ON ALTERNATING WEIGHTED BINOMIAL SUMS WITH FALLING FACTORIALS

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Abstract. In this paper, by inspiring from earlier recent works on weighted binomial sums, we introduce and compute new kinds of binomial sums including rising factorial of the summation indices.

1. Introduction

For $n > 0$, define second order linear sequences $\{U_n\}$ and $\{V_n\}$ by

$$U_n = pU_{n-1} + U_{n-2} \text{ and } V_n = pV_{n-1} + V_{n-2},$$

with $U_0 = 0$, $U_1 = 1$, and, $V_0 = 2$, $V_1 = p$, respectively (for more details, see [10] and references therein). When $p = 1$, $U_n = F_n$ (nth Fibonacci number) and $V_n = L_n$ (nth Lucas number).

The Binet formulae are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n,$$

where $\alpha, \beta = \left(p \pm \sqrt{\Delta}\right)/2$ and $\Delta = p^2 + 4$.

If $A(x)$ and $B(x)$ are the exponential generating functions of sequences $\{a_n\}$ and $\{b_n\}$, then the convolutions of them is defined as

$$A(x) \cdot B(x) = \sum_{n \geq 0} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right) \frac{x^n}{n!}.$$

In the literature, certain weighted binomial sums have been computed by several methods (we could refer to [1, 3, 6, 7, 8, 11, 12] and the references therein). For example, Haukkanen [3] and Prodinger [11] computed certain binomial sums by using the convolution of exponential generating functions and its applications.

Meanwhile there are some kinds of binomial sums that couldn't be derived via the convolution of exponential generating functions. For example, Kılıç [5, 9] considered and computed the generalized alternating binomial sums of the form

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i f(n, i, k, t) \text{ and } \sum_{i=0}^{n} \binom{n}{i} g(n, i, k, t),$$

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where $f(n,i,k,t)$ is $U_{kti}V_{kn-(t+2)ki}$ and $U_{kti}V_{(k+1)tn-(k+2)ti}$, and, $g(n,i,k,t)$ is $U_{ki}U_{k(tn+i)}$, $U_{ki}V_{k(tn+i)}$, $V_{ki}V_{k(tn+i)}$ and $V_{ki}U_{k(tn+i)}$ for positive integers t and m. For example, from [5], we recall that for odd m,

$$\sum_{i=0}^{n}\binom{n}{i}V_{ki}V_{k(mn+i)}=\Delta^{\lfloor\frac{n+1}{2}\rfloor}U_k^n\begin{cases}V_{(m+1)kn}, & \text{if } n \text{ is even},\\ U_{(m+1)kn} & \text{if } n \text{ is odd},\end{cases}$$

and for even m,

$$\sum_{i=0}^{n}\binom{n}{i}V_{ki}V_{k(mn+i)}=V_k^nV_{(m+1)kn}+2^nV_{kmn},$$

where Δ is defined as before.

Moreover the authors of [4] computed the weighted binomial sums including the powers of the summation index:

$$\sum_{i=0}^{n}\binom{n}{i}i^mU_{ti}^{2m+\varepsilon},\quad \sum_{i=0}^{n}\binom{n}{i}i^mV_{ti}^{2m+\varepsilon},$$
$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{n+i}\,i^mU_{ti}^{2m+\varepsilon},\quad \sum_{i=0}^{n}\binom{n}{i}(-1)^{n+i}\,i^mV_{ti}^{2m+\varepsilon},$$

where positive integer t and $\varepsilon\in\{0,1\}$.

In this paper, by inspiring from the works [4, 5, 9], we will take rising factorial of the summation index instead of its powers. Clearly we will consider and compute the generalized alternating weighted binomial sums:

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^i f(n,i,k,t),$$

where $f(n,i,k,t)$ as before and m is a nonnegative integer. These kind binomial sums (except some special cases of k and t) have not been considered according to our best literature acknowledgement. To compute the claimed sums, our approach is to use the Binet formula, the binomial theorem and a useful auxiliary sum formula will be given.

2. The main results

For later use, we start with recalling the result [2]:

Lemma 2.1. *For nonnegative integers n and m,*

$$\sum_{k=0}^{n}\binom{n}{k}k^{\underline{m}}a^k=a^mn^{\underline{m}}(1+a)^{n-m}\,[a\neq-1 \text{ and } m\neq n].$$

Now we present one of our main result.

Theorem 2.2. *For any integer t and nonnegative integer m,*

(i) For odd k,

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^iU_{kti}V_{k(n-i(t+2))}=n^{\underline{m}}\left[(-1)^{tn}U_{k(tn+m(t+1))}V_{k(t+1)}^{n-m}-U_{km}V_k^{n-m}\right].$$

(ii) For even k,

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}U_{kti}V_{k(n-i(t+2))}=n^{\underline{m}}(-1)^{m+1}\Delta^{(n-m-1)/2}$$
$$\times\begin{cases}\left[U_{k(tn+m(t+1))}U_{k(t+1)}^{n-m}-U_{k}^{n-m}U_{km}\right]\Delta^{1/2} & \text{if } n\equiv m \ (mod2),\\ U_{k}^{n-m}V_{km}-V_{k(tn+m(t+1))}U_{k(t+1)}^{n-m} & \text{if } n\equiv m+1 \ (mod2).\end{cases}$$

Proof. For odd k, consider

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}U_{kti}V_{k(n-(t+2)i)}=\frac{1}{\alpha-\beta}\times$$
$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}\left[\alpha^{k(n-2i)}-\beta^{k(n-2i)}-(-1)^{it}\left(\alpha^{k(n-2(t+1)i)}-\beta^{k(n-2(t+1)i)}\right)\right]$$
$$=\frac{\alpha^{kn}}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}\alpha^{-2ki}-\frac{\beta^{kn}}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}\beta^{-2ki}$$
$$-\frac{1}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i(t+1)}\left(\alpha^{k(n-2(t+1)i)}-\beta^{k(n-2(t+1)i)}\right),$$

which, by Lemma 2.1, equals

$$\frac{1}{\alpha-\beta}n^{\underline{m}}(-1)^{m}\left(\alpha^{-km}\left(\alpha^{k}+\beta^{k}\right)^{n-m}-\beta^{-km}\left(\alpha^{k}+\beta^{k}\right)^{n-m}\right)$$
$$-\frac{1}{\alpha-\beta}n^{\underline{m}}(-1)^{m(t+1)}\left(-\alpha^{-k(tn+mt+m)}\left(\alpha^{k(t+1)}+\beta^{k(t+1)}\right)^{n-m}\right.$$
$$\left.+\beta^{-k(tn+mt+m)}\left(\alpha^{k(t+1)}+\beta^{k(t+1)}\right)^{n-m}\right),$$

which, by the Binet formula, gives us the claimed result

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}U_{kti}V_{k(n-(t+2)i)}$$
$$=-n^{\underline{m}}\left((-1)^{m(k+1)}U_{km}V_{k}^{n-m}+(-1)^{tn}U_{k(tn+tm+m)}V_{k(t+1)}^{n-m}\right)$$
$$=n^{\underline{m}}\left((-1)^{tn}U_{k(tn+m(t+1))}V_{k(t+1)}^{n-m}-U_{km}V_{k}^{n-m}\right),$$

for odd k.

Now for the case k is even, consider

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}U_{kti}V_{k(n-(t+2)i)}=\frac{1}{\alpha-\beta}\times$$
$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}\left[\alpha^{k(n-2i)}-\beta^{k(n-2i)}+\left(\alpha^{k(n-2(t+1)i)}-\beta^{k(n-2(t+1)i)}\right)\right]$$
$$=\frac{\alpha^{kn}}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}\alpha^{-2ki}-\frac{\beta^{kn}}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}\beta^{-2ki}$$
$$+\frac{\alpha^{kn}}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}\alpha^{-2k(t+1)i}-\frac{\beta^{kn}}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}\beta^{-2k(t+1)i}.$$

By Lemma 2.1, we write

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}U_{kti}V_{k(n-(t+2)i)}$$
$$=\frac{n^{\underline{m}}}{\alpha-\beta}(-1)^{m}\Big(\alpha^{-km}\left(\alpha^{k}-\beta^{k}\right)^{n-m}-\beta^{-km}\left(\beta^{k}-\alpha^{k}\right)^{n-m}$$
$$+\alpha^{-k(tn+mt+m)}\left(\alpha^{k(t+1)}-\beta^{k(t+1)}\right)^{n-m}-\beta^{-k(tn+mt+m)}\left(\beta^{k(t+1)}-\alpha^{k(t+1)}\right)^{n-m}\Big)$$
$$=\frac{n^{\underline{m}}}{\alpha-\beta}(-1)^{m}\left(\left(\alpha^{-km}-\beta^{-km}\right)\left(\alpha^{k}-\beta^{k}\right)^{n-m}-\left(\alpha^{k(t+1)}-\beta^{k(t+1)}\right)^{n-m}\right)$$
$$\times\left((-1)^{n-m}(-\alpha)^{k(tn+mt+m)}-(-\beta)^{k(tn+mt+m)}\right),$$

which equals

$$-n^{\underline{m}}(-1)^{m}\Delta^{(n-m)/2}U_{km}U_{k}^{n-m}+n^{\underline{m}}\Delta^{(n-m)/2}(-1)^{m}U_{k(tn+m(t+1))}U_{k(t+1)}^{n-m}$$
$$=n^{\underline{m}}(-1)^{m}\Delta^{(n-m)/2}\left[U_{k(tn+m(t+1))}U_{k(t+1)}^{n-m}-U_{k}^{n-m}U_{km}\right]$$

if n and m have the same parity. Similarly, if n and m have the different parity, the claim is obtained. □

Similar to Theorem 2.2, we give the following result without proof.

Theorem 2.3. *For any integers k and t,*

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}U_{kti}V_{k(n-ti)}$$
$$=(-1)^{kn(t+1)+m}n^{\underline{m}}U_{kt}^{n-m}\begin{cases}\Delta^{(n-m)/2}U_{k(tn+tm-n)}, & \text{if } n\equiv m\ (mod2),\\ -\Delta^{(n-m-1)/2}V_{k(tn+tm-n)}, & \text{if } n\equiv m+1\ (mod2),\end{cases}$$

where m is nonnegative integer.

Theorem 2.4. *For any integers k and t,*
(i) for odd t,

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}U_{tki}V_{(k+1)tn-(k+2)ti}=n^{\underline{m}}\left[(-1)^{m}V_{t}^{n-m}U_{t(kn-m)}+V_{t(k+1)}^{n-m}U_{tm(k+1)}\right],$$

(ii) for even t,

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}U_{tki}V_{tn(k+1)-t(k+2)i}=(-1)^{m}n^{\underline{m}}\Delta^{(n-m-1)/2}$$
$$\times\begin{cases}\sqrt{\Delta}\left(U_{t}^{n-m}U_{t(kn-m)}+U_{t(k+1)}^{n-m}U_{tm(k+1)}\right) & \text{if } n\equiv m\ (mod2),\\ U_{t}^{n-m}V_{t(kn-m)}-U_{t(k+1)}^{n-m}V_{tm(k+1)} & \text{if } n\equiv m+1\ (mod2),\end{cases}$$

where m is nonnegative integer.

Proof. Consider

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}\left(-1\right)^{i}U_{kti}V_{(k+1)tn-(k+2)ti}$$
$$=\frac{1}{\alpha-\beta}\times\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}\left(-1\right)^{i}\Big(\alpha^{(k+1)tn-2ti}-\beta^{(k+1)tn-2ti}$$
$$-\left(-1\right)^{ikt}\left(\alpha^{(k+1)tn-2(k+1)ti}-\beta^{(k+1)tn-2(k+1)ti}\right)\Big),$$

which, by Lemma 2.1, equals

$$\frac{n^{\underline{m}}}{\alpha-\beta}\Big(\alpha^{(k+1)tn}(-\alpha^{-2t})^{m}(1-\alpha^{-2t})^{n-m}-\beta^{(k+1)tn}(-\beta^{-2t})^{m}(1-\beta^{-2t})^{n-m}$$
$$+\beta^{(k+1)tn}\left(-1\right)^{(tk+1)m}\beta^{-2tm(k+1)}\left(1+\left(-1\right)^{tk+1}\beta^{-2t(k+1)}\right)^{n-m}$$
$$-\alpha^{(k+1)tn}\left(-1\right)^{(tk+1)m}\alpha^{-2tm(k+1)}\left(1+\left(-1\right)^{tk+1}\alpha^{-2t(k+1)}\right)^{n-m}\Big)$$
$$=\frac{n^{\underline{m}}}{\alpha-\beta}\left(-1\right)^{m}\left(\alpha^{ktn-tm}\left(\alpha^{t}-\left(-1\right)^{t}\beta^{t}\right)^{n-m}-\beta^{ktn-tm}\left(\beta^{t}-\left(-1\right)^{t}\alpha^{t}\right)^{n-m}\right)$$
$$+\frac{n^{\underline{m}}}{\alpha-\beta}\left(-1\right)^{m(1-t)}\Big(\alpha^{tm(k+1)}\left(\left(-1\right)^{t+1}\alpha^{t(k+1)}+\beta^{t(k+1)}\right)^{n-m}$$
$$-\beta^{tm(k+1)}\left(\alpha^{t(k+1)}+\left(-1\right)^{t+1}\beta^{t(k+1)}\right)^{n-m}\Big).$$

Thus for even t, we write

$$\sum_{i=0}^{n}\binom{n}{i}\binom{i}{m}\left(-1\right)^{i}U_{kti}V_{(k+1)tn-(k+2)ti}$$
$$=\frac{n^{\underline{m}}}{\alpha-\beta}\left(-1\right)^{m}\left(\alpha^{t}-\beta^{t}\right)^{n-m}\left(\alpha^{t(kn-m)}-\left(-1\right)^{n-m}\beta^{t(kn-m)}\right)$$
$$+\frac{n^{\underline{m}}}{\alpha-\beta}\left(-1\right)^{m}\left(\left(-1\right)^{n-m}\alpha^{tm(k+1)}-\beta^{tm(k+1)}\right)\left(\alpha^{t(k+1)}-\beta^{t(k+1)}\right)^{n-m}.$$

For $n\equiv m\ (mod 2)$, we obtain

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}\left(-1\right)^{i}U_{kti}V_{(k+1)tn-(k+2)ti}$$
$$=\quad\left(-1\right)^{m}n^{\underline{m}}\Delta^{(n-m)/2}\left(U_{t}^{n-m}U_{t(kn-m)}+U_{t(k+1)}^{n-m}U_{tm(k+1)}\right).$$

The rest of the claims could be similarly proven. □

Theorem 2.5. *For even t,*

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}\left(-1\right)^{i}V_{2ti}$$
$$=\quad\left(-1\right)^{m}n^{\underline{m}}\begin{cases}\Delta^{(n-m)/2}U_{t}^{n-m}V_{t(n+m)}, & \text{if } n\equiv m\ (mod 2),\\ -\Delta^{(n-m+1)/2}U_{t}^{n-m}U_{t(n+m)}, & \text{if } n\equiv m+1\ (mod 2),\end{cases}$$

where m is nonnegative integer.

Proof. Consider

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}V_{2ti}=\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}\left(\left(-\alpha^{2t}\right)^{i}+\left(-\beta^{2t}\right)^{i}\right),$$

which, by Lemma 2.1 and even t, equals

$$\begin{aligned}&n^{\underline{m}}\left(-\alpha^{2t}\right)^{m}\left(1-\alpha^{2t}\right)^{n-m}+n^{\underline{m}}\left(-\beta^{2t}\right)^{m}\left(1-\beta^{2t}\right)^{n-m}\\&=n^{\underline{m}}(-1)^{m}\left(\alpha^{t(m+n)}\left(\beta^{t}-\alpha^{t}\right)^{n-m}+\beta^{t(m+n)}\left(\alpha^{t}-\beta^{t}\right)^{n-m}\right)\\&=n^{\underline{m}}(-1)^{m}\Delta^{(n-m)/2}U_{t}^{n-m}\left((-1)^{n-m}\alpha^{t(m+n)}+\beta^{t(m+n)}\right).\end{aligned}$$

Here if m and n are the same parity, then we obtain

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}V_{2ti}=n^{\underline{m}}(-1)^{m}\Delta^{(n-m)/2}U_{t}^{n-m}V_{t(m+n)}.$$

If m and n are the different parity, the claim is easily seen. □

Similar to the proof method of Theorem just above, we have the following identities without proof.

Theorem 2.6. *For any integer t and nonnegative integer m,*

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{ti}V_{2ti}&=(-1)^{tn}n^{\underline{m}}V_{t}^{n-m}V_{t(n+m)}\\\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{ti}U_{2ti}&=(-1)^{tn}n^{\underline{m}}V_{t}^{n-m}U_{t(n+m)}.\end{aligned}$$

Theorem 2.7. *For any integer t and nonnegative integer m,*

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}V_{t}^{i}U_{ti}&=n^{\underline{m}}V_{t}^{m}U_{t(2n-m)},\\\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}V_{t}^{i}V_{ti}&=n^{\underline{m}}V_{t}^{m}V_{t(2n-m)},\\\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}V_{t}^{i}U_{t(n-i)}&=(-1)^{n+1}n^{\underline{m}}V_{t}^{m}U_{t(n-m)},\\\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}(-1)^{i}V_{t}^{i}V_{t(n-i)}&=(-1)^{n}n^{\underline{m}}V_{t}^{m}V_{t(n-m)}.\end{aligned}$$

Proof. Consider the first claim

$$\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}V_{t}^{i}U_{ti}=\frac{1}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}}V_{t}^{i}\left(\alpha^{ti}-\beta^{ti}\right),$$

which, by Lemma 2.1, equals

$$\begin{aligned}&\frac{1}{\alpha-\beta}\left(n^{\underline{m}}V_{t}^{m}\alpha^{tm}\left(1+V_{t}\alpha^{t}\right)^{n-m}-n^{\underline{m}}V_{t}^{m}\beta^{tm}\left(1+V_{t}\beta^{t}\right)^{n-m}\right)\\&=\frac{n^{\underline{m}}}{\alpha-\beta}\left(V_{t}^{m}\alpha^{tm}\alpha^{2t(n-m)}-n^{\underline{m}}V_{t}^{m}\beta^{tm}\beta^{2t(n-m)}\right)=n^{\underline{m}}V_{t}^{m}U_{t(2n-m)},\end{aligned}$$

as claimed. The other claims are similarly obtained. □

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