ON FIBONOMIAL SUMS IDENTITIES WITH SPECIAL SIGN FUNCTIONS: ANALYTICALLY q-CALCULUS APPROACH

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Abstract. Recently Marques and Trojovsky [On some new identities for the Fibonomial coefficients, Math. Slovaca 64 (2014)] presented interesting two sum identities including the Fibonomial coefficients and Fibonacci numbers. These sums are *unusual* as they include a rare sign function and their upper bounds are odd. In this paper, we give generalizations of these sums including the Gaussian q-binomial coefficients. We also derive analogue q-binomial sums whose upper bounds are even. Finally we give q-binomial sums formulæ whose weighted functions are different from the earlier ones. To prove the claimed results, we analytically use q-calculus.

1. Introduction

For $n > 1$, define the second order linear sequences $\{U_n\}$ and $\{V_n\}$ by

$$U_n = pU_{n-1} + U_{n-2}\ ,\ \ U_0 = 0\ ,\ \ U_1 = 1,$$
$$V_n = pV_{n-1} + V_{n-2}\ ,\ \ V_0 = 2\ ,\ \ V_1 = p.$$

Falcon and Plaza named the previous sequences as k-Fibonacci numbers and k-Lucas numbers, see [2, 3].

For $n \geq k \geq 1$, define the generalized Fibonomial coefficients by

$$\left\{ {n \atop k} \right\}_U := \frac{U_1 U_2 \dots U_n}{(U_1 U_2 \dots U_k)(U_1 U_2 \dots U_{n-k})}$$

with $\left\{ {n \atop 0} \right\}_U = \left\{ {n \atop n} \right\}_U = 1$. When $p = 1$, we obtain the usual Fibonomial coefficients, denoted by $\left\{ {n \atop k} \right\}_F$. For more details about the Fibonomial and generalized Fibonomial coefficients, see [4, 6, 20].

The Binet forms are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n\ ,$$

where $\alpha, \beta = \left(p \pm \sqrt{p^2+4}\right)/2$.

Throughout this paper we will use the following notations: the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$ and the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

The link between the generalized Fibonomial and Gaussian q-binomial coefficients is

$$\left\{ {n \atop k} \right\}_U = \alpha^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q \quad \text{with} \quad q = -\alpha^{-2}.$$

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By taking $q=\beta/\alpha$, the Binet formulæ are reduced to the following forms:

$$U_n=\alpha^{n-1}\frac{1-q^n}{1-q} \quad \text{and} \quad V_n=\alpha^n(1+q^n),$$

where $\mathbf{i}=\sqrt{-1}=\alpha\sqrt{q}$. For later use note that q-form of the coefficient p in the recurrence relations of $\{U_n\}$ and $\{V_n\}$ is $(1+q)(-q)^{-1/2}$.

The Fibonomial coefficients surprisingly appear in several places in the literature (for more details, we refer to [1, 7, 8, 9, 10]). Nowadays interesting sums including the Fibonomial coefficients have been introduced and computed by several authors (see [11, 12, 13, 14, 15, 16, 17, 19, 21]).

Marques and Trojovsky [16] gave some sums formulæ including Fibonomial coefficients, Fibonacci and Lucas numbers. For example, for positive integers m and n, they showed that

$$\sum_{j=0}^{4m+2}(-1)^{\frac{j(j-1)}{2}}\left\{\begin{matrix}4m\\ j\end{matrix}\right\}_F F_{n+4m-j}=\frac{1}{2}F_{2m+n}\sum_{j=0}^{4m}(-1)^{\frac{j(j-1)}{2}}\left\{\begin{matrix}4m\\ j\end{matrix}\right\}_F L_{2m-j}.$$

Kılıç and Prodinger [13] gave a systematic approach to compute certain sums of squares of Fibonomial coefficients with finite products of generalized Fibonacci and Lucas numbers as coefficients. For example, if n is nonnegative integer, then they proved the following Gaussian q-binomial sums identity

$$\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\ k\end{bmatrix}_q^2(-1)^k q^{k^2-2kn-3k}(1-q^{2k})^2$$
$$=2(-1)^{n+1}q^{-n^2-2n-2}\frac{(1+q)(1-q^{2n+1})(1-q^{2n+1})}{(1+q^{2n})}\begin{bmatrix}2n+1\\ n\end{bmatrix}_{q^2}.$$

Recently, Marques and Trojovsky [17] derived various sums identities including the Fibonomial coefficients. Here we recall their main results for the reader's convenient: For any nonnegative integers l and n,

$$\sum_{j=0}^{4l+3}sgn(2l+1-j)\left\{\begin{matrix}4l+3\\ j\end{matrix}\right\}_F F_{n-j}=\frac{F_{2l}}{F_{4l+3}}\left\{\begin{matrix}4l+3\\ 2l+1\end{matrix}\right\}_F F_{n-4l-3} \tag{1.1}$$

and

$$\sum_{j=0}^{4l+1}sgn(2l-j)\left\{\begin{matrix}4l+1\\ j\end{matrix}\right\}_F F_{n-j}=-\frac{F_{2l-1}}{F_{4l+1}}\left\{\begin{matrix}4l+1\\ 2l\end{matrix}\right\}_F F_{n-4l-1}, \tag{1.2}$$

where $sgn(x)$ denotes the sign function of x, defined by $sgn(0)=0$ and $sgn(x)=x/|x|$, for $x\neq 0$. The authors mainly used the properties of the Fibonacci numbers and the induction method to prove their claims.

It would be valuable to note the following two facts to mention our purpose in this paper:

- The first one is that the sums identities given as the main results of [17] are *only valid* for the Fibonomial coefficients and the Fibonacci numbers that are not valid for the generalized Fibonomial coefficients and the generalized Fibonacci numbers.
- The second fact is that the upper bounds of these sums are only valid for odd integers of the forms $4l+3$ and $4l+1$.

In the present paper, we have two main purposes. The first one is to obtain generalizations of the results of [17] including the Gaussian q-binomial coefficients. By the way, we would obtain much more generalizations of the results of [17] because all the identities we will derive hold for general q, and results about the general Fibonomial coefficients, the generalized Fibonacci and Lucas numbers come out as corollaries for the special choice of q.

Our second purpose is to obtain complementary sums identities whose upper bounds are even. To prove the claimed results, we mainly and analytically use q-calculus. Finally we will give some analogue Gaussian q-binomial sums whose weight functions will be different the previous sums.

2. Odd Upper Bound Case

Now we mainly present two Gaussian q-binomial sums identities to give generalizations of the identities (1.1) and (1.2). Before this, we give three auxiliary lemmas and then give our first main result.

Lemma 2.1. *For nonnegative integer n, any nonzero constant c and any function f,*
(i)

$$\sum_{j=0}^{n} c^{[2\nmid j]} f(j) = \sum_{j=0}^{n} \left(\frac{c+1-(c-1)(-1)^j}{2} \right) f(j),$$

(ii)

$$\sum_{j=0}^{n} c^{[2|j]} f(j) = \sum_{j=0}^{n} \left(\frac{c+1+(c-1)(-1)^j}{2} \right) f(j),$$

where [ ] stands for the Iverson notation (see [5]).

Proof. For any integer j, since $[2|j] = \left(1+(-1)^j\right)/2$ and so

$$\begin{aligned} c^{[2\nmid j]} f(j) &= [2|j]\, f(j) + c\,[2\nmid j]\, f(j) = \left([2|j] + c\,(1-[2|j])\right) f(j) \\ &= \left(c - (c-1)\,[2|j]\right) f(j) = \left(c+1-(c-1)\,(-1)^j\right) \frac{f(j)}{2}, \end{aligned}$$

the first claim (i) follows. The latter is proven similar to the first one. □

Lemma 2.2. *For nonnegative integers n and k,*
(i)

$$\begin{aligned} &\sum_{j=0}^{2k} \begin{bmatrix} 4k+3 \\ j \end{bmatrix}_q \left(1-q^{n-j}\right) q^{-\frac{1}{2}j(4k-j+2)} (-1)^{\frac{1}{2}j(j-2)} \left(\frac{z+1-(z-1)(-1)^j}{2} \right) \\ &- \begin{bmatrix} 4k+3 \\ j+1 \end{bmatrix}_q \left(1-q^{j-4k+n-2}\right) q^{-\frac{1}{2}j(4k-j+2)} (-1)^{\frac{1}{2}j(j-2)} \left(\frac{z+1-(z-1)(-1)^j}{2} \right) \\ &= \left(1-q^{n-4k-3}\right) q^{1-2k} (1+q) \sum_{j=0}^{k-1} \begin{bmatrix} 4k+3 \\ 2j+1 \end{bmatrix}_q q^{2j(j-2k+1)} \frac{1-q^{4k-4j}}{1-q^{2j+2}} \\ &- q\left(1-q^{n-4k-3}\right) \sum_{j=0}^{k} \begin{bmatrix} 4k+3 \\ 2j \end{bmatrix}_q q^{2j(j-2k)} \frac{1-q^{4k+2-4j}}{1-q^{2j+1}}, \end{aligned}$$

(ii)

$$\begin{aligned} &\sum_{j=0}^{2k-1} q^{\frac{1}{2}j(j-4k)} (-1)^{j^2/2} \begin{bmatrix} 4k+1 \\ j \end{bmatrix}_q \left(1-q^{n-j}\right) \frac{\left(z+1+(z-1)(-1)^j\right)}{2} \\ &- \sum_{j=0}^{2k-1} q^{\frac{1}{2}j(j-4k)} (-1)^{j^2/2} \begin{bmatrix} 4k+1 \\ j+1 \end{bmatrix}_q \left(1-q^{n+j-4k}\right) \frac{\left(z+1+(z-1)(-1)^j\right)}{2} \end{aligned}$$

$$= \mathbf{i}q^{-\frac{1}{2}}q\left(1-q^{n-4k-1}\right)(1+q)\sum_{j=0}^{k-1}\begin{bmatrix}4k+1\\2j\end{bmatrix}_q q^{2j(j-2k+1)}\frac{1-q^{4k-4j}}{1-q^{2j+1}}$$
$$-\mathbf{i}q^{-\frac{1}{2}}q^{3-2k}\left(1-q^{n-4k-1}\right)\sum_{j=0}^{k-1}\begin{bmatrix}4k+1\\2j+1\end{bmatrix}_q q^{2j(j-2k+2)}\frac{1-q^{4k-4j-2}}{1-q^{2j+2}},$$

where $z=-\mathbf{i}\left(1+q\right)q^{-1/2}$.

Proof. We only give a proof for the first claim. The latter is similar. To prove the claim (i), note that for any two functions $F(j)$ and $G(j)$ of j, the following equality holds

$$\sum_{j=0}^{2n}\left[F(j)-G(j)\right]=\sum_{j=0}^{n}\left[F(2j)-G(2j)\right]+\sum_{j=0}^{n-1}\left[F(2j+1)-G(2j+1)\right]. \tag{2.1}$$

By this fact we write the LHS of the claim (i) as

$$\sum_{j=0}^{k}q^{-j(4k-2j+2)}\begin{bmatrix}4k+3\\2j\end{bmatrix}_q\left(1-q^{n-2j}\right)-\sum_{j=0}^{k}q^{-j(4k-2j+2)}\begin{bmatrix}4k+3\\2j+1\end{bmatrix}_q\left(1-q^{2j-4k+n-2}\right)$$
$$-(1+q)\,q^{-2k-1}\sum_{j=0}^{k-1}q^{2j(j-2k)}\left(\begin{bmatrix}4k+3\\2j+1\end{bmatrix}_q\left(1-q^{n-(2j+1)}\right)-\begin{bmatrix}4k+3\\2j+2\end{bmatrix}_q\left(1-q^{2j-4k+n-1}\right)\right)$$

which, by using the well-known identity

$$\begin{bmatrix}n+1\\k+1\end{bmatrix}_q=\frac{1-q^{n-k+1}}{1-q^{k+1}}\begin{bmatrix}n+1\\k\end{bmatrix}_q,$$

equals

$$q^{-(4k+2)}(q^n-q^{4k+3})\sum_{j=0}^{k}\begin{bmatrix}4k+3\\2j\end{bmatrix}_q q^{2j(j-2k)}\frac{1-q^{4k+2-4j}}{1-q^{2j+1}}$$
$$-q^{-6k-2}\left(q^n-q^{4k+3}\right)(1+q)\sum_{j=0}^{k-1}\begin{bmatrix}4k+3\\2j+1\end{bmatrix}_q q^{2j(j-2k+1)}\frac{1-q^{4k-4j}}{1-q^{2j+2}},$$

as claimed. □

Lemma 2.3. *For positive integer m and any integer c,*

(i)

$$\sum_{j=0}^{m}\begin{bmatrix}4k+2c+3\\2j\end{bmatrix}_q q^{2j(j-2k-c)}\frac{1-q^{4k-4j+2c+2}}{1-q^{2j+1}}=q^{-2m(2k+c-m)}\begin{bmatrix}4k+2c+2\\2m+1\end{bmatrix}_q,$$

(ii)

$$\sum_{j=0}^{m}\begin{bmatrix}4k+2c+3\\2j+1\end{bmatrix}_q q^{2j(j-2k-c+1)}\frac{1-q^{4k-4j+2c}}{1-q^{2j+2}}=-q^{4k+2c}+q^{-2m(2k+c-m-1)}\begin{bmatrix}4k+2c+2\\2m+2\end{bmatrix}_q.$$

Proof. We give a proof for the first claim (i). The other is similar. Denote the sums

$$\sum_{j=0}^{m}\begin{bmatrix}4k+2c+3\\2j\end{bmatrix}_q q^{2j(j-2k-c)}\frac{1-q^{4k-4j+2c+2}}{1-q^{2j+1}}$$

and

$$\sum_{j=0}^{m}\begin{bmatrix}4k+2c+3\\2j+1\end{bmatrix}_q q^{2j(j-2k-c+1)}\frac{1-q^{4k-4j+2c}}{1-q^{2j+2}}$$

by $S_1(m,k,c)$ and $S_2(m,k,c)$, or briefly S_1 and S_2, resp.

For $m\geq 2k+c+1$, the sum S_1 is a whole sum and equals 0. To see this, consider

$$\begin{aligned}S_1&=\sum_{j\geq 0}\begin{bmatrix}4k+2c+3\\2j\end{bmatrix}_q q^{2j(j-2k-c)}\frac{1-q^{4k-4j+2c+2}}{1-q^{2j+1}}\\&=\sum_{j\geq 0}\begin{bmatrix}4k+2c+3\\j\end{bmatrix}_q q^{j\left(\frac{1}{2}j-2k-c\right)}\frac{1-q^{4k-2j+2c+2}}{1-q^{j+1}}\left(\frac{1+(-1)^j}{2}\right)\\&=\frac{1}{1-q^{4k+2c+4}}\sum_{j\geq 0}\begin{bmatrix}4k+2c+4\\j+1\end{bmatrix}_q q^{j\left(\frac{1}{2}j-2k-c\right)}\left(1-q^{4k-2j+2c+2}\right)\left(\frac{1+(-1)^j}{2}\right)\\&=\frac{1}{1-q^{4k+2c+4}}\sum_{j\geq 1}\begin{bmatrix}4k+2c+4\\j\end{bmatrix}_q q^{-\frac{1}{2}(j-1)(4k+2c-j+1)}\left(1-q^{4k-2j+2c+4}\right)\left(\frac{1-(-1)^j}{2}\right).\end{aligned}$$

By taking $4k+2c+4-j$ instead of j, we write that

$$\begin{aligned}S_1&=\frac{1}{1-q^{4k+2c+4}}\sum_{j\geq 1}\begin{bmatrix}4k+2c+4\\j\end{bmatrix}_q q^{-\frac{1}{2}(j-1)(4k+2c-j+1)}\left(1-q^{4k-2j+2c+4}\right)\left(\frac{1-(-1)^j}{2}\right)\\&=\frac{1}{1-q^{4k+2c+4}}\sum_{j\geq 0}\begin{bmatrix}4k+2c+4\\j\end{bmatrix}_q q^{-\frac{1}{2}(j-3)(2c-j+4k+3)}\left(1-q^{2j-2c-4k-4}\right)\left(\frac{1-(-1)^j}{2}\right)\\&=-\frac{1}{1-q^{4k+2c+4}}\sum_{j\geq 0}\begin{bmatrix}4k+2c+4\\j\end{bmatrix}_q q^{-\frac{1}{2}(j-1)(4k+2c-j+1)}\left(1-q^{4k-2j+2c+4}\right)\left(\frac{1-(-1)^j}{2}\right)\\&=-S_1.\end{aligned}$$

Since $S_1=-S_1$, we get $S_1=0$.

The similar direction could be derived for S_2 that is for $m\geq 2k+c$ the sum S_2 is a whole sum and equals 0.

Define

$$G(k,j):=q^{2j^2-4kj-2cj}\begin{bmatrix}4k+2c+2\\2j+1\end{bmatrix}_q.$$

Then we have

$$G(k,j)=S_1,$$

which follows from

$$\begin{aligned}&G(k,j)-G(k,j-1)\\&=q^{-2j(c-j+2k)}\begin{bmatrix}4k+2c+2\\2j+1\end{bmatrix}_q-q^{-2(j-1)(c-j+2k+1)}\begin{bmatrix}4k+2c+2\\2j-1\end{bmatrix}_q\\&=q^{-2j(c-j+2k)}\frac{(q;q)_{4k+2c+2}}{(q;q)_{2j+1}(q;q)_{4k+2c+1-2j}}-q^{-2(j-1)(2k-j+1)}\frac{(q;q)_{4k+2c+2}}{(q;q)_{2j-1}(q;q)_{4k+2c+3-2j}}\\&=\frac{(q;q)_{4k+2c+2}}{(q;q)_{2j-1}(q;q)_{4k+2c+1-2j}}\left[\frac{q^{-2j(c-j+2k)}}{(1-q^{2j})(1-q^{2j+1})}-\frac{q^{-2(j-1)(2k-j+1)}}{(1-q^{4k+2c+3-2j})(1-q^{4k+2c+2-2j})}\right]\end{aligned}$$

$$= \frac{(q;q)_{4k+2}}{(q;q)_{2j-1}\,(q;q)_{4k+2c+1-2j}} \frac{q^{2j(j-2k-c)}\left(1-q^{4k+2c+3}\right)\left(1-q^{4k+2c+2-4j}\right)}{(1-q^{2j+1})\,(1-q^{2j})\,(1-q^{4k+3-2j})\,(1-q^{4k+2-2j})}$$
$$= q^{2j(j-2k)}\frac{1-q^{4k+2c+2-4j}}{1-q^{2j+1}}\begin{bmatrix}4k+2c+3\\2j\end{bmatrix}_q,$$

as claimed. The second claim could be similarly proven. □

Theorem 2.1. *For nonnegative integers n and k,*

(i)

$$\sum_{j=0}^{4k+3} sgn(2k+1-j)\begin{bmatrix}4k+3\\j\end{bmatrix}_q (-1)^{\frac{1}{2}j(j-2)}\,q^{-\frac{1}{2}j(4k-j+2)}(1-q^{n-j})z^{[2\nmid j]}$$
$$= q^{2-2k^2}\frac{1-q^{2k}}{1-q^{4k+3}}\begin{bmatrix}4k+3\\2k+1\end{bmatrix}_q(1-q^{n-4k-3}),$$

(ii)

$$\sum_{j=0}^{4k+1} sgn(2k-j)\begin{bmatrix}4k+1\\j\end{bmatrix}_q (-1)^{\frac{1}{2}j^2}\,q^{\frac{1}{2}j(j-4k)}(1-q^{n-j})z^{[2|j]}$$
$$= -(-q)^{\frac{4k+3-4k^2}{2}}\frac{1-q^{2k-1}}{1-q^{4k+1}}\begin{bmatrix}4k+1\\2k\end{bmatrix}_q(1-q^{n-4k-1}),$$

where $[\ ]$ stands for the Iverson notation and $z=(1+q)\,(-q)^{-1/2}$.

Proof. By the signum function, we rewrite the first claim will be proven in the form

$$\sum_{j=0}^{2k}\begin{bmatrix}4k+3\\j\end{bmatrix}_q\left(1-q^{n-j}\right)(-1)^{\frac{1}{2}j(j-2)}\,q^{-\frac{1}{2}j(4k-j+2)}\left(\frac{z+1-(z-1)(-1)^j}{2}\right)$$
$$-\sum_{j=2k+2}^{4k+3}\begin{bmatrix}4k+3\\j\end{bmatrix}_q\left(1-q^{n-j}\right)(-1)^{\frac{1}{2}j(j-2)}\,q^{-\frac{1}{2}j(4k-j+2)}\left(\frac{z+1-(z-1)(-1)^j}{2}\right)$$
$$= q^{2-2k^2}\frac{1-q^{2k}}{1-q^{4k+3}}\begin{bmatrix}4k+3\\2k+1\end{bmatrix}_q(1-q^{n-4k-3}). \tag{2.2}$$

The second sum on the LHS of the equation just above, by taking $4k+3-j$ instead of j, equals

$$\sum_{j=0}^{2k+1}\begin{bmatrix}4k+3\\j\end{bmatrix}_q\left(1-q^{j-4k+n-3}\right)(-1)^{\frac{1}{2}(j^2-1)}\,q^{-\frac{1}{2}(j-1)(4k-j+3)}\left(\frac{z+1+(z-1)(-1)^j}{2}\right),$$

which, by an arrangement, equals

$$q^{2k+1}\left(q^{n-4k-3}-1\right)(1+q)$$
$$+\sum_{j=1}^{2k+1}\begin{bmatrix}4k+3\\j\end{bmatrix}_q\left(1-q^{j-4k+n-3}\right)(-1)^{\frac{1}{2}(j^2-1)}\,q^{-\frac{1}{2}(j-1)(4k-j+3)}\left(\frac{z+1+(z-1)(-1)^j}{2}\right),$$

which, by taking $j+1$ instead of j in the sum, equals

$$q^{2k+1}\left(q^{n-4k-3}-1\right)(1+q)$$
$$+\sum_{j=0}^{2k}\begin{bmatrix}4k+3\\j+1\end{bmatrix}_q\left(1-q^{j-4k+n-2}\right)(-1)^{\frac{1}{2}j(j+2)}\,q^{-\frac{1}{2}j(4k-j+2)}\left(\frac{z+1-(z-1)(-1)^j}{2}\right). \tag{2.3}$$

By (2.3), the claimed equality (2.2) takes the form

$$\sum_{j=0}^{2k} \begin{bmatrix} 4k+3 \\ j \end{bmatrix}_q \left(1-q^{n-j}\right) (-1)^{\frac{1}{2}j(j-2)} q^{-\frac{1}{2}j(4k-j+2)} \left(\frac{z+1-(z-1)(-1)^j}{2}\right)$$
$$-\sum_{j=0}^{2k} \begin{bmatrix} 4k+3 \\ j+1 \end{bmatrix}_q \left(1-q^{j-4k+n-2}\right) (-1)^{\frac{1}{2}j(j+2)} q^{-\frac{1}{2}j(4k-j+2)} \left(\frac{z+1-(z-1)(-1)^j}{2}\right)$$
$$+q^{2k+1}\left(1-q^{n-4k-3}\right)(1+q)$$
$$= q^{2-2k^2} \frac{1-q^{2k}}{1-q^{4k+3}} \begin{bmatrix} 4k+3 \\ 2k+1 \end{bmatrix}_q (1-q^{n-4k-3}),$$

which, by Lemma 2.2(i), equals

$$-q\left(1-q^{n-4k-3}\right) \sum_{j=0}^{k} \begin{bmatrix} 4k+3 \\ 2j \end{bmatrix}_q q^{2j(j-2k)} \frac{1-q^{4k+2-4j}}{1-q^{2j+1}}$$
$$+\left(1-q^{n-4k-3}\right)(1+q)\left(q^{1-2k}\sum_{j=0}^{k-1} \begin{bmatrix} 4k+3 \\ 2j+1 \end{bmatrix}_q q^{2j(j-2k+1)} \frac{1-q^{4k-4j}}{1-q^{2j+2}} - q^{2k+1}\right)$$
$$= q^{2-2k^2} \frac{1-q^{2k}}{1-q^{4k+3}} \begin{bmatrix} 4k+3 \\ 2k+1 \end{bmatrix}_q (1-q^{n-4k-3}).$$

By Lemma 2.3, we rewrite the LHS of the equation just above as

$$-\left(1-q^{n-4k-3}\right)\left(q^{1-2k^2} \begin{bmatrix} 4k+2 \\ 2k+1 \end{bmatrix}_q + q^{2k+1}(1+q) - (1+q)\, q^{1-2k^2} \begin{bmatrix} 4k+2 \\ 2k \end{bmatrix}_q\right)$$
$$+q^{2k+1}\left(1-q^{n-4k-3}\right)(1+q)$$
$$= -q^{1-2k^2}\left(1-q^{n-4k-3}\right)\left(\begin{bmatrix} 4k+2 \\ 2k+1 \end{bmatrix}_q - (1+q) \begin{bmatrix} 4k+2 \\ 2k \end{bmatrix}_q\right)$$
$$= q^{2-2k^2} \frac{1-q^{2k}}{1-q^{4k+3}} \begin{bmatrix} 4k+3 \\ 2k+1 \end{bmatrix}_q (1-q^{n-4k-3}),$$

as claimed. □

Corollary 1. *For nonnegative integers n and k,*

(i)

$$\sum_{j=0}^{4k+3} sgn(2k+1-j) p^{[2\nmid j]} \begin{Bmatrix} 4k+3 \\ j \end{Bmatrix}_U U_{n-j} = \frac{U_{2k}}{U_{4k+3}} \begin{Bmatrix} 4k+3 \\ 2k+1 \end{Bmatrix}_U U_{n-4k-3},$$

(ii)

$$\sum_{j=0}^{4k+1} sgn(2k-j) p^{[2|j]} \begin{Bmatrix} 4k+1 \\ j \end{Bmatrix}_U U_{n-j} = -\frac{U_{2k-1}}{U_{4k+1}} \begin{Bmatrix} 4k+1 \\ 2k \end{Bmatrix}_U U_{n-4k-1},$$

where [ ] stands for the Iverson notation.

Following the proof of Theorem 2.1 and after some simple arrangements, we derive that

$$\sum_{j=0}^{4k+3} sgn(2k+1-j) \begin{bmatrix} 4k+3 \\ j \end{bmatrix}_q (-1)^{\frac{1}{2}j(j-2)} q^{-\frac{1}{2}j(4k-j+2)} (1-q^{n-j}) z^{[2\nmid j]}$$

$$= q^{-(2k+2)}(1+q) + \sum_{j=0}^{k-1} \begin{bmatrix} 4k+3 \\ 2j \end{bmatrix}_q \frac{q^{-2(j+1)(2k-j+1)}}{(1-q^{2j+1})(1-q^{2j+2})}$$
$$\times \left(\left(1-q^{4k+3-2j}\right)\left(1-q^{4k-4j}\right) q^{2j-2k}(1+q) - \left(1-q^{4k+2-4j}\right)\left(1-q^{2j+2}\right) \right),$$

which equals

$$q^{2-2k^2} \frac{1-q^{2k}}{1-q^{4k+3}} \begin{bmatrix} 4k+3 \\ 2k+1 \end{bmatrix}_q (1-q^{n-4k-3}).$$

Thus we derive an equivalent sum identity to the statement of Theorem 2.1 (i):

$$\sum_{j=0}^{k-1} \begin{bmatrix} 4k+3 \\ 2j \end{bmatrix}_q q^{-2(j+1)(2k-j+1)} \frac{1}{(1-q^{2j+1})(1-q^{2j+2})}$$
$$\times \left[\left(1-q^{4k+3-2j}\right)\left(1-q^{4k-4j}\right) q^{2j-2k}(1+q) - \left(1-q^{4k+2-4j}\right)\left(1-q^{2j+2}\right) \right]$$
$$= q^{-(2k+2)}(1+q) + q^{2-2k^2} \frac{1-q^{2k}}{1-q^{4k+3}} \begin{bmatrix} 4k+3 \\ 2k+1 \end{bmatrix}_q (1-q^{n-4k-3}).$$

By Lemma 2.3, now we present a generalization of Theorem 2.1 with two additional parameters by the following Theorem 2.2 without proof.

Theorem 2.2. *For nonnegative integer m, any integer r and any real c,*

(i)

$$\sum_{j=0}^{m} \begin{bmatrix} 4k+3 \\ 2j \end{bmatrix}_q q^{-2(j+1)(2k-j+1)} \frac{1}{(1-q^{2j+1})(1-q^{2j+2})}$$
$$\times \left[\left(1-q^{4k+3-2j}\right)\left(1-q^{4k-4j}\right) q^{2j-2k}(1+cq^r) - \left(1-q^{4k+2-4j}\right)\left(1-q^{2j+2}\right) \right]$$
$$= q^{-2-6k-4km+2m^2} \frac{q^{2m}(1-q^r) - q^{2k}\left(1+q^{2k+1}\right) + q^{2k+2m+2}\left(1+q^{2k+r-2m-1}\right)}{1-q^{4k+3}} \begin{bmatrix} 4k+3 \\ 2m+2 \end{bmatrix}_q$$
$$- q^{-2-2k}(1+cq^r),$$

(ii)

$$\sum_{j=0}^{m} \begin{bmatrix} 4k+1 \\ 2j \end{bmatrix}_q q^{2j(j-2k+1)} \frac{1}{(1-q^{2j+1})(1-q^{2j+2})}$$
$$\times \left[\left(1-q^{2j+2}\right)\left(1-q^{4k-4j}\right)(1+cq^r) - q^{2j-2k+2}\left(1-q^{4k+1-2j}\right)\left(1-q^{4k-2-4j}\right) \right]$$
$$= q^{2k} - q^{-2k+2m-4km+2m^2} \frac{q^{2m+2}\left(1+q^{2k}\right) - q^{2k}(1+q^{2k+1}) - cq^{2k+r}(1-q^{2m+2})}{1-q^{2m+2}} \begin{bmatrix} 4k \\ 2m+1 \end{bmatrix}_q.$$

When $m = k-1$, $c = r = 1$, the result of Theorem 2.2 gives us the result of Theorem 2.1.

3. Even Upper Bound Case

In the previous Section, we gave two sums identities whose upper bounds are odd numbers. Now we present two sums identities whose upper bounds will be even numbers as analogue of Theorem 2.1:

Theorem 3.1. *For nonnegative integer k,*

(i)

$$\sum_{j=0}^{4k} sgn(2k-j) \begin{bmatrix} 4k \\ j \end{bmatrix}_q \left(1-q^{n-j}\right)(-1)^{\frac{1}{2}j(j-2)} q^{-\frac{1}{2}j(4k-j)} \left(\frac{z+1-(z-1)(-1)^j}{2} \right)$$

$$= -q^{-2k^2-2k+n-1} \begin{bmatrix} 4k+1 \\ 2k-1 \end{bmatrix}_q \frac{\left(1-q^{2k+1}\right)\left(1-q^{2k+2}\right)\left(1-q^{2k+1}\right)}{\left(1-q^{4k-1}\right)\left(1-q^{4k+1}\right)}$$

(ii)

$$\sum_{j=0}^{4k+2} sgn\left(2k+1-j\right) \begin{bmatrix} 4k+2 \\ j \end{bmatrix}_q \left(1-q^{n-j}\right)(-1)^{\frac{1}{2}j(j-2)} q^{-\frac{1}{2}j(4k-j+2)} \left(\frac{z+1-(z-1)(-1)^j}{2}\right)$$
$$= -q^{-2k^2-4k+n-1} \begin{bmatrix} 4k+2 \\ 2k \end{bmatrix}_q \frac{\left(1-q^{2k-1}\right)\left(1-q^{2k+2}\right)}{1-q^{4k+1}},$$

where z is defined as before.

Similar to Theorem 2.2, we give a general case of Theorem 3.1 with two additional parameters without proof .

Theorem 3.2. *For nonnegative integers k and m,*

(i)

$$\sum_{j=0}^{m} \begin{bmatrix} 4k \\ 2j \end{bmatrix}_q q^{2j(j-2k-1)} \frac{\left(1-q^{2j+1}\right)\left(q^{4j-4k}-1\right)-q^{2j-2k-1}(1+q)\left(1-q^{4k-2j}\right)\left(1-q^{4j+2-4k}\right)}{1-q^{2j+1}}$$
$$= -q^{-1-6k+2m-4km+2m^2}\left(-q^{2k+1}\left(1-q^{2m+1}\right)-q^{4k}\left(1+q\right)\left(1-q^{2m-4k+2}\right)\right)\frac{1-q^{4k}}{1-q^{4k-1}} \begin{bmatrix} 4k-1 \\ 2m+1 \end{bmatrix}_q,$$

(ii)

$$\sum_{j=0}^{m} \begin{bmatrix} 4k+2 \\ 2j \end{bmatrix}_q q^{2j(j-2k-2)} \frac{1}{1-q^{2j+1}}$$
$$\times \left[\left(1-q^{2j+1}\right)\left(q^{4j-4k-2}-1\right)-q^{2j-2k-2}(1+q)\left(1-q^{4k-2j+2}\right)\left(1-q^{4j-4k}\right)\right]$$
$$= -q^{-3-6k-4km+2m^2}\left(1-q^{4k+2}\right)\frac{q^{2k}\left(1-q^{2m+3}\right)+q^{4k}\left(1+q\right)\left(1-q^{2m+2-4k}\right)}{(1-q^{4k+1})(1-q^{4k-1})} \begin{bmatrix} 4k+1 \\ 2m+1 \end{bmatrix}_q.$$

We don't give proofs for Theorems 3.1 and 3.2 to don't bother readers. However, for their proof, similar to the proof of Theorem 2.1, we present related auxiliary lemmas without proof.

Lemma 3.1. *For positive integer k,*

(i)

$$\sum_{j=0}^{2k-1} \begin{bmatrix} 4k \\ j \end{bmatrix} q^{\frac{1}{2}j(j-4k-2)} (-1)^{\frac{j(j-2)}{2}} \left(q^{2j-4k}-1\right) \frac{\left(z+1-(z-1)(-1)^j\right)}{2}$$
$$= \sum_{j=0}^{k-1} \begin{bmatrix} 4k \\ 2j \end{bmatrix} q^{2j(j-2k-1)}(q^{4j-4k}-1) - q^{-2k-1}(1+q)\sum_{j=0}^{k-1} \begin{bmatrix} 4k \\ 2j+1 \end{bmatrix} q^{2j(j-2k)}(q^{4j-4k+2}-1),$$

(ii)

$$\sum_{j=0}^{2k} \begin{bmatrix} 4k+2 \\ j \end{bmatrix} q^{\frac{1}{2}j(j-4k-4)} (-1)^{\frac{j(j-2)}{2}} \left(q^{2j-4k-2}-1\right) \frac{\left(z+1-(z-1)(-1)^j\right)}{2}$$
$$= \sum_{j=0}^{k} \begin{bmatrix} 4k+2 \\ 2j \end{bmatrix} q^{2j(j-2k-2)}(q^{4j-4k-2}-1) - q^{-2k-2}(1+q)\sum_{j=0}^{k-1} \begin{bmatrix} 4k+2 \\ 2j+1 \end{bmatrix} q^{2j(j-2k-1)}(q^{4j-4k}-1).$$

Lemma 3.2. *For positive integer m and any integer c,*

(i)

$$\sum_{j=0}^{m}\begin{bmatrix}4k+2c\\2j+1\end{bmatrix}_q q^{-2j(c-j+2k)}\left(1-q^{2-2c+4j-4k}\right)=-q^{-2(m+1)(c+2k-m-1)}\left(1-q^{4k+2c}\right)\begin{bmatrix}4k+2c-2\\2m+1\end{bmatrix}_q,$$

(ii)

$$\sum_{j=0}^{m}\begin{bmatrix}4k+2c\\2j\end{bmatrix}_q q^{-2j(c-j+2k+1)}\left(1-q^{-2c+4j-4k}\right)=-q^{-2(m+1)(c+2k-m)}\left(1-q^{4k+2c}\right)\begin{bmatrix}4k+2c-2\\2m\end{bmatrix}_q.$$

Corollary 2. *For nonnegative integers n and k,*

(i)

$$\sum_{j=0}^{4k+2} sgn(2k+1-j)p^{[2\nmid j]}\begin{Bmatrix}4k+2\\j\end{Bmatrix}_U(-\alpha)^jU_{n-j}=(-1)^n\alpha^{4k-n+2}\frac{U_{2k+2}^2}{U_{4k+1}}\begin{Bmatrix}4k+2\\2k\end{Bmatrix}_U,$$

(ii)

$$\sum_{j=0}^{4k} sgn(2k-j)p^{[2\nmid j]}\begin{Bmatrix}4k\\j\end{Bmatrix}_U\alpha^jU_{n-j}=(-1)^n\alpha^{4k-n}\frac{U_{2k+1}^2U_{2k+2}}{U_{4k+1}U_{4k-1}}\begin{Bmatrix}4k+1\\2k-1\end{Bmatrix}_U,$$

where [ ] stands for the Iverson notation.

4. Additional Sums Formulæ

In this Section, we will give new sums formulæ whose weighted functions depend on z are different from the earlier sums.

(1)

$$\sum_{j=0}^{4k+3} sgn\left(2k+1-j\right)\begin{bmatrix}4k+3\\j\end{bmatrix}_q(1-q^{n-j})q^{\frac{1}{2}j(j-4k-2)}\left(-1\right)^{\frac{1}{2}j(j-2)}\left(\frac{z-1+(z+1)(-1)^j}{2}\right)$$
$$=\mathbf{i}q^{\frac{1}{2}-2k^2}\frac{1-q^{2k+3}}{1-q^{4k+3}}(1-q^{n-4k-3})\begin{bmatrix}4k+3\\2k+1\end{bmatrix}_q,$$

(2)

$$\sum_{j=0}^{4k+1} sgn(2k-j)\begin{bmatrix}4k+1\\j\end{bmatrix}_q(1-q^{n-j})q^{\frac{1}{2}j(j-4k)}\left(-1\right)^{\frac{1}{2}j^2}\left(\frac{z-1-(z+1)(-1)^j}{2}\right)$$
$$=-q^{-2k(k-1)}(1-q^{n-4k-1})\frac{1-q^{2k+2}}{1-q^{4k+1}}\begin{bmatrix}4k+1\\2k\end{bmatrix}_q,$$

(3)

$$\sum_{j=0}^{4k} sgn\left(2k-j\right)\begin{bmatrix}4k\\j\end{bmatrix}_q\left(1-q^{n-j}\right)\left(-1\right)^{\frac{1}{2}j(j-2)}q^{-\frac{1}{2}j(4k-j)}\left(\frac{z-1+(z+1)(-1)^j}{2}\right)$$
$$=-\mathbf{i}q^{-2k^2-2k+n+\frac{1}{2}}\frac{\left(1-q^{2k+1}\right)\left(1-q^{2k+2}\right)\left(1-q^{2k-2}\right)}{\left(1-q^{4k-1}\right)\left(1-q^{4k+1}\right)}\begin{bmatrix}4k+1\\2k-1\end{bmatrix}_q,$$

(4)

$$\sum_{j=0}^{4k+2} sgn\,(2k+1-j) \begin{bmatrix} 4k+2 \\ j \end{bmatrix}_q (1-q^{n-j})\,(-1)^{\frac{1}{2}j(j-2)}\, q^{-\frac{1}{2}j(4k-j+2)} \left(\frac{z-1+(z+1)(-1)^j}{2}\right)$$
$$= -\mathbf{i}q^{-2k^2-4k+n-\frac{5}{2}} \frac{\left(1-q^{2k+2}\right)\left(1-q^{2k+2}\right)}{1-q^{4k+1}} \begin{bmatrix} 4k+2 \\ 2k \end{bmatrix}_q .$$

One could derive many the generalized Fibonomial-Fibonacci-Lucas corollaries from our general results just above by choosing specific values of q.

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