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Factorizations and representations of the backward second-order linear recurrences

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Abstract

We show the relationships between the determinants and permanents of certain tridiagonal matrices and the negatively subscripted terms of second-order linear recurrences. Also considering how to the negatively subscripted terms of second-order linear recurrences can be connected to Chebyshev polynomials by determinants of these matrices, we give factorizations and representations of these numbers.

Keywords: Second-order linear recurrences; Determinant; Permanent; Factorization

1. Introduction

The well-known Fibonacci, Lucas and Pell numbers can be generalized as follows: Let A and B be nonzero, relatively prime integers such that $D = A^2 - 4B \neq 0$. Define sequences $\{u_n\}$ and $\{v_n\}$ by, for all $n \geqslant 2$ (see [24]),

$$u_n = Au_{n-1} - Bu_{n-2},$$
$$v_n = Av_{n-1} - Bv_{n-2},$$

where $u_0 = 0$, $u_1 = 1$ and $v_0 = 2$, $v_1 = A$. If $A = 1$ and $B = -1$, then $u_n = F_n$ (the nth Fibonacci number) and $v_n = L_n$ (the nth Lucas number). If $A = 2$ and $B = -1$, then $u_n = P_n$ (the nth Pell number).

Let the roots of the equation $t^2 - At + B = 0$ be α and β. Then for $n \geqslant 0$:

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad v_n = \alpha^n + \beta^n$$

and

$$u_{-n} = \frac{\alpha^{-n} - \beta^{-n}}{\alpha - \beta}, \quad v_{-n} = \alpha^{-n} + \beta^{-n}.$$

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For example, when $A=1$ and $B=-1$, then $u_{-n}=F_{-n}$ (the nth negatively subscripted Fibonacci number). In [1], the author consider the sequence of $\{F_{-n}\}$ and proves that any integer is uniquely representable as a sum of distinct nonconsecutive Fibonacci numbers of nonpositive indices $\{F_{-n}\}$. When $A=2$ and $B=-1$, then $u_{-n}=P_{-n}$ (the nth negatively subscripted Pell number). In [9], the author proves the following theorem: for any n in $\mathbb{Z}$ there exists a unique representation of the form $\sum_{i=1}^{\infty} a_i P_{-i}$, where a_i in $\{0, 1, 2\}$ and 2 is always followed by 0. In [8], the author consider the above work [9], and then describes an alternative approach to the proof of the above theorem.

In [29,7], the authors give complex factorizations of the Fibonacci numbers by considering the roots of Fibonacci polynomials as follow:

$$F_n = \prod_{k=1}^{n-1}\left(1 - 2\mathrm{i}\,\cos\frac{\pi k}{n}\right), \quad n \geqslant 2.$$

In [23,19], the authors establish the following forms:

$$F_n = \mathrm{i}^{n-1}\frac{\sin\left(n\cos^{-1}\left(-\frac{\mathrm{i}}{2}\right)\right)}{\sin\left(\cos^{-1}\left(-\frac{\mathrm{i}}{2}\right)\right)}, \quad n \geqslant 1$$

and

$$L_n = 2\mathrm{i}^n\cos\left(n\cos^{-1}\left(-\frac{\mathrm{i}}{2}\right)\right), \quad n \geqslant 1.$$

Also in [7], the authors obtain the factorization of the generalized Fibonacci polynomials by the roots of these polynomials.

In [11], the authors give the factorizations of the second-order linear recurrences $\{u_n\}$ and $\{v_n\}$ by considering how to these sequences can be connected to Chebyshev polynomials by determinants of the certain matrices. Also the authors give interesting representations of these sequences is as follows:

$$u_{n+1} = \prod_{j=1}^{n}\left(A - 2\mathrm{i}\sqrt{-B}\cos\frac{\pi j}{n+1}\right),$$

$$u_{n+1} = (\mathrm{i}\sqrt{-B})^n\frac{\sin\left((n+1)\cos^{-1}\left(\frac{-\mathrm{i}A}{2\sqrt{-B}}\right)\right)}{\sin\left(\cos^{-1}\left(\frac{-\mathrm{i}A}{2\sqrt{-B}}\right)\right)},$$

$$v_n = \prod_{k=1}^{n}\left(A - 2\mathrm{i}\sqrt{-B}\cos\frac{\pi(k-1/2)}{n}\right)$$

and

$$v_n = 2(\mathrm{i}\sqrt{-B})^n\cos\left(n\cos^{-1}\left(\frac{-\mathrm{i}A}{2\sqrt{-B}}\right)\right).$$

In [5], the authors show that

$$L_n = \prod_{k=1}^{n}\left(1 - 2\mathrm{i}\,\cos\frac{\pi(k-1/2)}{n}\right), \quad n \geqslant 1,$$

which is a special case of our above result.

In this paper, considering our earlier work [11], we give some new results for the backward three-terms recurrences.

The *permanent* of an n-square matrix $A=(a_{ij})$ is defined by

$$\operatorname{per} A=\sum_{\sigma\in S_n}\prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation extends over all permutations σ of the symmetric group S_n. Also one can find applications of permanents in [21].

The permanent of a matrix is analogous to the determinant, where all of the signs used in the Laplace expansion of minors are positive.

There are many connections between permanents or determinants of tridiagonal matrices and the Fibonacci and Lucas numbers. For example, in [14], the authors present a result involving the permanent of an tridiagonal $(-1, 0, 1)$-matrix and the Fibonacci number F_{n+1}. The authors then explore similar directions involving the positive subscripted Fibonacci and Lucas numbers as well as their uncommon negatively subscripted counterparts. Finally the authors explore the generalized order-k Lucas numbers, (see [28,13] for more details on the generalized Fibonacci and Lucas numbers), and their permanents. Minc [20] define an $n\times n$ super-diagonal $(0, 1)$-matrix $F(n, k)$ for $n+1\geqslant k$, and shows that the permanent of $F(n, k)$ equals a generalized order-k Fibonacci number. Note that when $k=2$, the matrix $F(n, 2)$ is reduced to the tridiagonal matrix and its permanent equals a usual Fibonacci number. Also in [25,27], the authors define a family of tridiagonal matrices $M(n)$ and show that the determinants of $M(n)$ are the Fibonacci numbers F_{2n+2}. In [4,3], the family of tridiagonal matrices $H(n)$ is defined and the authors show that the determinants of $H(n)$ are the Fibonacci numbers F_n. In a similar family of matrices, the $(1, 1)$ element of $H(n)$ is replaced with a 3, then the determinants, [2], now generate the Lucas sequence L_n. In [18], Lehmer discussed the relationships between permanent of tridiagonal matrices, recurrence relations, and continued fractions. Recently, in [15], the authors defined two tridiagonal matrices and then gave the relationships of the permanents and determinants of these matrices and the second-order linear recurrences $\{u_n\}$ and $\{v_n\}$. In [16] the authors show that the permanents of certain generalized doubly stochastic matrices satisfy a second-order linear recurrence. In [12,13], the authors defined certain $(0, 1)$-matrices and then showed that the relations involving the sums of the usual and generalized Fibonacci and Lucas numbers, and the permanents of these matrices.

In [17], the authors define some Hessenberg matrices and then show that the determinants or permanents of these matrices are equal to the terms of the second-order recurrence, $\{u_n\}$, u_n, u_{2n+1} and u_{2n}.

2. On the backward three-term recurrences

We recall that the Binet formula for the backward three-term recurrence $\{u_{-n}\}$ is given by, for $n>0$:

$$u_{-n}=\frac{\sigma^{-n}-\gamma^{-n}}{\sigma-\gamma},$$

or, by some arrangements,

$$u_{-n}=-\frac{\sigma^{n}-\gamma^{n}}{(\sigma\gamma)^{n}(\sigma-\gamma)}. \tag{1}$$

Let α and β be nonzero real or complex numbers such that $\alpha\beta\neq 0$ and $(\alpha+\beta)^2\neq 4\alpha\beta$.

Definition 1. For $n\geqslant 2$, let $H_n(\alpha, \beta)=[h_{ij}]$ denote the $n\times n$ tridiagonal matrix with $h_{k,k}=(\alpha+\beta)/\alpha\beta$ for $2\leqslant k\leqslant n$, $h_{k,k+1}=1/\beta$ for $2\leqslant k\leqslant n-1$, $h_{k+1,k}=1/\alpha$ for $1\leqslant k\leqslant n-1$, $h_{12}=-1/\alpha\beta^2$ and $h_{11}=-(\alpha+\beta)/(\alpha\beta)^2$.

That is,

$$H_n(\alpha,\beta)=\begin{pmatrix} -\dfrac{\alpha+\beta}{(\alpha\beta)^2} & -\dfrac{1}{\alpha\beta^2} & & & 0 \\ \dfrac{1}{\alpha} & \dfrac{\alpha+\beta}{\alpha\beta} & \dfrac{1}{\beta} & & \\ & \dfrac{1}{\alpha} & \dfrac{\alpha+\beta}{\alpha\beta} & \ddots & \\ & & \ddots & \ddots & \dfrac{1}{\beta} \\ 0 & & & \dfrac{1}{\alpha} & \dfrac{\alpha+\beta}{\alpha\beta} \end{pmatrix}.$$

Then we have the following theorem.

Theorem 2. *Let $H_n(\alpha,\beta)$ be as in Definition 1. Then for all $n\geqslant 2$:*

$$\det H_n(\alpha,\beta)=-\left(\sum_{j=0}^{n}\alpha^{n-j}\beta^j\right)\Big/(\alpha\beta)^{n+1}.$$

Proof (*Induction on n*). If $n=2$, then we have

$$\det H_2(\alpha,\beta)=\begin{vmatrix} -\dfrac{\alpha+\beta}{(\alpha\beta)^2} & -\dfrac{1}{\alpha\beta^2} \\ \dfrac{1}{\alpha} & \dfrac{\alpha+\beta}{\alpha\beta}\end{vmatrix} = -\left(\frac{(\alpha+\beta)^2-\alpha\beta}{(\alpha\beta)^3}\right) = -\left(\sum_{j=0}^{2}\alpha^{2-j}\beta^j\right)\Big/(\alpha\beta)^3.$$

We suppose that the equation holds for k. Now we show that the equation holds for $k+1$. Thus computing all determinants by the Laplace expansion of determinant with respect to last column, we obtain

$$\det H_{k+1}(\alpha,\beta)=\frac{\alpha+\beta}{\alpha\beta}$$
$$\times\begin{vmatrix} -\dfrac{\alpha+\beta}{(\alpha\beta)^2} & -\dfrac{1}{\alpha\beta^2} & & & 0 \\ \dfrac{1}{\alpha} & \dfrac{\alpha+\beta}{\alpha\beta} & \dfrac{1}{\beta} & & \\ & \dfrac{1}{\alpha} & \dfrac{\alpha+\beta}{\alpha\beta} & \ddots & \\ & & \ddots & \ddots & \dfrac{1}{\beta} \\ 0 & & & \dfrac{1}{\alpha} & \dfrac{\alpha+\beta}{\alpha\beta} \end{vmatrix} - \frac{1}{\beta}\begin{vmatrix} -\dfrac{\alpha+\beta}{(\alpha\beta)^2} & -\dfrac{1}{\alpha\beta^2} & & & & 0 \\ \dfrac{1}{\alpha} & \dfrac{\alpha+\beta}{\alpha\beta} & \dfrac{1}{\beta} & & & \\ & \dfrac{1}{\alpha} & \dfrac{\alpha+\beta}{\alpha\beta} & \ddots & & \\ & & \ddots & \ddots & & \\ & & & \dfrac{1}{\alpha} & \dfrac{\alpha+\beta}{\alpha\beta} & \dfrac{1}{\beta} \\ 0 & & & & 0 & \dfrac{1}{\alpha} \end{vmatrix},$$

which, by the Laplace expansion of determinant and the definition of the matrix $H_n(\alpha,\beta)$, satisfy

$$\det H_{k+1}(\alpha,\beta)=\frac{\alpha+\beta}{\alpha\beta}\det H_k(\alpha,\beta)-\frac{1}{\alpha\beta}\det H_{k-1}(\alpha,\beta).$$

By our assumption, we write

$$\det H_{k+1}(\alpha,\beta)=\frac{\alpha+\beta}{\alpha\beta}\left(-\frac{\sum_{j=0}^{k}\alpha^{k-j}\beta^j}{(\alpha\beta)^{k+1}}\right)-\frac{1}{\alpha\beta}\left(-\frac{\sum_{j=0}^{k-1}\alpha^{k-1-j}\beta^j}{(\alpha\beta)^k}\right)$$
$$=\frac{-(\alpha+\beta)(\sum_{j=0}^{k}\alpha^{k-j}\beta^j)}{(\alpha\beta)^{k+2}}+\frac{(\sum_{j=0}^{k-1}\alpha^{k-1-j}\beta^j)}{(\alpha\beta)^{k+1}},$$

or

$$\det H_{k+1}(\alpha,\beta)=\frac{-(\alpha+\beta)(\sum_{j=0}^{k}\alpha^{k-j}\beta^{j})}{(\alpha\beta)^{k+2}}+\frac{(\alpha\beta)(\sum_{j=0}^{k-1}\alpha^{k-1-j}\beta^{j})}{(\alpha\beta)^{k+2}},$$

which satisfy

$$\det H_{k+1}(\alpha,\beta)=-\left(\sum_{j=0}^{k+1}\alpha^{k+1-j}\beta^{j}\right)\Bigg/(\alpha\beta)^{k+2}.$$

So the proof is complete. □

Corollary 3. *Suppose that A and B satisfy* $A^2-4B\neq 0$ *and* $t^2-At+B=0$ *has roots* α *and* β. *Let* u_{-n} *be given by* (1). *Then for* $n\geqslant 2$:

$$\det H_n(\alpha,\beta)=u_{-(n+1)}.$$

Proof. From Theorem 1, we have that

$$\det H_n(\alpha,\beta)=-\left(\sum_{j=0}^{n}\alpha^{n-j}\beta^{j}\right)\Bigg/(\alpha\beta)^{n+1}.$$

If we multiple and divide the value of $\det H_n(\alpha,\beta)$ by $(\alpha-\beta)$, then we have

$$\begin{aligned}\det H_n(\alpha,\beta)&=-\frac{(\alpha^n+\alpha^{n-1}\beta^1+\cdots+\alpha^1\beta^{n-1}+\beta^n)(\alpha-\beta)}{(\alpha-\beta)(\alpha\beta)^{n+1}}\\&=-\frac{\alpha^{n+1}-\beta^{n+1}}{(\alpha\beta)^{n+1}(\alpha-\beta)}.\end{aligned}$$

So, by (1), the proof is readily seen. □

Now we recall that the Binet formula for the backward three-term recurrence $\{v_{-n}\}$ is given by, for $n>0$:

$$v_{-n}=\sigma^{-n}+\gamma^{-n}=\frac{\sigma^n+\gamma^n}{(\sigma\gamma)^n}. \tag{2}$$

Definition 4. For $n\geqslant 2$, let $G_n(\alpha,\beta)=[g_{ij}]$ denote the $n\times n$ tridiagonal matrix with $g_{ii}=(\alpha+\beta)/\alpha\beta$ for $1\leqslant k\leqslant n$, $g_{k,k+1}=1/\beta$ for $2\leqslant k\leqslant n-1$, $g_{k+1,k}=1/\alpha$ for $1\leqslant k\leqslant n-1$ and $g_{12}=2/\beta$.

That is,

$$G_n(\alpha,\beta)=\begin{pmatrix}\frac{\alpha+\beta}{\alpha\beta} & \frac{2}{\beta} & & & 0\\ \frac{1}{\alpha} & \frac{\alpha+\beta}{\alpha\beta} & \frac{1}{\beta} & & \\ & \frac{1}{\alpha} & \frac{\alpha+\beta}{\alpha\beta} & \ddots & \\ & & \ddots & \ddots & \frac{1}{\beta}\\ 0 & & & \frac{1}{\alpha} & \frac{\alpha+\beta}{\alpha\beta}\end{pmatrix}.$$

Then we have the following theorem.

Theorem 5. *Let $G_n(\alpha, \beta)$ be as in Definition 2. Then for all $n \geqslant 2$:*

$$\det G_n(\alpha, \beta) = \frac{\alpha^n + \beta^n}{(\alpha\beta)^n}.$$

Proof. The proof is very similar to the proof of Theorem 2. □

Then, by (2), we have the following corollary without proof.

Corollary 6. *Suppose that A and B satisfy $A^2 - 4B \neq 0$ and $t^2 - At + B = 0$ has roots α and β. Let v_{-n} be given by* (2). *Then for $n \geqslant 2$:*

$$\det G_n(\alpha, \beta) = v_{-n}.$$

We note that it is clear that the value of following determinant is independent of x (see [30, p. 105]):

$$\begin{vmatrix} a & x & & 0 \\ \frac{1}{x} & a & \ddots & \\ & \ddots & \ddots & x \\ 0 & & \frac{1}{x} & a \end{vmatrix}.$$

If A and B satisfy $A^2 - 4B \neq 0$ and $t^2 - At + B = 0$ has roots α and β, then we have that $A = \alpha + \beta$ and $B = \alpha\beta$. Thus considering the above results and the result of Zhang, we can easily obtain the following facts: we now define the following $n \times n$ tridiagonal matrices:

$$C_n(A, B) = \begin{pmatrix} -\frac{A}{B^2} & -\frac{1}{B} & & & 0 \\ \frac{1}{B} & \frac{A}{B} & 1 & & \\ & \frac{1}{B} & \frac{A}{B} & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & \frac{1}{B} & \frac{A}{B} \end{pmatrix} \tag{3}$$

and

$$D_n(A, B) = \begin{pmatrix} \frac{A}{B} & 2 & & & 0 \\ \frac{1}{B} & \frac{A}{B} & 1 & & \\ & \frac{1}{B} & \frac{A}{B} & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & \frac{1}{B} & \frac{A}{B} \end{pmatrix}. \tag{4}$$

Then we have the following corollaries.

Corollary 7. *Suppose that A and B satisfy $A^2 - 4B \neq 0$ and $t^2 - At + B = 0$ has roots α and β. Let u_{-n} and $C_n(A, B)$ be given by* (1) *and* (3), *respectively. Then for $n \geqslant 2$:*

$$\det C_n(A, B) = u_{-(n+1)}.$$

Corollary 8. *Suppose that A and B satisfy* $A^2-4B\neq 0$ *and* $t^2-At+B=0$ *has roots* α *and* β. *Let* v_{-n} *and* $D_n(A,B)$ *be given by* (2) *and* (4), *respectively. Then for* $n\geqslant 2$:

$$\det D_n(A,B)=v_{-n}.$$

For example, if we take $\alpha=\frac{1+\sqrt{5}}{2}$ and $\beta=\frac{1-\sqrt{5}}{2}$, then, by the above results, we have

$$\det\begin{pmatrix} -1 & 1 & & & \\ -1 & -1 & 1 & & \\ & -1 & -1 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & -1 & -1 \end{pmatrix}=F_{-(n+1)}.$$

A matrix is said to be a $(1,-1)$-*matrix* if each of its entries is either 1 or -1.

Definition 9. A matrix A is called *convertible* if there is an $n\times n$ $(1,-1)$-matrix H such that $\text{per}\, A=\det(A\circ H)$, where $A\circ H$ denotes the Hadamard product of A and H.

Such a matrix H is called a *converter* of A.

Let S be a $(1,-1)$-matrix of order n, defined by

$$S=\begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ -1 & 1 & 1 & \dots & 1 & 1 \\ 1 & -1 & 1 & \dots & 1 & 1 \\ 1 & 1 & -1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \dots & -1 & 1 \end{bmatrix}.$$

We denote the $n\times n$ matrices $C_n(A,B)\circ S$ and $D_n(A,B)\circ S$ by $E_n(A,B)$ and $W_n(A,B)$, respectively. Thus

$$E_n(A,B)=\begin{pmatrix} -\frac{A}{B^2} & -\frac{1}{B} & & & 0 \\ -\frac{1}{B} & \frac{A}{B} & 1 & & \\ & -\frac{1}{B} & \frac{A}{B} & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & -\frac{1}{B} & \frac{A}{B} \end{pmatrix} \tag{5}$$

and

$$W_n(A,B)=\begin{pmatrix} \frac{A}{B} & 2 & & & 0 \\ -\frac{1}{B} & \frac{A}{B} & 1 & & \\ & -\frac{1}{B} & \frac{A}{B} & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & -\frac{1}{B} & \frac{A}{B} \end{pmatrix}. \tag{6}$$

Then we have the following theorems.

Theorem 10. *Suppose that A and B satisfy* $A^2-4B\neq 0$ *and* $t^2-At+B=0$ *has roots* α *and* β. *Let* u_{-n} *be given by* (1) *and let* $E_n(A,B)$ *be given by* (5). *Then for* $n>1$:

$$\text{per}\, E_n(A,B)=u_{-(n+1)}.$$

Proof. The proof is similar to the proof of Theorem 2. □

Theorem 11. *Suppose that A and B satisfy $A^2-4B \neq 0$ and $t^2-At+B=0$ has roots α and β. Let v_{-n} be given by* (2) *and let $W_n(A,B)$ be given by* (6). *Then for* $n>1$:

$$\operatorname{per} W_n(A,B)=v_{-n}.$$

Proof. We will use the induction method to prove that the per $W_n(A,B)=v_{-n}$. If $n=2$, then we have

$$\operatorname{per} W_2(A,B)=\operatorname{per}\begin{pmatrix} \frac{A}{B} & 2 \\ -\frac{1}{B} & \frac{A}{B}\end{pmatrix}=\frac{A^2-2B}{B^2}=v_{-2}.$$

We suppose that the equation holds for n. Then we show that the equation holds for $n+1$. Computing the per $W_{n+1}(A,B)$ by the Laplace expansion of permanent with respect to the last column gives

$$\operatorname{per} W_{n+1}(A,B)=\frac{A}{B}\operatorname{per}$$
$$\times\begin{pmatrix} \frac{A}{B} & 2 & & & & 0 \\ -\frac{1}{B} & \frac{A}{B} & 1 & & & \\ & -\frac{1}{B} & \frac{A}{B} & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & -\frac{1}{B} & \frac{A}{B} & 1 \\ 0 & & & & -\frac{1}{B} & \frac{A}{B}\end{pmatrix}+\operatorname{per}\begin{pmatrix} \frac{A}{B} & 2 & & & & 0 \\ -\frac{1}{B} & \frac{A}{B} & 1 & & & \\ & -\frac{1}{B} & \frac{A}{B} & \ddots & & \\ & & \ddots & \ddots & 1 & 0 \\ & & & -\frac{1}{B} & \frac{A}{B} & 1 \\ 0 & & & & 0 & -\frac{1}{B}\end{pmatrix},$$

which, by the definition of the matrix $W_n(A,B)$, satisfy

$$\operatorname{per} W_{n+1}(A,B)=\frac{A}{B}\operatorname{per} W_n(A,B)-\frac{1}{B}\operatorname{per} W_{n-1}(A,B).$$

By our assumption, we may write

$$\operatorname{per} W_{n+1}(A,B)=\frac{Av_{-n}-v_{-(n-1)}}{B},$$

which, by the recurrence relations of the sequence $\{v_{-n}\}$, satisfy

$$\operatorname{per} W_{n+1}(A,B)=v_{-(n+1)}.$$

Thus the proof is complete. □

Furthermore, from [21], it well known that let A be a tridiagonal matrix, and let $\hat{A}=(\hat{a}_{ij})$ be defined by $\hat{a}_{st}=\mathrm{i}a_{st}$ if $s\neq t$ and $\hat{a}_{ss}=a_{ss}$, for all s and t ($\mathrm{i}=\sqrt{-1}$). Then we have

$$\operatorname{per}(A)=\det\left(\widehat{A}\right).$$

Now we consider the $n \times n$ following matrices:

$$\hat{E}_n(A,B) = \begin{pmatrix} -\frac{A}{B^2} & -\frac{i}{B} & & & 0 \\ -\frac{i}{B} & \frac{A}{B} & i & & \\ & -\frac{i}{B} & \frac{A}{B} & \ddots & \\ & & \ddots & \ddots & i \\ 0 & & & -\frac{i}{B} & \frac{A}{B} \end{pmatrix}, \tag{7}$$

$$\hat{W}_n(A,B) = \begin{pmatrix} \frac{A}{B} & 2i & & & 0 \\ -\frac{i}{B} & \frac{A}{B} & i & & \\ & -\frac{i}{B} & \frac{A}{B} & \ddots & \\ & & \ddots & \ddots & i \\ 0 & & & -\frac{i}{B} & \frac{A}{B} \end{pmatrix}. \tag{8}$$

Then we have the following corollaries.

Corollary 12. *Suppose that A and B satisfy* $A^2 - 4B \neq 0$ *and* $t^2 - At + B = 0$ *has roots* α *and* β. *Let* u_{-n} *and* $\hat{E}_n(A,B)$ *be given by* (1) *and* (7), *respectively. Then for* $n \geqslant 2$:

$$\det \hat{E}_n(A,B) = u_{-(n+1)}.$$

Corollary 13. *Suppose that A and B satisfy* $A^2 - 4B \neq 0$ *and* $t^2 - At + B = 0$ *has roots* α *and* β. *Let* v_{-n} *and* $\hat{W}_n(A,B)$ *be given by* (2) *and* (8), *respectively. Then for* $n \geqslant 2$:

$$\det \hat{W}_n(A,B) = v_{-n}.$$

Finally, using the above corollaries, the result of Zhang and by simple calculations, the following facts hold:

$$\det Q_n(A,B) = \begin{vmatrix} \frac{A}{B} & \frac{i}{\sqrt{-B}} & & 0 \\ \frac{i}{\sqrt{-B}} & \frac{A}{B} & \ddots & \\ & \ddots & \ddots & \frac{i}{\sqrt{-B}} \\ 0 & & \frac{i}{\sqrt{-B}} & \frac{A}{B} \end{vmatrix}_{n\times n} = (-B).u_{-(n+1)} \tag{9}$$

and

$$\det Z_n(A,B) = \begin{vmatrix} \frac{A}{2B} & \frac{i}{\sqrt{-B}} & & & 0 \\ \frac{i}{\sqrt{-B}} & \frac{A}{B} & \frac{i}{\sqrt{-B}} & & \\ & \frac{i}{\sqrt{-B}} & \frac{A}{B} & \ddots & \\ & & \ddots & \ddots & \frac{i}{\sqrt{-B}} \\ 0 & & & \frac{i}{\sqrt{-B}} & \frac{A}{B} \end{vmatrix}_{n\times n} = \frac{v_{-n}}{2}, \tag{10}$$

where u_{-n} and v_{-n} are given by (1) and (2), respectively, and A and B be as before.

Also we note that

$$\det Q_n(A,B) = (-B)\operatorname{per}(E_n(A,B))$$

and

$$\det Z_n(A,B) = \frac{\operatorname{per}(W_n(A,B))}{2}.$$

3. Factorizations

In this section, we consider the sequence $\{u_{-n}\}$ and then we give its factorizations. There are a variety of ways to compute the matrix determinant (see [6,10] for more details). In addition to the method of cofactor expansion, the determinant of a matrix can be found by taking the product of its eigenvalues. Therefore, we will compute the spectrum of $Q_n(A,B)$ to find an alternative representation of $\det Q_n(A,B)$.

We define an $n \times n$ tridiagonal toeplitz matrix $V_n = [v_{ij}]$ with $v_{ii} = 0$ for $1 \leqslant i \leqslant n$ and $v_{i,i-1} = v_{i-1,i} = 1$ for $2 \leqslant i \leqslant n$ and 0 otherwise. Clearly

$$V_n = \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{bmatrix}. \tag{11}$$

So it is clear that $Q_n(A,B) = (A/B)I + (\mathrm{i}/\sqrt{-B})V_n$ where $Q_n(A,B)$ is given by (9). Then we give the following theorem.

Theorem 14. *Let u_{-n} be the nth negatively subscripted term of the sequence $\{u_n\}$. Then, for $n \geqslant 2$:*

$$u_{-(n+1)} = -\frac{1}{B}\prod_{j=1}^{n}\left(\frac{A}{B} - \frac{2\mathrm{i}}{\sqrt{-B}}\cos\frac{\pi j}{n+1}\right).$$

Proof. Let λ_j, $j = 1, 2, \ldots, n$, be the eigenvalues of V_n with respect to eigenvectors x_j. Then, for all j

$$\begin{aligned} Q_n(A,B)x_j &= \left(\frac{A}{B}I + \frac{\mathrm{i}}{\sqrt{-B}}V_n\right)x_j = \frac{A}{B}Ix_j + \frac{\mathrm{i}}{\sqrt{-B}}V_n x_j \\ &= \frac{A}{B}x_j + \frac{\mathrm{i}}{\sqrt{-B}}\lambda_j x_j = \left(\frac{A}{B} + \frac{\mathrm{i}}{\sqrt{-B}}\lambda_j\right)x_j. \end{aligned}$$

Therefore, $\mu_j = (A/B) + (\mathrm{i}/\sqrt{-B})\lambda_j$, $j = 1, 2, \ldots, n$, are the eigenvalues of the matrix $Q_n(A,B)$. Hence, for $n \geqslant 2$:

$$\det Q_n(A,B) = \prod_{j=1}^{n}\left(\frac{A}{B} + \frac{\mathrm{i}}{\sqrt{-B}}\lambda_j\right). \tag{12}$$

To compute the λ_j's, we recall that each λ_j is a zero of the characteristic polynomial $p_n(\lambda) = |V_n - \lambda I|$. Since $V_n - \lambda I$ is a tridiagonal toeplitz matrix, i.e.,

$$V_n - \lambda I = \begin{pmatrix} -\lambda & 1 & & \\ 1 & -\lambda & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -\lambda \end{pmatrix}, \tag{13}$$

we can easily establish a recursive formula for the characteristic polynomials V_n:

$$p_1(\lambda) = -\lambda,$$
$$p_2(\lambda) = \lambda^2 - 1,$$
$$p_n(\lambda) = -\lambda p_{n-1}(\lambda) - p_{n-2}(\lambda).$$

This family of characteristic polynomials can be transformed into another family $\{U_n(x), n \geqslant 1\}$ by taking $\lambda \equiv -2x$:

$$U_1(x) = 2x,$$
$$U_2(x) = 4x^2 - 1,$$
$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x).$$

The family $\{U_n(x),\ n \geqslant 1\}$ is the set of Chebyshev polynomials of second kind. It is a well-known fact (see [23]) that defining $x \equiv \cos\theta$ allows the Chebyshev polynomials of the second kind to be written as

$$U_n(x) = \frac{\sin[(n+1)\theta]}{\sin\theta}. \tag{14}$$

From (14), we can see that the roots of $U_n(x) = 0$ are given by $\theta_k = \pi k/(n+1),\ k = 1, 2, \ldots, n$, or $x_k = \cos\theta_k = \cos[\pi k/(n+1)],\ k = 1, 2, \ldots, n$. Applying the transformation $\lambda \equiv -2x$, we have the eigenvalues of V_n:

$$\lambda_k = -2\cos\frac{\pi k}{n+1}, \quad k = 1, 2, \ldots, n. \tag{15}$$

Considering (9), (12) and (15), we obtain

$$u_{-(n+1)} = -\frac{1}{B}\det Q_n(A, B) = -\frac{1}{B}\prod_{j=1}^{n}\left(\frac{A}{B} - \frac{2\mathrm{i}}{\sqrt{-B}}\cos\frac{\pi j}{n+1}\right),$$

which is desired.

We note that an alternative proof of Theorem 14 can be obtained from [26,22].

For example, when $A = 2$ and $B = -1$, the sequence $\{u_{-n}\}$ is reduced to the negatively subscripted Pell sequence $\{P_{-n}\}$ and, by Theorem 14, then we obtain that

$$P_{-(n+1)} = \prod_{j=1}^{n}\left(-2 - 2\mathrm{i}\cos\frac{\pi j}{n+1}\right).$$

Corollary 15. *Let u_{-n} be the nth negatively subscripted term of the sequence $\{u_n\}$. Then for $n \geqslant 2$ and n even*

$$u_{-(n+1)} = -\frac{1}{B}\prod_{k=1}^{n/2}\left(\frac{A^2}{B^2} - \frac{4}{B}\cos^2\frac{k\pi}{n+1}\right)$$

and, for n odd

$$u_{-(n+1)} = -\frac{A}{B^2}\prod_{k=1}^{(n-1)/2}\left(\frac{A^2}{B^2} - \frac{4}{B}\cos^2\frac{k\pi}{n+1}\right).$$

Proof. This is an immediate consequence of Theorem 14, since, for $1 \leqslant k < n/2$:

$$\cos\frac{k\pi}{n} = -\frac{\cos(n-k)\pi}{n}. \qquad \square$$

Theorem 16. *Let u_{-n} be nth negatively subscripted term of the sequence $\{u_n\}$. Then, for $n \geqslant 2$:*

$$u_{-(n+1)} = -\frac{1}{B}\left(\frac{\mathrm{i}}{\sqrt{-B}}\right)^n \frac{\sin\left((n+1)\cos^{-1}\left(\frac{A\mathrm{i}}{2\sqrt{-B}}\right)\right)}{\sin\left(\cos^{-1}\left(\frac{A\mathrm{i}}{2\sqrt{-B}}\right)\right)}.$$

Proof. From (13), we can think of Chebyshev polynomials of the second kind as being generated by determinants of successive matrices of the form

$$K_n(x) = \begin{bmatrix} 2x & 1 & & \\ 1 & 2x & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2x \end{bmatrix},$$

where $K_n(x)$ is $n \times n$. If we denote that $Q_n(A, B) = (\mathrm{i}/\sqrt{-B})K_n\left(A\mathrm{i}/2\sqrt{-B}\right)$, then we obtain:

$$\det Q_n(A, B) = \left(\frac{\mathrm{i}}{\sqrt{-B}}\right)^n \det K_n\left(\frac{A\mathrm{i}}{2\sqrt{-B}}\right) = \left(\frac{\mathrm{i}}{\sqrt{-B}}\right)^n U_n\left(\frac{A\mathrm{i}}{2\sqrt{-B}}\right). \tag{16}$$

Combining (9), (14) and (16) yields, for $n \geqslant 1$

$$u_{-(n+1)} = -\frac{1}{B}\left(\frac{\mathrm{i}}{\sqrt{-B}}\right)^n \frac{\sin\left((n+1)\cos^{-1}\left(\frac{A\mathrm{i}}{2\sqrt{-B}}\right)\right)}{\sin\left(\cos^{-1}\left(\frac{A\mathrm{i}}{2\sqrt{-B}}\right)\right)}.$$

The proof is complete. □

When $A = 1$ and $B = -1$, the sequence $\{u_{-n}\}$ is reduced to the negatively subscripted Fibonacci sequence $\{F_{-n}\}$ and by Theorem 16, we obtain

$$F_{-(n+1)} = \mathrm{i}^n \frac{\sin\left((n+1)\cos^{-1}\left(\frac{\mathrm{i}}{2}\right)\right)}{\sin\left(\cos^{-1}\left(\frac{\mathrm{i}}{2}\right)\right)}.$$

Now we consider another backward sequence $\{v_{-n}\}$ and then we give its factorizations. We start with the following theorem.

Theorem 17. *Let v_{-n} be the nth negatively subscripted term of the sequence $\{v_n\}$. Then for $n \geqslant 2$:*

$$v_{-n} = \prod_{k=1}^{n}\left(\frac{A}{B} - \frac{2\mathrm{i}}{\sqrt{-B}}\cos\frac{\pi(k-\frac{1}{2})}{n}\right).$$

Proof. From (10), we have that $2\det Z_n(A, B) = v_{-n}$. We will not compute the spectrum of $Z_n(A, B)$ directly. Instead, we will note the following ($\det(I + e_1 e_1^T) = 2$):

$$\det Z_n(A, B) = \tfrac{1}{2}\det((I + e_1 e_1^T)Z_n(A, B)), \tag{17}$$

where e_j is the jth column of the identity matrix. Thus we can write the right side of (17) as follows:

$$\frac{1}{2}\det((I + e_1 e_1^T)Z_n(A, B)) = \frac{1}{2}\det\left(\frac{A}{B}I + \frac{\mathrm{i}}{\sqrt{-B}}(V_n + e_1 e_2^T)\right),$$

where the matrix V_n is given by (11). Let γ_j, $j=1,2,\ldots,n$, be the eigenvalues of $V_n+e_1e_2^T$ with respect to eigenvectors y_j. Then, for all j

$$\left(\frac{A}{B}I+\frac{\mathrm{i}}{\sqrt{-B}}(V_n+e_1e_2^T)\right)y_j=\frac{A}{B}Iy_j+\frac{\mathrm{i}}{\sqrt{-B}}(V_n+e_1e_2^T)y_j$$
$$=\frac{A}{B}y_j+\frac{\mathrm{i}}{\sqrt{-B}}\gamma_jy_j=\left(\frac{A}{B}+\frac{\mathrm{i}}{\sqrt{-B}}\gamma_j\right)y_j.$$

Thus

$$\frac{1}{2}\det\left(\frac{A}{B}I+\frac{\mathrm{i}}{\sqrt{-B}}(V_n+e_1e_2^T)\right)=\frac{1}{2}\prod_{k=1}^{n}\left(\frac{A}{B}+\frac{\mathrm{i}}{\sqrt{-B}}\gamma_j\right).$$

To compute the γ_j's, we recall that all γ is a zero of the characteristic polynomial $t_n(\gamma)=\det(V_n+e_1e_2^T-\gamma I)$. Since $\det(I-\frac{1}{2}e_1e_1^T)=\frac{1}{2}$, we can alternately write the characteristic polynomial as

$$t_n(\gamma)=2\det[(I-\tfrac{1}{2}e_1e_1^T)(V_n+e_1e_2^T-\gamma I)].$$

Since $t_n(\gamma)$ is twice the determinant of a tridiagonal matrix, that is,

$$t_n(\gamma)=2\det\begin{bmatrix}-\frac{\gamma}{2} & 1 & & & \\ 1 & -\gamma & 1 & & \\ & 1 & -\gamma & \ddots & \\ & & \ddots & \ddots & 1\\ & & & 1 & -\gamma\end{bmatrix}, \tag{18}$$

one can derive a recursive formula for $t_n(\gamma)/2$:

$$\frac{t_1(\gamma)}{2}=-\frac{\gamma}{2},$$
$$\frac{t_2(\gamma)}{2}=\frac{\gamma^2}{2}-1,$$
$$\frac{t_n(\gamma)}{2}=-\gamma\frac{t_{n-1}(\gamma)}{2}-\frac{t_{n-2}(\gamma)}{2}.$$

This family of polynomials can be transformed into another family $\{T_n(x), n\geqslant 1\}$ by taking $\gamma\equiv -2x$:

$$T_1(x)=x,$$
$$T_2(x)=2x^2-1,$$
$$T_n(x)=2xT_{n-1}(x)-T_{n-2}(x).$$

The family $\{T_n(x), n\geqslant 1\}$ is the set of Chebyshev polynomials of first kind. In [23], Rivlin presents that defining $x\equiv\cos\theta$ allows the Chebyshev polynomials of the first kind to be written as

$$T_n(x)=\cos n\theta. \tag{19}$$

From (19), one can see that the roots of $T_n(x)=0$ are given by

$$\theta_k=\frac{\pi(k-\frac{1}{2})}{n}\quad\text{or}\quad x_k=\cos\theta_k=\cos\frac{\pi(k-\frac{1}{2})}{n}\quad\text{for } k=1,2,\ldots,n.$$

Applying the transformation $\gamma\equiv -2x$ and considering the roots of the $t_n(\gamma)$ are also roots of $\det(V_n+e_1e_2^T-\gamma I)=0$, we have the eigenvalues of $V_n+e_1e_2^T$:

$$\gamma_k=-2\cos\frac{\pi(k-\frac{1}{2})}{n}\quad\text{for } k=1,2,\ldots,n. \tag{20}$$

From (10), (17) and (20), we obtain

$$v_{-n}=\prod_{k=1}^{n}\left(\frac{A}{B}-\frac{2\mathrm{i}}{\sqrt{-B}}\cos\frac{\pi(k-\frac{1}{2})}{n}\right).$$

So the proof is complete. □

We note that an alternative proof of Theorem 17 can be obtained from [26,22].

When $A=1$ and $B=-1$, then the backward sequence $\{v_{-n}\}$ is reduced to the negatively subscripted Lucas sequence and by Theorem 17, we have that

$$L_{-n}=\prod_{k=1}^{n}\left(-1-2\mathrm{i}\cos\frac{\pi(k-\frac{1}{2})}{n}\right).$$

Theorem 18. *Let v_{-n} be the nth negatively subscripted term of the sequence $\{v_n\}$. Then, for $n\geqslant 2$:*

$$v_{-n}=2\left(\frac{\mathrm{i}}{\sqrt{-B}}\right)^{n}\cos\left(n\cos^{-1}\left(\frac{\mathrm{i}A}{2\sqrt{-B}}\right)\right).$$

Proof. From (18), we think of Chebyshev polynomials of the first kind as being generated by determinants of successive matrices of the form

$$M_n(x)=\begin{bmatrix} x & 1 & & & \\ 1 & 2x & 1 & & \\ & 1 & 2x & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 2x \end{bmatrix}_{n\times n}.$$

We note that $\det Z_n(A,B)=(\mathrm{i}/\sqrt{-B})M_n(\mathrm{i}A/2\sqrt{-B})$, thus

$$\det Z_n(A,B)=\left(\frac{\mathrm{i}}{\sqrt{-B}}\right)^{n}\det M_n\left(\frac{\mathrm{i}A}{2\sqrt{-B}}\right)=\left(\frac{\mathrm{i}}{\sqrt{-B}}\right)^{n}T_n\left(\frac{\mathrm{i}A}{2\sqrt{-B}}\right). \tag{21}$$

From (10), (19) and (21), we obtain

$$v_{-n}=2\left(\frac{\mathrm{i}}{\sqrt{-B}}\right)^{n}\cos\left(n\cos^{-1}\left(\frac{\mathrm{i}A}{2\sqrt{-B}}\right)\right).$$

The proof is complete. □

If $A=1$ and $B=-1$, then the sequence $\{v_{-n}\}$ is reduced to the negatively subscripted Lucas sequence and by using the Theorem 18, we have that

$$L_{-n}=2\mathrm{i}^{n}\cos\left(n\cos^{-1}\left(\frac{\mathrm{i}}{2}\right)\right).$$

Corollary 19. *Let v_{-n} be the nth negatively subscripted term of the sequence $\{v_n\}$. Then, for $n\geqslant 2$ and n even*

$$v_{-n}=\prod_{k=1}^{n/2}\left(\frac{A^2}{B^2}-\frac{4}{B}\cos^2\frac{(k-\frac{1}{2})\pi}{n}\right).$$

and, for n odd

$$v_{-n}=\frac{A}{B}\prod_{k=1}^{(n-1)/2}\left(\frac{A^2}{B^2}-\frac{4}{B}\cos^2\frac{(k-\frac{1}{2})\pi}{n}\right).$$

Proof. This is an immediate consequence of Theorem 18, since, for $1 \leqslant k < n/2$:

$$\cos \frac{k\pi}{n} = -\frac{\cos(n-k)\pi}{n}.$$

In the above results, we give the complex factorizations of the sequences $\{u_{-n}\}$ and $\{v_{-n}\}$. Now we give another factorizations of these sequences.

From Section 2, we can easily derive the following facts: suppose that A and B be nonzero, relatively prime integers such that $D = A^2 - 4B \neq 0$. Then we have, for $n \geqslant 2$:

$$\det \begin{bmatrix} \frac{A}{B} & \frac{1}{\sqrt{B}} & & & \\ \frac{1}{\sqrt{B}} & \frac{A}{B} & \frac{1}{\sqrt{B}} & & \\ & \frac{1}{\sqrt{B}} & \frac{A}{B} & \ddots & \\ & & \ddots & \ddots & \frac{1}{\sqrt{B}} \\ & & & \frac{1}{\sqrt{B}} & \frac{A}{B} \end{bmatrix}_{n\times n} = (-B)u_{-(n+1)}$$

and

$$\det \begin{bmatrix} \frac{A}{2B} & \frac{1}{\sqrt{B}} & & & \\ \frac{1}{\sqrt{B}} & \frac{A}{B} & \frac{1}{\sqrt{B}} & & \\ & \frac{1}{\sqrt{B}} & \frac{A}{B} & \ddots & \\ & & \ddots & \ddots & \frac{1}{\sqrt{B}} \\ & & & \frac{1}{\sqrt{B}} & \frac{A}{B} \end{bmatrix}_{n\times n} = \frac{v_{-n}}{2}.$$

Then we have the following corollaries.

Corollary 20. *Let u_{-n} be the nth negatively subscripted terms of the backward sequence $\{u_{-n}\}$. Then for $n \geqslant 2$:*

$$u_{-(n+1)} = \left(\frac{-1}{B}\right) \prod_{k=1}^{n} \left(\frac{A}{B} - \frac{2}{\sqrt{B}} \cos \frac{\pi k}{n+1}\right).$$

Corollary 21. *Let v_{-n} be the nth negatively subscripted term of the backward sequence $\{v_{-n}\}$. Then for $n \geqslant 2$:*

$$v_{-n} = \prod_{k=1}^{n} \left(\frac{A}{B} - \frac{2}{\sqrt{B}} \cos \frac{\pi(k-\frac{1}{2})}{n}\right).$$

References

[1] F.J. Bunder, Zeckendorf representations using negative Fibonacci numbers, Fibonacci Quart. 30 (2) (1992) 111–115.
[2] P.F. Byrd, Problem B-12: a Lucas determinant, Fibonacci Quart. 1 (4) (1963) 78.
[3] N.D. Cahill, D.A. Narayan, Fibonacci and Lucas numbers as tridiagonal matrix determinants, Fibonacci Quart. 42 (3) (2004) 216–221.
[4] N.D. Cahill, J.R. D'Errica, D.A. Narayan, J.Y. Narayan, Fibonacci determinants, College Math. J. 3 (3) (2002) 221–225.
[5] N. Cahill, J.R. D'Ericco, J.P. Spence, Complex factorizations of the Fibonacci and Lucas numbers, Fibonacci Quart. 41 (1) (2003) 13–19.
[6] G. Golub, C. Van Loan, Matrix Computations, third ed., vol. 310, Johns Hopkins University Press, Baltimore, 1996, pp. 50–51.
[7] V. Hoggatt Jr., C. Long, Divisibility properties of generalized Fibonacci polynomials, Fibonacci Quart. 12 (2) (1974) 113–120.
[8] A.F. Horadam, An alternative proof of a unique representation theorem, Fibonacci Quart. 32 (5) (1994) 409–411.

[9] A.F. Horadam, Unique minimal representation of integers by negatively subscripted Pell numbers, Fibonacci Quart. 32 (3) (1994) 202–206.
[10] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, New York, 1990.
[11] E. Kilic, Factorizations and representations of the second order linear recurrences, submitted for publication.
[12] E. Kilic, D. Tasci, On families of Bipartite graphs associated with sums of Fibonacci and Lucas numbers, Ars Combin., to appear.
[13] E. Kilic, D. Tasci, On the generalized order-k Fibonacci and Lucas numbers, Rocky Mountain J. Math., to appear.
[14] E. Kilic, D. Tasci, On the permanents of some tridiagonal matrices with applications to the Fibonacci and Lucas numbers, Rocky Mountain J. Math., to appear.
[15] E. Kilic, D. Tasci, On the second order linear recurrences by tridiagonal matrices, Ars Combin., to appear.
[16] E. Kilic, D. Tasci, On the second order linear recurrence satisfied by the permanent of generalized doubly stochastic matrices, Ars Combin., to appear.
[17] E. Kilic, D. Tasci, On the generalized Fibonacci and Pell sequences by Hessenberg matrices, Ars Combin., to appear.
[18] D. Lehmer, Fibonacci and related sequences in periodic tridiagonal matrices, Fibonacci Quart. 13 (1975) 150–158.
[19] J. Margado, Note on the Chebyshev polynomials and applications to the Fibonacci numbers, Port. Math. 52 (1995) 363–378.
[20] H. Minc, Permanents of (0, 1)-circulants, Canad. Math. Bull. 7 (2) (1964) 253–263.
[21] H. Minc, Permanents, Encyclopedia of Mathematics and its Applications, Addison-Wesley, New York, 1978.
[22] M. Püschel, J.M.F. Moura, The algebraic approach to the discrete cosine and sine transforms and their fast algorithms, SIAM J. Comput. 32 (5) (2003) 1280–1316.
[23] T. Rivlin, Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory, second ed., Wiley, New York, 1990.
[24] N. Robbins, Beginning Number Theory, Dubuque, Wm. C. Brown Publishers, Iowa, 1993.
[25] G. Strang, Introduction to Linear Algebra, second ed., Wellesley-Cambridge, Wellesley MA, 1998.
[26] G. Strang, The discrete cosine transform, SIAM Rev. 41 (1999) 135–147.
[27] G. Strang, K. Borre, Linear Algebra, Geodesy and GPS, Wellesley-Cambridge, Wellesley MA, 1997 pp. 555–557.
[28] D. Tasci, E. Kilic, On the order-k generalized Lucas numbers, Appl. Math. Comput. 155 (3) (2004) 637–641.
[29] W.A. Webb, E.A. Parberry, Divisibility properties of Fibonacci polynomials, Fibonacci Quart. 7 (5) (1969) 457–463.
[30] F. Zhang, Matrix Theory, Springer, New York, Berlin, Heidelberg, 1999.