SYLVESTER-TRIDIAGONAL MATRIX WITH ALTERNATING MAIN DIAGONAL ENTRIES AND ITS SPECTRA

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ABSTRACT. Recently some generalizations of Sylvester type tridiagonal matrices have been considered with their spectra. We introduce a new kind generalization of Sylvester type tridiagonal matrix by considering its main diagonal entries. Then we compute its spectra and determinant.

1. Introduction

The Sylvester type tridiagonal matrix $M_n(x)$ of order $(n+1)$ is defined by

$$M_n(x) = \begin{bmatrix} x & 1 & 0 & & & & 0 \\ n & x & 2 & 0 & & & \\ 0 & (n-1) & x & 3 & 0 & & \\ & 0 & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & 3 & x & (n-1) & 0 \\ & & & 0 & 2 & x & n \\ 0 & & & & 0 & 1 & x \end{bmatrix}$$

and its determinant is found by Sylvester [6, p. 305] (cf. [4, p. 64] also) as

$$\det M_n(x) = \prod_{k=0}^{n} (x+n-2k).$$

In order to compute the determinant of $M_n(x)$, Askey shows two ways, one matrix-theoretic and another based on orthogonal polynomials. In addition, he explores their connection to orthogonal polynomials. For the relationships between orthogonal polynomials and other determinants of Sylvester type matrices related to Krawtchouk, Hahn and Racah polynomials, we refer to [1, 2, 5]. In [5], the author shows how the determinants from [2] can be evaluated by left or right eigenvectors of the corresponding matrices coupled with a simple similarity trick.

2000 *Mathematics Subject Classification.* 15A36, 15A18, 15A15.
Key words and phrases. Sylvester-tridiagonal matrix, spectrum, left-eigenvector, determinant.

In [3], the author considers the following generalization of tridiagonal-Sylvester matrix:

$$M_n(x,y) = \begin{bmatrix} x & 1 & 0 & & & & & 0 \\ n & x+y & 2 & 0 & & & & \\ 0 & n-1 & x+2y & 3 & 0 & & & \\ & 0 & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & 3 & x+(n-2)y & n-1 & 0 & \\ & & & 0 & 2 & x+(n-1)y & n & \\ 0 & & & & 0 & 1 & x+ny & \end{bmatrix}$$

and evaluates the determinant of $M_n(x,y)$ as

$$\det M_n(x,y) = \prod_{k=0}^{n}\left(x + \frac{ny}{2} + \frac{n-2k}{2}\sqrt{4+y^2}\right)$$

by constructing left eigenvectors of $M_n(x,y)$ via the generalized Fibonacci sequences.

In this paper, we consider a new kind generalization of tridiagonal-Sylvester matrix. Then we compute its spectra and determinants.

2. A Generalized tridiagonal-Sylvester matrix

Define the generalized tridiagonal-Sylvester matrix of order $(n+1)$ with two parameters $A_n(x)$ as follows

$$A_n(x) = \begin{bmatrix} x & n & 0 & & & & 0 \\ 1 & -x & (n-1) & 0 & & & \\ 0 & 2 & x & (n-2) & 0 & & \\ & 0 & 3 & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & (-1)^{n-1}x & 2 & 0 \\ & & & 0 & (n-1) & (-1)^n x & 1 \\ 0 & & & & 0 & n & (-1)^{n+1}x \end{bmatrix}.$$

In this paper, our main purpose is to prove that the determinant of matrix $A_n(x)$ is given by for even n,

$$\det A_n(x) = (-1)^{\frac{n}{2}}\, x \prod_{t=1}^{n/2}\left(x^2 + (2t)^2\right),$$

and, for odd n

$$\det A_n(x) = (-1)^{\lceil \frac{n}{2} \rceil} \prod_{t=0}^{\lfloor n/2 \rfloor}\left(x^2 + (2t+1)^2\right).$$

We will frequently denote the matrix $A_n(x)$ by A_n for brevity.

Let $\alpha = x - \delta$ and $\beta = x + \delta$ where $\delta = \sqrt{x^2+n^2}$.

For the matrix A_n with even dimension, n is an odd number, we define the pair of following vectors:

$$z^+ := \begin{bmatrix} -\binom{n}{0} & \frac{1}{n}\binom{n}{1}\alpha & -\binom{n}{2} & \frac{1}{n}\binom{n}{3}\alpha & -\binom{n}{4} & \frac{1}{n}\binom{n}{5}\alpha & -\binom{n}{6} & \cdots & -\binom{n}{n-1} & \frac{1}{n}\binom{n}{n}\alpha \end{bmatrix}$$

and

$$z^- := \begin{bmatrix} -\binom{n}{0} & \frac{1}{n}\binom{n}{1}\beta & -\binom{n}{2} & \frac{1}{n}\binom{n}{3}\beta & -\binom{n}{4} & \frac{1}{n}\binom{n}{5}\beta & -\binom{n}{6} & \cdots & -\binom{n}{n-1} & \frac{1}{n}\binom{n}{n}\beta \end{bmatrix}.$$

For the matrix A_n with odd dimensional, n is an even number, we define the pair of following vectors:

$$s^+ := \begin{bmatrix} -\binom{n}{0} & \frac{1}{n}\binom{n}{1}\alpha & -\binom{n}{2} & \frac{1}{n}\binom{n}{3}\alpha & -\binom{n}{4} & \frac{1}{n}\binom{n}{5}\alpha & -\binom{n}{6} & \cdots & \frac{1}{n}\binom{n}{n-1}\alpha & -\binom{n}{n} \end{bmatrix}$$

and

$$s^- := \begin{bmatrix} -\binom{n}{0} & \frac{1}{n}\binom{n}{1}\beta & -\binom{n}{2} & \frac{1}{n}\binom{n}{3}\beta & -\binom{n}{4} & \frac{1}{n}\binom{n}{5}\beta & -\binom{n}{6} & \cdots & \frac{1}{n}\binom{n}{n-1}\beta & -\binom{n}{n} \end{bmatrix}.$$

We start with our first Lemmas:

Lemma 1. *For any odd integer n, the matrix $A_n(x)$ has the eigenvalues δ and $-\delta$ with the corresponding left eigenvectors z^+ and z^-, respectively.*

Proof. To prove this claim it is sufficient to examine the two vectors z^+A_n and z^-A_n. We only examine first case for odd n such that $n = 2k+1$ and the other cases can be done similarly. From the definition of A_n, we should prove that the kth components of $z^{\pm}A_n$ are

$$\begin{aligned} z_0^{\pm}x + z_1^{\pm} &= (\pm\delta)\, z_0^{\pm}, \text{ for } k = 0, \\ z_{n-1}^{\pm} + (-1)^n x z_n^{\pm} &= (\pm\delta)\, z_n^{\pm}, \text{ for } k = n \end{aligned}$$

and for $0 < k < n$,

$$z_{k-1}^{\pm}(n-k+1) + (-1)^k x z_k^{\pm} + z_{k+1}^{\pm}(k+1) = (\pm\delta)\, z_k^{\pm}.$$

Consider the case $k = 0$, thus by $\delta = x - \alpha$ and $z_0^{\pm} = 1$, we get the claimed equality $z_0^{\pm}x + z_1^{\pm} = (\pm\delta)\, z_0^{\pm}$. Next consider the case $k = n$, by the definition of $z_n^{\pm}$, we write for odd n,

$$\begin{aligned} z_{n-1}^{+} + (-1)^n x z_n^{+} &= -\binom{n}{n-1} - x\frac{1}{n}\binom{n}{n}\alpha \\ &= -n - \frac{1}{n}x(x-\delta) \\ &= -\frac{1}{n}\left(n^2 + x^2 - \delta x\right) \\ &= -\frac{1}{n}\left(\delta^2 - \delta x\right) \\ &= -\frac{1}{n}\delta(\delta - x) \\ &= \delta\frac{1}{n}\binom{n}{n}\alpha = \delta z_n^{+}. \end{aligned}$$

Finally consider the last case $0 < k < n$, thus we must show that

$$z_{k-1}^{\pm}(n-k+1) + (-1)^k x z_k^{\pm} + z_{k+1}^{\pm}(k+1) = (\pm\delta)\, z_k^{\pm}$$

Again by examining the case z^+A_n for odd $n = 2k+1$ and odd k, we write

$$\begin{aligned} & z_{k-1}(n-k+1) - x z_k + z_{k+1}(k+1) \\ = \;& -(n-k+1)\binom{n}{k-1} - x\frac{1}{n}\binom{n}{k}\alpha - (k+1)\binom{n}{k}, \end{aligned}$$

which, by the well known identity $\binom{n}{k}=\binom{n}{k-1}\frac{n-k+1}{k}$, satisfies

$$\begin{aligned} &= -k\tbinom{n}{k}-x\frac{1}{n}\tbinom{n}{k}\alpha-(k+1)\tbinom{n}{k} \\ &= -\tbinom{n}{k}\left(2k+1-\frac{1}{n}x\alpha\right), \end{aligned}$$

which, since $n=2k+1$ and $\alpha=x-\delta$, gives us

$$\begin{aligned} &= -\tbinom{n}{k}\left(n-\frac{1}{n}x\alpha\right)=-\frac{1}{n}\tbinom{n}{k}\left(n^2-x\left(x-\delta\right)\right) \\ &= -\frac{1}{n}\tbinom{n}{k}\left(n^2+x^2-\delta x\right)=-\frac{1}{n}\tbinom{n}{k}\left(\delta^2-\delta x\right) \\ &= -\frac{1}{n}\tbinom{n}{k}\left(\delta^2-\delta x\right)=-\frac{1}{n}\tbinom{n}{k}\delta\left(\delta-x\right) \\ &= \frac{1}{n}\tbinom{n}{k}\delta\alpha=\delta z_k^+, \end{aligned}$$

as claimed.

We leave the proof to the reader for the case k is even for examining z^+A_n as well as the whole case z^-A_n are left to the reader.

For any even integer n, the matrix $A_n(x)$ has the eigenvalues δ and $-\delta$ with the corresponding left eigenvectors s^+ and s^-, respectively. The proof can be done similar to the proof of Lemma 1. □

For an odd integer n, define an $(n+1)\times(n+1)$ matrix $T_n(x)$ as

$$T_n=\left[\begin{array}{cc|ccccc} & z_0^+ & \vdots & z_1^+ & z_2^+ & \dots & z_n^+ \\ & z_0^- & \vdots & z_1^- & z_2^- & \dots & z_n^- \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & \vdots & & & & \\ & 0_{(n-1)\times 2} & \vdots & & I_{n-1} & & \\ & & \vdots & & & & \\ & & \vdots & & & & \end{array}\right]$$

where $0_{(n-1)\times 2}$ is the zero matrix of order $(n-1)\times 2$ and I_n is the identity matrix of order n.

According to the definition of matrix T_n, we can easily obtain the inverse of the matrix T_n of order $(n+1)$ as follows: for odd n

$$T_n^{-1}=\left[\begin{array}{cccccccccccccc} -\frac{\beta}{2\delta} & \frac{\alpha}{2\delta} & \vdots & \binom{n}{2} & 0 & \binom{n}{4} & 0 & \binom{n}{6} & 0 & \dots & 0 & \binom{n}{n-1} & 0 \\ -\frac{1}{2\delta} & -\frac{1}{2\delta} & \vdots & 0 & \frac{1}{n}\binom{n}{3} & 0 & \frac{1}{n}\binom{n}{5} & 0 & \frac{1}{n}\binom{n}{7} & \dots & \frac{1}{n}\binom{n}{n-2} & 0 & \frac{1}{n}\binom{n}{n} \\ \dots & \dots & \dots & \dots & \dots & & & & & \dots & & & \dots \\ & 0_{(n-1)\times 2} & \vdots & & I_{n-1} & & & & & & & & \\ & & \vdots & & & & & & & & & & \\ & & \vdots & & & & & & & & & & \end{array}\right]$$

So we can see that the matrix A_n is similar to matrix $E_n := T_n A_n T_n^{-1}$ via the matrix T_n as shown

$$T_n A_n T_n^{-1} = \left[\begin{array}{cc|c} \delta & 0 & 0_{2\times(n-1)} \\ 0 & -\delta & \\ \hline -\frac{1}{\delta} & \frac{1}{\delta} & \\ 0_{(n-2)\times 2} & & W_{n-1} \end{array}\right],$$

where the matrix W_{n-1} of order $(n-1)$ is given by

$$W_{n-1} = \begin{bmatrix} x & \left(2-\frac{2}{n}\binom{n}{3}\right) & 0 & -\frac{2}{n}\binom{n}{5} & 0 & -\frac{2}{n}\binom{n}{7} & 0 & \cdots & 0 & -\frac{2}{n}\binom{n}{n} \\ 3 & -x & (n-3) & 0 & \cdots & & & & \cdots & 0 \\ 0 & 4 & x & (n-4) & \ddots & & & & & \vdots \\ \vdots & 0 & 5 & -x & \ddots & & & & & \\ & & \ddots & \ddots & \ddots & & & & & \\ & & & & & & & \ddots & & \\ & & & & & & & \ddots & 0 & \vdots \\ & & & & & \ddots & \ddots & \ddots & 2 & 0 \\ \vdots & & & & & & 0 & n-1 & x & 1 \\ 0 & \cdots & & & & & \cdots & 0 & n & -x \end{bmatrix}.$$

Considering the 2×2 principal submatrix of E_n, we see that it has two eigenvalues $\lambda_1 = \delta$ and $\lambda_2 = -\delta$.

Up to now, we study the matrix A_n for odd n and give some results about its two eigenvalues. Now we consider the matrix A_n for even n and then will determine its two eigenvalues. Then we consider the case n is even integer and define an $(n+1)\times(n+1)$ matrix Y_n as

$$Y_n = \left[\begin{array}{c|cccc} s_0^+ & s_1^+ & s_2^+ & \cdots & s_n^+ \\ s_0^- & s_1^- & s_2^- & \cdots & s_n^- \\ \hline & & & & \\ 0_{(n-1)\times 2} & & I_{n-1} & & \\ & & & & \end{array}\right]$$

where $0_{(n-1)\times 2}$ is the zero matrix of order $(n-1)\times 2$ and I_n is the identity matrix of order n.

Thus we can obtain the inverse of Y_n of order $(n+1)$ as follows: for even n,

$$Y_n^{-1} = \begin{bmatrix} -\frac{\beta}{2\delta} & \frac{\alpha}{2\delta} & \vdots & \binom{n}{2} & 0 & \binom{n}{4} & 0 & \binom{n}{6} & 0 & \dots & \binom{n}{n-2} & 0 & \binom{n}{n} \\ -\frac{1}{2\delta} & -\frac{1}{2\delta} & \vdots & 0 & \frac{1}{n}\binom{n}{3} & 0 & \frac{1}{n}\binom{n}{5} & 0 & \frac{1}{n}\binom{n}{7} & \dots & 0 & \frac{1}{n}\binom{n}{n-1} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & & & & & \cdots & & & \cdots \\ & 0_{(n-1)\times 2} & \vdots & & I_{n-1} & & & & & & & & \\ & & \vdots & & & & & & & & & & \\ & & \vdots & & & & & & & & & & \end{bmatrix}$$

So the matrix A_n is similar to matrix $E_n := Y_n A_n Y_n^{-1}$ via the matrix Y_n as shown

$$Y_n A_n Y_n^{-1} = \begin{bmatrix} \delta & 0 & \vdots & 0_{2\times(n-1)} \\ 0 & -\delta & \vdots & \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{\delta} & \frac{1}{\delta} & \vdots & \\ 0_{(n-2)\times 2} & & \vdots & Q_{n-1} \end{bmatrix},$$

where the matrix Q_{n-1} of order $(n-1)$ is given by

$$Q_{n-1} = \begin{bmatrix} x & \left(2-\frac{2}{n}\binom{n}{3}\right) & 0 & -\frac{2}{n}\binom{n}{5} & 0 & -\frac{2}{n}\binom{n}{7} & 0 & \dots & -\frac{2}{n}\binom{n}{n} & 0 \\ 3 & -x & (n-3) & 0 & \cdots & & & \cdots & \cdots & 0 \\ 0 & 4 & x & (n-4) & \ddots & & & & & \vdots \\ \vdots & 0 & 5 & -x & \ddots & & & & & \\ & & \ddots & \ddots & \ddots & & & & & \\ & & & & & & & \ddots & & \\ & & & & & & & \ddots & 0 & \vdots \\ & & & & & \ddots & \ddots & \ddots & 2 & 0 \\ \vdots & & & & & & 0 & n-1 & -x & 1 \\ 0 & \cdots & & & & & \cdots & 0 & n & x \end{bmatrix}.$$

Thus we have two eigenvalues of matrix A_n for both even and odd n. For the remaining eigenvalues of matrix A_n for even n, we will give some results.

Define an upper triangular matrix U_n of order $(n+1)$ as follows

$$U_n = \frac{1}{\binom{n+2}{2}} \begin{bmatrix} \binom{2}{2} & 0 & -\binom{n}{2} & 0 & \cdots & 0 \\ & \binom{3}{2} & 0 & -\binom{n-1}{2} & \ddots & \vdots \\ & & 2\binom{4}{2} & \ddots & \ddots & 0 \\ & & & \ddots & 0 & -\binom{2}{2} \\ & & & & \binom{n}{2} & 0 \\ 0 & & & & & \binom{n+2}{2} \end{bmatrix}.$$

Then both the matrices W_n and Q_n are similar to the same tridiagonal matrix

$$G_n := U_n^{-1} W_n U_n \text{ and } G_n := U_n^{-1} Q_n U_n$$

which can be found as

$$G_n = \begin{bmatrix} x & (n-1) & 0 & 0 & 0 & 0 \\ 1 & -x & (n-2) & 0 & 0 & 0 \\ 0 & 2 & x & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 2 & 0 \\ 0 & 0 & 0 & (n-2) & (-1)^{n-1} x & 1 \\ 0 & 0 & 0 & 0 & (n-1) & (-1)^n x \end{bmatrix}$$

For further required computations, we define an $(n+1) \times (n+1)$ matrix U as

$$U = \begin{bmatrix} I_2 & \vdots & 0_{2\times(n-1)} \\ \cdots & \vdots\cdots & \cdots \\ & \vdots & \\ 0_{(n-1)\times 2} & \vdots & U_{n-1} \end{bmatrix},$$

then it is not hard to compute that

$$U^{-1} = \begin{bmatrix} I_2 & \vdots & 0_{2\times(n-1)} \\ \cdots & \vdots\cdots & \cdots \\ & \vdots & \\ 0_{(n-1)\times 2} & \vdots & U_{n-1}^{-1} \end{bmatrix}.$$

Thus for both even and odd cases, that is, for the matrices W_n and Q_n, we have that

$$U^{-1} E_n U = \begin{bmatrix} \delta & 0 & \vdots & 0_{2\times(n-1)} \\ 0 & -\delta & \vdots & \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{\delta}\binom{n}{2} & \frac{1}{\delta}\binom{n}{2} & \vdots & \\ 0_{(n-2)\times 2} & & \vdots & U_{n-1}^{-1} W_{n-1} U_{n-1} \end{bmatrix}$$

and

$$U^{-1} E_n U = \begin{bmatrix} \delta & 0 & \vdots & 0_{2\times(n-1)} \\ 0 & -\delta & \vdots & \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{\delta}\binom{n}{2} & \frac{1}{\delta}\binom{n}{2} & \vdots & \\ 0_{(n-2)\times 2} & & \vdots & U_{n-1}^{-1} Q_{n-1} U_{n-1} \end{bmatrix}.$$

In general, we obtain that $\det E_n$ reduced to a block lower-triangular form

$$U^{-1}E_nU=\begin{bmatrix} \delta & 0 & \vdots & 0_{2\times(n-1)} \\ 0 & -\delta & \vdots & \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{\delta}\binom{n}{2} & \frac{1}{\delta}\binom{n}{2} & \vdots & \\ 0_{(n-2)\times 2} & & \vdots & G_{n-1} \end{bmatrix}, \tag{2.1}$$

where G_n is defined as before.

Up to now, we derive some results and now summarize them as

$$\begin{aligned} E_n &= TA_nT^{-1} \text{ for odd } n, \\ E_n &= YA_nY^{-1} \text{ for even } n, \\ G_n &= U_n^{-1}W_nU_n. \end{aligned}$$

Since clearly $G_n=A_{n-1}$, therefore A_n reduces to a block lower-triangular form

$$\begin{bmatrix} Diag\,(\delta,-\delta) & 0 \\ * & A_{n-2} \end{bmatrix}$$

and so from (2.1) we have the following recursion

$$\det A_n=-\delta^2\det A_{n-2}$$

where $\delta=\sqrt{x^2+n^2}$ for $n>0$.

It is clear that x is the eigenvalue of the matrix $A_0\,(x)$. Therefore we have the spectrum of the matrix A_n as shown: for even n,

$$\lambda\,(A_n)=\left\{\mp\sqrt{x^2+(2k)^2}\right\}_{k=1}^{n/2}\cup\{x\}$$

and, for odd n,

$$\lambda\,(A_n)=\left\{\mp\sqrt{x^2+(2k+1)^2}\right\}_{k=0}^{\lfloor n/2\rfloor}.$$

Thus the determinant of matrix $A_n\,(x)$ is determined as for even n

$$\det A_n\,(x)=(-1)^{\frac{n}{2}}\,x\prod_{t=1}^{n/2}\left(x^2+(2t)^2\right),$$

and, for odd n

$$\det A_n\,(x)=(-1)^{\lceil\frac{n}{2}\rceil}\prod_{t=0}^{\lfloor n/2\rfloor}\left(x^2+(2t+1)^2\right).$$

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