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ON THE PERMANENTS OF SOME TRIDIAGONAL MATRICES WITH APPLICATIONS TO THE FIBONACCI AND LUCAS NUMBERS

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ABSTRACT. In this paper, we derive some interesting relationships between the permanents of some tridiagonal matrices with applications to the negatively and positively subscripted usual Fibonacci and Lucas numbers. Also, we give a relation involving the generalized order-k Lucas number and permanent of a matrix.

1. Introduction. The Fibonacci sequence, $\{F_n\}$, is defined by the recurrence relation, for $n \geq 1$

$$F_{n+1} = F_n + F_{n-1} \tag{1.1}$$

where $F_0 = 0$, $F_1 = 1$. The Lucas sequence, $\{L_n\}$, is defined by the recurrence relation, for $n \geq 1$

$$L_{n+1} = L_n + L_{n-1} \tag{1.2}$$

where $L_0 = 2$, $L_1 = 1$.

Rules (1.1) and (1.2) can be used to extend the sequences backward, respectively, thus

$$\begin{aligned} F_{-1} &= F_1 - F_0, \qquad F_{-2} = F_0 - F_{-1} \\ L_{-1} &= L_1 - L_{0,} \qquad L_{-2} = L_0 - L_{-1}, \dots , \end{aligned}$$

and so on. Clearly,

$$F_{-n} = F_{-n+2} - F_{-n+1} = (-1)^{n+1} F_n, \tag{1.3}$$

$$L_{-n} = L_{-n+2} - L_{-n+1} = (-1)^{n} L_n. \tag{1.4}$$

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In [**2**] Er defined k sequences of the generalized *order-k* Fibonacci numbers as shown:

$$g_n^i = \sum_{j=1}^{k} g_{n-j}^i, \quad \text{for} \quad n > 0 \quad \text{and} \quad 1 < i \le k, \tag{1.5}$$

with boundary conditions for $1 - k \le n \le 0$,

$$g_n^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise,} \end{cases}$$

where g_n^i is the nth term of the ith sequence. For example, if $k = 2$, then $\{g_n^2\}$ is the usual Fibonacci sequence, $\{F_n\}$, and, if $k = 4$, then the fourth sequence of the generalized *order*-4 Fibonacci number is

$$1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, \dots .$$

In [**6**] the authors defined k sequences of the generalized *order-k* Lucas numbers as shown:

$$l_n^i = \sum_{j=1}^{k} l_{n-j}^i, \quad \text{for} \quad n > 0 \quad \text{and} \quad 1 < i \le k,$$

with boundary conditions for $1 - k \le n \le 0$,

$$l_n^i = \begin{cases} -1 & \text{if } n = 1 - i, \\ 2 & \text{if } n = 2 - i, \\ 0 & \text{otherwise,} \end{cases}$$

where l_n^i is the nth term of the ith sequence. For example, if $k = 2$, then $\{l_n^2\}$ is the usual Lucas sequence, $\{L_n\}$, and, if $k = 4$, then the fourth sequence of the generalized *order*-4 Lucas numbers is

$$1, 3, 4, 8, 16, 31, 59, 114, 220, 424, 817, 1575, 30636, \dots .$$

In [**3**], we gave the generalized Binet formula, the combinatorial representations and some relations involving the generalized order-k Fibonacci and Lucas numbers. In particular, we showed that, for $k \ge 2$

$$l_n^k = g_n^k + 2g_{n-1}^k \tag{1.6}$$

where l_n^k and g_n^k are the generalized *order-k* Lucas and Fibonacci numbers, respectively, for $i = k$. The above result is a well-known relation that, for $k = 2$,

$$L_n = F_n + 2F_{n-1} \quad \text{(see [\textbf{7}, page 176])}.$$

The *permanent* of an n-square matrix $A = (a_{ij})$ is defined by

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ of the symmetric group S_n.

The permanent of a matrix is analogous to the determinant, where all of the signs used in the Laplace expansion of minors are positive.

A matrix is said to be a $(0,1)$-*matrix* if each of its entries is either 0 or 1.

In [**5**], Minc constructed the $n \times n$ $(0,1)$-*matrix* $F(n,k)$ as follows:

$$(1.7) \qquad F(n,k) = \begin{bmatrix} 1 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 1 & 1 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & 1 & 1 & \dots & 1 & 0 \\ 0 & \dots & \dots & 0 & 0 & 1 & \dots & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \dots & 0 & 1 & 1 & 1 \\ 0 & \dots & \dots & 0 & \dots & 0 & 0 & 1 & 1 \end{bmatrix}$$

where, $k \leq n+1$, $F(n,k)$ denote the n-square $(0,1)$-*matrix* with 1 in the (i,j) position for $i-1 \leq j \leq i+k-1$ and 0 otherwise. Also, he showed that

$$(1.8) \qquad \operatorname{per} F(n,k) = g_{n+1}^k$$

where g_n^k is the nth generalized order-k Fibonacci number, for $i = k$.

Also, Lee defined the matrix $\mathcal{L}_n$ as follows [**4**]:

$$\mathcal{L}_n = \begin{bmatrix} 1 & 0 & 1 & 0 & & & & \\ 1 & 1 & 1 & 0 & \ddots & & & \\ 0 & 1 & 1 & 1 & \ddots & 0 & & \\ & \ddots & \ddots & \ddots & \ddots & 0 & 0 & \\ & & 0 & 1 & 1 & 1 & 0 & 0 \\ & & & 0 & 1 & 1 & 1 & 0 \\ & & & & 0 & 1 & 1 & 1 \\ & & & & & 0 & 1 & 1 \end{bmatrix}$$

and showed that

$$\operatorname{per} \mathcal{L}_n = L_{n-1}$$

where L_n is the nth usual Lucas number.

A matrix is said to be a $(-1,0,1)$-*matrix* if each of its entries are $-1, 0$ or 1.

The purpose of this paper is to develop relations involving the positively and negatively subscripted Fibonacci and Lucas numbers and the permanents of some $(-1,0,1)$-tridiagonal matrices.

Let $A = [a_{ij}]$ be an $m \times n$ real matrix row vectors $\alpha_1, \alpha_2, \dots, \alpha_m$. We say A is *contractible on column*, respectively *row*, k *if column*, respectively *row*, k contains exactly two nonzero entries. Suppose A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1)\times(n-1)$ matrix $A_{ij:k}$ obtained from A by replacing row i with $a_{jk}\alpha_i + a_{ik}\alpha_j$ and deleting row j and column k is called the *contraction of A on column k relative to rows i and j*. If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = \left[A^T_{ij:k}\right]^T$ is called the *contraction of A on row k relative to columns i and j*. Every contraction used in this paper will be on the first column using the first and second rows.

We say that A can be contracted to a matrix B if either $B = A$ or there exist matrices $A_0, A_1, \dots A_t$, $t \geq 1$, such that $A_0 = A$, $A_t = B$ and A_r is a contraction of A_{r-1} for $r = 1, 2, \dots, t$.

2. On the contraction of some tridiagonal matrices. Let us consider the following Lemma, see [**1**].

Lemma 1. *Let A be a nonnegative integral matrix of order $n > 1$, and let B be a contraction of A. Then*

$$\operatorname{per} A = \operatorname{per} B.$$

Let $C = [c_{ij}]$ be an $n \times n$ tridiagonal matrix as follows:

(2.1)

$$C = \begin{bmatrix} c_{11} & -c_{12} & 0 & & & & \\ -c_{21} & c_{22} & -c_{23} & \ddots & & & \\ 0 & -c_{32} & c_{33} & \ddots & & & \\ & \ddots & \ddots & \ddots & & & \\ & & & & c_{n-2,n-2} & -c_{n-2,n-1} & 0 \\ & & & & -c_{n-1,n-2} & c_{n-1,n-1} & -c_{n-1,n} \\ & & & & 0 & -c_{n,n-1} & c_{n,n} \end{bmatrix}$$

where the c_{ij}s are arbitrary integers and $c_{ij} \neq 0$.

So give an extension of Lemma 1 for the matrix C, by the following Theorem.

Theorem 1. *Let C be as in* (2.1), *and let B be a contraction of C. Then*

$$\operatorname{per} C = \operatorname{per} B.$$

Proof. It suffices to consider the case where B is the contraction of C on column 1 relative to rows 1 and 2. Thus, C and B have the form

$$C = \begin{bmatrix} c_{11} & -c_{12} \\ -c_{21} & c_{22} \\ 0 & D \end{bmatrix}, \qquad B = \begin{bmatrix} c_{11}c_{22} + c_{12}c_{21} \\ D \end{bmatrix}$$

where $c_{ij} \neq 0$.

Using the Laplace expansion of the permanent with respect to column 1, we obtain

$$\operatorname{per} C = c_{11}\operatorname{per} \begin{bmatrix} c_{22} \\ D \end{bmatrix} + c_{21}\operatorname{per} \begin{bmatrix} c_{12} \\ D \end{bmatrix}$$

Hence, by linearity of the permanent, $\operatorname{per} C = \operatorname{per} B$. □

Now we give an application of the above result. We introduce an n-square $(-1,0,1)$-tridiagonal Toeplitz matrix whose permanent and principal subpermanents are Fibonacci numbers of prescribed order.

Let A_n denote an $n \times n$ $(-1,0,1)$-tridiagonal Toeplitz matrix as follows: for $n \geq 3$

$$A_n = \begin{bmatrix} 1 & -1 & & & & 0 \\ -1 & 1 & -1 & & & \\ & -1 & 1 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 1 & -1 \\ 0 & & & & -1 & 1 \end{bmatrix} \tag{2.2}$$

Corollary 1. *Let A_n be the $n \times n$ $(-1,0,1)$-tridiagonal Toeplitz matrix as in* (2.2)*. Then, for $n \geq 3$*

$$\operatorname{per} A_n = F_{n+1}$$

where F_n is the nth Fibonacci number.

Proof. If $n = 3$, then we have

$$A_3 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

and hence $\operatorname{per} A_3 = 3 = F_4$.

Let A_n^p be the pth contraction of A_n, $1 \leq p \leq n-2$. From the definition of A_n, the matrix A_n can be contracted on column 1 so that

$$A_n^1 = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 1 & -1 & & & \\ & -1 & 1 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 1 & -1 \\ & & & & -1 & 1 \end{bmatrix}.$$

Since the matrix A_n^1 can be contracted on column 1 and $F_4 = 3$, $F_3 = 2$,

$$A_n^2 = \begin{bmatrix} 3 & -2 & & & & \\ -1 & 1 & -1 & & & \\ & -1 & 1 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 1 & -1 \\ & & & & -1 & 1 \end{bmatrix} = \begin{bmatrix} F_4 & -F_3 & & & & \\ -1 & 1 & -1 & & & \\ & -1 & 1 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 1 & -1 \\ & & & & -1 & 1 \end{bmatrix}.$$

Furthermore, the matrix A_n^2 can be contracted on column 1 so that

$$A_n^3 = \begin{bmatrix} 5 & -3 & & & & \\ -1 & 1 & -1 & & & \\ & -1 & 1 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 1 & -1 \\ & & & & -1 & 1 \end{bmatrix}$$

where $F_5 = 5$, $F_4 = 3$. Continuing this process, we obtain

$$A_n^r = \begin{bmatrix} F_{r+2} & -F_{r+1} & & & & \\ -1 & 1 & -1 & & & \\ & -1 & 1 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 1 & -1 \\ & & & & -1 & 1 \end{bmatrix}$$

for $3 \leq r \leq n-4$. Hence,

$$A_n^{n-3} = \begin{bmatrix} F_{n-1} & -F_{n-2} & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

which, by contraction of A_n^{n-4} on column 1, gives

$$A_n^{n-2} = \begin{bmatrix} F_{n-2}+F_{n-1} & -F_{n-1} \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} F_n & -F_{n-1} \\ -1 & 1 \end{bmatrix}$$

By applying Theorem 1, we obtain

$$\operatorname{per} A_n = \operatorname{per} A_n^{n-2} = F_{n+1}.$$

So the proof is complete. □

As a different application of the contraction, we give the following Lemma for negatively subscripted Lucas numbers.

Lemma 2. *Let V_n be an $n \times n$ matrix of the form:*

$$V_n = \begin{bmatrix} -1 & 2 & 0 & & & & & \\ 1 & -1 & 1 & 0 & & & & \\ 0 & 1 & -1 & 1 & 0 & & & \\ & 0 & 1 & -1 & 1 & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & 0 & \\ & & & 0 & 1 & -1 & 1 & 0 \\ & & & & 0 & 1 & -1 & 1 \\ & & & & & 0 & 1 & -1 \end{bmatrix},$$

and let V_n^m denote the mth contraction of the matrix V_n. Then

$$\operatorname{per} V_n^{n-2} = L_{-n}$$

where L_{-n} is the nth negatively subscripted Lucas number.

Proof. From the definition of the matrix V_n, the matrix V_n can be contracted on column 1, so that

$$V_n^1 = \begin{bmatrix} 3 & -1 & 0 & & & \\ 1 & -1 & 1 & \ddots & & \\ 0 & 1 & -1 & \ddots & 0 & \\ & \ddots & \ddots & \ddots & 1 & 0 \\ & & 0 & 1 & -1 & 1 \\ & & & 0 & 1 & -1 \end{bmatrix}$$

Since the matrix V_n^1 can be contracted on column 1 and $L_{-3} = -4$, $L_{-2} = 3$,

$$V_n^2 = \begin{bmatrix} -4 & 3 & 0 & & & \\ 1 & -1 & 1 & \ddots & & \\ 0 & 1 & -1 & \ddots & 0 & \\ & \ddots & \ddots & \ddots & 1 & 0 \\ & & 0 & 1 & -1 & 1 \\ & & & 0 & 1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} L_{-3} & L_{-2} & 0 & & & \\ 1 & -1 & 1 & 0 & & \\ 0 & 1 & -1 & 1 & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ & & 0 & 1 & -1 & 1 \\ & & & 0 & 1 & -1 \end{bmatrix}.$$

Furthermore, the matrix V_n^2 can be contracted on column 1 and $L_{-4} = 7$ so that

$$V_n^3 = \begin{bmatrix} 7 & -4 & 0 & & & \\ 1 & -1 & 1 & \ddots & & \\ 0 & 1 & -1 & \ddots & 0 & \\ & \ddots & \ddots & \ddots & 1 & 0 \\ & & 0 & 1 & -1 & 1 \\ & & & 0 & 1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} L_{-4} & L_{-3} & 0 & & & \\ 1 & -1 & 1 & \ddots & & \\ 0 & 1 & -1 & \ddots & 0 & \\ & \ddots & \ddots & \ddots & 1 & 0 \\ & & 0 & 1 & -1 & 1 \\ & & & 0 & 1 & -1 \end{bmatrix}.$$

Continuing this process, we reach

$$V_n^k = \begin{bmatrix} L_{-(k+1)} & L_{-k} & 0 & & & \\ 1 & -1 & 1 & \ddots & & \\ 0 & 1 & -1 & \ddots & 0 & \\ & \ddots & \ddots & \ddots & 1 & 0 \\ & & 0 & 1 & -1 & 1 \\ & & & 0 & 1 & -1 \end{bmatrix}$$

for $3 \leq k \leq n-4$. Hence,

$$V_n^{n-3} = \begin{bmatrix} L_{-(n-2)} & L_{-(n-3)} & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

which, by contraction of V_n^{n-4}, gives

$$V_n^{n-2} = \begin{bmatrix} L_{-(n-1)} & L_{-(n-2)} \\ 1 & -1 \end{bmatrix}.$$

Then we calculate

$$\operatorname{per} V_n^{n-2} = L_{-n+2} - L_{-n+1}$$

and, using formula (1.4), we obtain

$$\operatorname{per} V_n^{n-2} = L_{-n}.$$

So the proof is complete. □

3. On the permanents relations of some tridiagonal matrices. In this section, we introduce the sequences of $n \times n$ tridiagonal matrices $\{C_1(n),\ n = 1, 2, \dots\}$ and $\{C_2(n),\ n = 1, 2, \dots\}$, where $C_1(n)$ and $C_2(n)$ matrices of the forms:

$$(3.1) \qquad C_1(n) = \begin{bmatrix} c_{1,1} & c_{1,2} & & & & \\ c_{2,1} & c_{2,2} & c_{2,3} & & & \\ & c_{3,2} & c_{3,3} & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & c_{n-1,n-1} & c_{n-1,n} \\ & & & & c_{n,n-1} & c_{n,n} \end{bmatrix}$$

and

$$(3.2)\qquad C_2(n) = \begin{bmatrix} -c_{1,1} & c_{1,2} & & & & \\ c_{2,1} & -c_{2,2} & c_{2,3} & & & \\ & c_{3,2} & -c_{3,3} & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & -c_{n-1,n-1} & c_{n-1,n} \\ & & & & c_{n,n-1} & -c_{n,n} \end{bmatrix}$$

where the signs of the main diagonals of the matrices $C_1(n)$ and $C_2(n)$ are positive and negative, respectively.

Now we give some recursive formula for the permanents of the matrices $C_1(n)$ and $C_2(n)$. We start with the following Lemma.

Lemma 3. *Let the sequence* $\{C_1(n),\ n = 1, 2, \dots\}$ *be as in* (3.1). *Then the successive permanents of* $C_1(n)$ *are given by the recursive formula*

$$(3.3)\qquad \begin{aligned} \operatorname{per} C_1(1) &= c_{1,1}, \\ \operatorname{per} C_1(2) &= c_{1,1}c_{2,2} + c_{1,2}c_{2,1}, \\ \operatorname{per} C_1(n) &= c_{n,n} perC_1(n-1) + c_{n-1,n}c_{n,n-1} perC_1(n-2). \end{aligned}$$

Proof. We prove Lemma 3 by the second principle of finite induction, computing all permanents by the Laplace expansion of the permanent with respect to the last column. For the basis step, we have:

$$\begin{aligned} \operatorname{per} C_1(1) &= c_{1,1}, \\ \operatorname{per} C_1(2) &= \operatorname{per} \begin{bmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{bmatrix} = c_{1,1}c_{2,2} + c_{1,2}c_{2,1}, \\ \operatorname{per} C_1(3) &= \operatorname{per} \begin{bmatrix} c_{1,1} & c_{1,2} & 0 \\ c_{2,1} & c_{2,2} & c_{2,3} \\ 0 & c_{3,2} & c_{3,3} \end{bmatrix} \\ &= c_{3,3} \operatorname{per} C_1(2) + c_{2,3} \operatorname{per} \begin{bmatrix} c_{1,1} & c_{1,2} \\ 0 & c_{3,2} \end{bmatrix} \\ &= c_{3,3} \operatorname{per} C_1(2) + c_{2,3}c_{3,2} \operatorname{per} C_1(1). \end{aligned}$$

For the inductive step, we assume that

$$\operatorname{per} C_1(k) = c_{k,k}\operatorname{per} C_1(k-1) + c_{k-1,k}c_{k,k-1}\operatorname{per} C_1(k-2)$$

for $3 \le k \le n$. Now we show that the equation is true for $k+1$. Then

$$\begin{aligned}
&\operatorname{per} C_1(k+1) \\
&= \operatorname{per}\begin{bmatrix} c_{1,1} & c_{1,2} & & & & & \\ c_{2,1} & c_{2,2} & c_{2,3} & & & & \\ & c_{3,2} & c_{3,3} & \ddots & & & \\ & & \ddots & \ddots & & & \\ & & & & c_{k-1,k-1} & c_{k-1,k} & \\ & & & & c_{k,k-1} & c_{k,k} & c_{k,k+1} \\ & & & & & c_{k+1,k} & c_{k+1,k+1} \end{bmatrix} \\
&= c_{k+1,k+1}\operatorname{per} C_1(k) \\
&\quad + c_{k,k+1}\operatorname{per}\begin{bmatrix} c_{1,1} & c_{1,2} & 0 & & & \\ c_{2,1} & c_{2,2} & c_{2,3} & & & \\ 0 & c_{3,2} & c_{3,3} & \ddots & & \\ & & \ddots & \ddots & c_{k-2,k-1} & 0 \\ & & & & c_{k-1,k-1} & c_{k-1,k} \\ & & & & 0 & c_{k+1,k} \end{bmatrix} \\
&= c_{k+1,k+1}\operatorname{per} C_1(k) + c_{k,k+1}c_{k+1,k}\operatorname{per} C_1(k-1).
\end{aligned}$$

So the proof is complete. □

If we take $c_{ij} = 1$, $i-1 \le j \le i+1$, in the above Lemma, then $\operatorname{per} C_1(1) = 1$, $\operatorname{per} C_1(2) = 2$ and $\operatorname{per} C_1(n) = \operatorname{per} C_1(n-1) + \operatorname{per} C_1(n-2)$, which is exactly the Fibonacci recurrence.

Lemma 4. *Let the sequence* $\{C_2(n),\ n = 1, 2, \dots\}$ *be as in* (3.2). *Then the successive permanents of* $C_2(n)$ *is given by the recursive formula*

$$\begin{aligned}
&\operatorname{per} C_2(1) = -c_{1,1}, \\
&\operatorname{per} C_2(2) = c_{1,1}c_{2,2} + c_{1,2}c_{2,1}, \qquad (3.4) \\
&\operatorname{per} C_2(n) = -c_{n,n}\operatorname{per} C_2(n-1) + c_{n-1,n}c_{n,n-1}\operatorname{per} C_2(n-2).
\end{aligned}$$

Proof. By the similar method in Lemma 3, the proof is readily seen. □

If we take $c_{ij} = 1$, $i-1 \leq j \leq i+1$, in the above Lemma, then $\operatorname{per} C_2(1) = -1$, $\operatorname{per} C_2(2) = 2$ and $\operatorname{per} C_2(n) = -\operatorname{per} C_2(n-1) + \operatorname{per} C_2(n-2)$, which is exactly the negatively subscripted Fibonacci recurrence.

Combining Lemmas 3 and 4, we give the following Theorem.

Theorem 2. *Let the sequences*

$$\{C_1(n),\ n = 1,2,\dots\} \quad \textit{and} \quad \{C_2(n),\ n = 1,2,\dots\}$$

be as in (3.1) *and* (3.2)*, respectively. Then, for* $n \geq 1$,

$$(-1)^n \operatorname{per} C_2(n) = \operatorname{per} C_1(n).$$

Proof. We will use induction method to prove that $(-1)^n \operatorname{per} C_2(n) = \operatorname{per} C_1(n)$. If $n = 1$, then

$$(-1)^1 \operatorname{per} C_2(1) = c_{1,1} = \operatorname{per} C_1(1).$$

Suppose that the equation holds for n. So we have

$$(-1)^n \operatorname{per} C_2(n) = \operatorname{per} C_1(n). \tag{3.5}$$

Now we show that the equation is true for $n+1$. From equation (3.4), we write that

$$\begin{aligned}
&(-1)^{n+1} \operatorname{per} C_2(n+1) \\
&= (-1)^{n+1} \left(-c_{n+1,n+1} \operatorname{per} C_2(n) + c_{n,n+1} c_{n+1,n} \operatorname{per} C_2(n-1)\right) \\
&= (-1)^n c_{n+1,n+1} \operatorname{per} C_2(n) + (-1)^{n+1} c_{n,n+1} c_{n+1,n} \operatorname{per} C_2(n-1)
\end{aligned}$$

and by using equation (3.5), we may write

$$(-1)^{n+1} \operatorname{per} C_2(n+1) = c_{n+1,n+1} \operatorname{per} C_1(n) + c_{n,n+1} c_{n+1,n} \operatorname{per} C_1(n-1).$$

Also, from Lemma 3, the last equation can be written as

$$(-1)^{n+1}\operatorname{per} C_2(n+1) = \operatorname{per} C_1(n+1).$$

So the proof is complete. □

In the above Theorem, we obtain the relation $(-1)^{n+1}\operatorname{per} C_2(n+1) = \operatorname{per} C_1(n+1)$ which is exactly the relation between the negatively and positively subscripted Fibonacci numbers.

As a result of Theorem 2, we give following Corollary for the negatively subscripted Fibonacci numbers.

Corollary 2. *Let $\{T(n),\ n = 1, 2, \dots\}$ be a sequence of $n \times n$ tridiagonal matrices of the form*:

$$T(n) = \begin{bmatrix} -1 & 1 & 0 & 0 & & & \\ 1 & -1 & 1 & 0 & & & \\ 0 & 1 & -1 & 1 & \ddots & & \\ & & & \ddots & \ddots & & \\ & & & & 1 & -1 & 1 \\ & & & & 0 & 1 & -1 \end{bmatrix}. \tag{3.6}$$

Then the permanent of the matrix $T(n)$ is $F_{-(n+1)}$ where F_{-n} is the nth negatively subscripted Fibonacci number.

Proof. When $k = 2$ in (1.7), then $F(n,k)$ is reduced to the matrix

$$F(n,2) = \begin{bmatrix} 1 & 1 & 0 & & & & \\ 1 & 1 & 1 & 0 & & & \\ 0 & 1 & 1 & 1 & \ddots & & \\ & \ddots & \ddots & \ddots & \ddots & 0 & \\ & & 0 & 1 & 1 & 1 & 0 \\ & & & 0 & 1 & 1 & 1 \\ & & & & 0 & 1 & 1 \end{bmatrix}.$$

Also, when $k = 2$ in the definition of the k-generalized Fibonacci sequence $\{g_n^k\}$, then the sequence is reduced to the usual Fibonacci sequence $\{F_n\}$. Then by (1.8) we have

$$\operatorname{per} F(n,2) = g_{n+1}^2 = F_{n+1}. \tag{3.7}$$

It is also seen that the matrix $T(n)$ and $F(n,2)$ are elements of the sequences $\{C_2(n)\}$ and $\{C_1(n)\}$, respectively. Using Theorem 2, we have that

$$\operatorname{per} F(n,2) = (-1)^n \operatorname{per} T(n)$$

or

$$\operatorname{per} T(n) = (-1)^n \operatorname{per} F(n,2).$$

From equation (3.7), we write

$$\operatorname{per} T(n) = (-1)^n F_{n+1}.$$

Thus, we obtain

$$\operatorname{per} T(n) = F_{-(n+1)}.$$

So the proof is complete. □

4. On generalized order-k Lucas numbers. Let $H(n+1,k) = [h_{ij}]$ be a $(n+1)\times(n+1)$ matrix as the form:

$$H(n+1,k) = \begin{bmatrix} 1 & 2 & 0 & \dots & 0 \\ 1 & & & & \\ 0 & & F(n,k) & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix} \tag{4.1}$$

where $F(n,k)$ is the $n\times n$ $(0,1)$-*matrix* as in (1.7).

Now we give a relation between the generalized order-k Lucas number, l_n^i, for $i=k$ and permanent of the matrix $H(n+1,k)$ by the following theorem.

Theorem 3. *Let the matrix* $H(n+1,k)$ *be as in* (4.1). *Then*

$$\operatorname{per} H(n+1,k) = l_{n+1}^k$$

where l_n^k *is the nth element of kth sequence of the generalized order-k Lucas numbers.*

Proof. Using the Laplace expansion of the permanent for the matrix $H(n+1,k)$ with respect to row 1, we have

$$\operatorname{per} H(n+1,k) = \operatorname{per} F(n,k) + 2\operatorname{per} \begin{bmatrix} 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & F(n-1,k) & & & & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{bmatrix}.$$

Let

$$C = \begin{bmatrix} 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & F(n-1,k) & & & & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{bmatrix};$$

then we may write

$$\operatorname{per} C = \operatorname{per} F(n-1,k).$$

Thus,

$$\operatorname{per} H(n+1,k) = \operatorname{per} F(n,k) + 2\operatorname{per} F(n-1,k).$$

From equations (1.8) and (1.6), we obtain

$$\operatorname{per} H(n+1,k) = g_{n+1}^k + 2g_n^k = l_{n+1}^k.$$

So the proof is complete. □

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