i}\tbinom{k}{j}\tbinom{(k+1)/2+i-j}{2i+1}\tfrac{k-2j}{(k+1)/2+i-j}(-1)^{\frac{k-1}{2}+i-j}\\ &\quad\times q^{k^n\left(\frac{k-1}{2}-i\right)}V_{k^n}^{2i+1}.\end{aligned}$$

For the sum

$$\sum_{j=1}^{\frac{k-1}{2}-i}(-1)^j\tbinom{k}{j}\tbinom{(k+1)/2+i-j}{2i+1}\tfrac{k-2j}{(k+1)/2+i-j},$$

if we take $m=(k-1)/2-i$ in (2.5), we obtain the claimed result. □

We now give a nonlinear first order recurrence relation for the sequence $\{V_{k^n}\}$ for even k.

Theorem 3. *For $n > 0$ and even $k > 1$,*

$$V_{k^{n+1}} = V_{k^{n+1}} = V_{k^n}^k - \sum_{i=0}^{k/2-1} (-1)^{i+\frac{k}{2}} \binom{i+k/2-1}{2i} \frac{2k}{2i-k} q^{\left(\frac{k}{2}-i\right)k^n} V_{k^n}^{2i}.$$

Proof. By the binomial theorem, we have that for even k,

$$V_{k^{n+1}} = V_{k^n}^k + \tbinom{k}{k/2} q^{\frac{k^{n+1}}{2}} - \sum_{j=1}^{\frac{k}{2}} \tbinom{k}{j} q^{k^n j} V_{(k-2j)k^n}.$$

From *(iv)* in Lemma 1, we write

$$\begin{aligned} V_{k^{n+1}} &= V_{k^n}^k - \sum_{i=0}^{\frac{k}{2}} \sum_{j=1}^{\frac{k}{2}+1-i} \tbinom{k}{j} \frac{(k-2j)}{\frac{k}{2}-j+i} \binom{\frac{k}{2}-j+i}{2i} (-1)^{\frac{k}{2}-j-i} \\ &\quad \times q^{\left(\frac{k}{2}-i\right)k^n} V_{k^n}^{2i}. \end{aligned}$$

After reversing the summation order and by using (2.5), we get

$$V_{k^{n+1}} = V_{k^n}^k - \sum_{i=0}^{\frac{k}{2}-1} (-1)^{i+\frac{k}{2}} \binom{i+\frac{k}{2}-1}{2i} \frac{2k}{2i-k} q^{\left(\frac{k}{2}-i\right)k^n} V_{k^n}^{2i},$$

as claimed. □

For example, when $k = 6$, we have that

$$V_{6^{n+1}} = V_{6^n}^6 - 6q^{6^n} V_{6^n}^4 + 9q^{6^n 2} V_{6^n}^2 - 2q^{6^n 3}. \tag{2.6}$$

3. A Polynomial Representation

We give a polynomial representation for the Lucas number V_{k^n} in terms of the generalized Fibonacci numbers U_{k^n} for even k.

Theorem 4. *For even $k > 0$, $n \geq 0$ and*

$$V_{k^{n+1}} = \sum_{i=0}^{k/2} \frac{2k}{k+2i} \binom{i+k/2}{2i} \triangle^i U_{k^n}^{2i} q^{k^n(k/2-i)}.$$

Proof. Consider

$$\begin{aligned} &U_{k^n}^k \\ = \quad & \frac{1}{\triangle^{k/2}} \sum_{j=0}^{k} \tbinom{k}{j} (-1)^j \beta^{jk^n} \alpha^{(k-j)k^n} \\ = \quad & \frac{1}{\triangle^{k/2}} \left(V_{k^{n+1}} - (-1)^{\frac{k}{2}} q^{\frac{k^{n+1}}{2}} \tbinom{k}{k/2} + \sum_{j=1}^{k/2} (-1)^j \tbinom{k}{j} V_{(k-2j)k^n} q^{jk^n} \right). \end{aligned}$$

By *(ii)* in Lemma 1 and reversing the summation order of the equation above, we write

$$U_{k^n}^k = \frac{1}{\triangle^{k/2}}\Big(V_{k^{n+1}} + (-1)^{\frac{k}{2}} q^{k^{n+1}/2}\tbinom{k}{k/2} + \sum_{j=1}^{\frac{k-2}{2}}\sum_{i=0}^{\frac{k}{2}-j}\tbinom{k}{j} \times (-1)^j \tfrac{k-2j}{k/2-j+i}\tbinom{k/2-j+i}{2i}\triangle^i q^{k^n(k/2-i)}U_{k^n}^{2i}\Big),$$

which becomes,

$$= \frac{1}{\triangle^{k/2}}\left(V_{k^{n+1}} + \sum_{i=0}^{\frac{k-2}{2}}\sum_{j=1}^{\frac{k}{2}-i}(-1)^j \tfrac{k-2j}{k/2-j+i}\tbinom{k}{j}\tbinom{k/2-j+i}{2i} q^{k^n(k/2-i)}\triangle^i U_{k^n}^{2i}\right).$$

If we take $m = \frac{k}{2} - i$ in (2.5) for $1 \le m \le k/2$, the last equation takes the form:

$$U_{k^n}^k = \frac{1}{\triangle^{k/2}}\left(V_{k^{n+1}} - \sum_{i=0}^{\frac{k-2}{2}} \frac{2k}{k+2i}\binom{i+k/2}{2i}\triangle^i U_{k^n}^{2i} q^{k^n(k/2-i)}\right),$$

as claimed. □

When $k = 6$, we get

$$V_{6^{n+1}} = \triangle^3 U_{6^n}^6 + 6\triangle^2 U_{6^n}^4 q^{6^n} + 9\triangle U_{6^n}^2 q^{6^n 2} + 2q^{6^n 3}. \tag{3.1}$$

Notice that even the coefficients of the formula in (3.1) and (2.6) appears to be the terms of the sequence *A034807* in the OEIS.

Conclusions

Throughout the paper, we obtain recurrence relations for the sequences $\{U_{k^n}\}$ and $\{V_{k^n}\}$ for certain k's (not all k's) and obtain a polynomial representation for the generalized Lucas number V_{k^n} in terms of generalized Fibonacci numbers U_{k^n} of degree k for even k. In order to clear how the remaining cases could not be obtained, we note some facts here. Since we never reach at the statement $U_{k^{n+1}}$ when we expand the k^{th} powers of the statements U_{k^n} and V_{k^n} by the binomial theorem for even integer k, we can't give a recurrence relation for $U_{k^{n+1}}$ for even k. As a second remaining case, is there a polynomial representation of $U_{k^{n+1}}$ in terms of V_{k^n} for odd k ? Related with this question, we note that while doing required operations, there is a problem (in reversing the summation order) so that we couldn't find a representation for the term $U_{k^{n+1}}$ in terms of V_{k^n}.

Acknowledgement

The author would thank to the anonymous referee for his/her valuable suggestions.

References

[1] P. Filipponi, On the Fibonacci numbers whose subscript is a power, The Fibonacci Quart. 34(3) (1996), 271–276.

[2] E. Kılıç, Second order linear recursions whose subscripts are a power, Miskolc Math. Notes, 11(2) (2010), 151–161.

[3] H. Prodinger, On a sum of Melham and its variants, The Fibonacci Quart. 46/47 (2008/2009), 207-215.

[4] J. Riordan, Combinatorial identities. Wiley, New York, 1979.

[5] A. Stinchcomb, Problem B-769: First order cubic recurrence relation, The Fibonacci Quart. 32(4) (1994) 373.

[6] M. N. S. Swamy, On certain identities involving Fibonacci and Lucas numbers. The Fibonacci Quart. 35(3) (1997) 230–232.

[7] D. C. Terr, Solution B-769: The recurrence of F_{3^n}, The Fibonacci Quart. 33(5) (1995) 468.

TOBB Economics and Technology University Mathematics Department 06560 Ankara Turkey

E-mail address: ekilic@etu.edu.tr