Three binomial sums weighted by falling and rising factorials

Emrah KILIÇ

ABSTRACT. In the paper, we will investigate and evaluate, in closed forms, three binomial sums weighted by falling and rising factorials. We first use the relationships between the rising, falling factorials and the binomial coefficients. Then we rewrite the claimed identities in terms of generalized hypergeometric functions to prove the claimed results.

1. Introduction

Many kinds of binomial sums with certain weight functions have been considered and computed by several techniques (for their large collection, we could refer to [3]). Spivey [10] presented a new approach to evaluate combinatorial sums by using finite differences. He extended his approach to handle alternating and aerated binomial sums of the form

$$\sum_k (-1)^k \binom{n}{k} a_k, \quad \sum_k \binom{n}{2k} a_k \quad \text{and} \quad \sum_k \binom{n}{2k+1} a_k$$

for any sequence $\{a_k\}$. Meanwhile, Spivey considered the following interesting binomial sum whose weight function is the falling factorial and computed it as

$$\sum_{k=0}^{n} \binom{n}{k} k^{\underline{m}} = n^{\underline{m}} 2^{n-m} \quad \text{for } m \neq n,$$

where the falling factorial is defined as $k^{\underline{m}} = k(k-1)(k-2)\cdots(k-m+1)$. For later use, we also note that the rising factorial is defined as $k^{\overline{m}} = k(k+1)(k+2)\cdots(k+m-1)$.

Meanwhile Shapiro [8] derived the following triangle similar to Pascal's triangle with half binomial entries given by

$$B_{n,k} = \frac{k}{n} \binom{2n}{n-k},$$

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which is called *Catalan triangle* because the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ are the entries in the first column.

Shapiro derived sums identities from the Catalan triangle. For example, he gave the following identities:

$$\sum_{p=1}^{n} (B_{n,p})^2 = C_{2n-1} \quad \text{and} \quad \sum_{p=1}^{n} B_{n,p}B_{n+1,p} = C_{2n}.$$

We also refer to [4, 7] and references therein for other examples.

Much recently, Chu [2] investigated the following moments on quadratic product of binomial coefficients and evaluated them in closed forms

$$\Theta_\gamma^\delta (m,n) = \sum_{k\geq\delta} \left(k - \frac{\delta}{2}\right)^\gamma \binom{2m+\delta}{m+k}\binom{2n+\delta}{n+k},$$

where $\delta = 0$, 1 and γ, m, n are three natural numbers.

As mentioned earlier, some authors compute various sums including the falling factorial or half binomial coefficients.

Much recently, Slavik [9] presented a recursive method to find closed forms for the sums

$$S_m(n) = \sum_{k=0}^{n-1} k^m \binom{2n}{k}\binom{2n}{k}, \qquad n \geq 1$$

and

$$T_m(n) = \sum_{k=0}^{n-2} k^m \binom{2n-1}{k}\binom{2n-1}{k}, \qquad n \geq 2$$

for a fixed integer $m \geq 0$.

After this, Kılıç and Prodinger [6] provided explicit evaluations of these sums. The computations were reduced to the instance $m = 0$; this is, however, simple, since, by symmetry, the sums are basically half of the full sum which is evaluated by the Vandermonde convolution. This reduction was achieved by replacing the powers k^m by a linear combination of $k^{\underline{h}}k^{\underline{h}}$ and $k^{\underline{h}}k^{\underline{h-1}}$. The coefficients that appear here resemble the Stirling subset numbers (Stirling numbers of the second kind). The authors evaluated following four auxiliary sums using the above mentioned symmetry argument:

$$A_m(n) = \sum_{k=0}^{n-1} k^{\underline{m}}k^{\underline{m}} \binom{2n}{k}\binom{2n}{k}, \; B_m(n) = \sum_{k=0}^{n-1} k^{\underline{m}}k^{\underline{m-1}} \binom{2n}{k}\binom{2n}{k},$$

$$C_m(n) = \sum_{k=0}^{n-2} k^{\underline{m}}k^{\underline{m}} \binom{2n-1}{k}\binom{2n-1}{k}$$

and

$$D_m(n) = \sum_{k=0}^{n-2} k^{\underline{m}} k^{\underline{m-1}} \binom{2n-1}{k} \binom{2n-1}{k}.$$

Especially, the authors expressed the sums $C_m(n)$ and $D_m(n)$ in terms of $A_m(n)$ and $B_{m+1}(n)$.

In this paper, by inspired by the results of the works [6, 8, 9, 10] and combining the ideas of using half binomial coefficients as well as their squares, and, both rising and falling factorials rather than one of them in a sum as summand, we consider and compute three kinds of binomial sums of the forms:

$$\sum_{k=-n}^{n} k^{\underline{m}} k^{\overline{m}} \binom{2n}{n+k} \text{ and } \sum_{k=-n}^{n} k^{\underline{m}} k^{\overline{m}} \binom{2n}{n+k}^2$$

and

$$\sum_{k=-n}^{n} (-1)^k k^{\underline{m}} k^{\overline{m}} \binom{2n}{n+k}^3.$$

We first use the relationships between the rising, falling factorials and the binomial coefficients. Then we rewrite the claimed identities in terms of generalized hypergeometric functions to prove the claimed results.

2. The Main Results

Now we present our first main result:

Theorem 1. *For all positive integers n and m, the following identities hold:*

(i)

$$\sum_{k=-n}^{n} k^{\underline{m}} k^{\overline{m}} \binom{2n}{n+k} = 2^{2n-2m} (m!)^2 \binom{n}{m} \binom{2m}{m},$$

(ii)

$$\sum_{k=-n}^{n} k^{\underline{m}} k^{\overline{m}} \binom{2n}{n+k}^2 = m!^2 \binom{n}{m} \binom{2m}{m} \binom{2n-m}{n} \binom{4n-2m}{2n-m} \Big/ \binom{2n}{n}.$$

Proof. Denote, for brevity, the first sum by S' and then rewrite it as

$$S' = 2(-1)^m m!^2 \sum_{k=m}^{n} \binom{2n}{n+k} \binom{k}{m} \binom{-k}{m}.$$

Here note that $k^{\underline{m}}$ vanishes if k is positive and $m > k$, $k^{\overline{m}}$ vanishes if k is negative and $m > -k$, and $k^{\underline{m}} k^{\overline{m}} \binom{2n}{n+k}$ is unchanged if k is replaced by $-k$. Thus, we get $S^{'} = 2\sum_{k=m}^{n} k^{\underline{m}} k^{\overline{m}} \binom{2n}{n+k}$.

Making the replacement $k \to m+i$ on the summation index and then utilizing the relations

$$\begin{aligned} \binom{2n}{n+k} &\implies (-1)^i \binom{2n}{m+n} \frac{(m-n)_i}{(m+n+1)_i}, \\ \binom{k}{m}\binom{-k}{m} &\implies \frac{(-1)^m}{2} \binom{2m}{m} \frac{(m+1)_i (2m)_i}{i!(m)_i}; \end{aligned}$$

we can express S' in terms of hypergeometric series

$$S' = (2m)! \binom{2n}{m+n} {}_3F_2 \left[\begin{matrix} 2m, & m-n, & 1+m \\ & 1+m+n, & m \end{matrix} \,\middle|\, -1 \right],$$

where $(x)_n$ stands for the Pochhammer notation defined by $(x)_n = x(x+1)\ldots(x+n-1)$. It is seen that the Pochhammer notation coincides the rising factorial.

Recall the following useful Dougall sum on well–poised series (cf. Bailey [§4.3], [1] or see the formula 16.4.9 in [5], under the name "RogersDougall Very Well-Poised Sum")

(2.1)

$${}_5F_4 \left[\begin{matrix} a, 1+\frac{a}{2}, \ b, \ d, \ -n \\ \frac{a}{2}, 1+a-b, 1+a-d, 1+a+n \end{matrix} \,\middle|\, 1 \right] = \frac{(1+a)_n (1+a-b-d)_n}{(1+a-b)_n (1+a-d)_n}.$$

Its particular case specified by $n \to n-m, a \to 2m, b \to m+\frac{1}{2}, d \to \infty$ can be employed to evaluate

$$\begin{aligned} &{}_3F_2 \left[\begin{matrix} 2m, & m-n, & 1+m \\ & 1+m+n, & m \end{matrix} \,\middle|\, -1 \right] \\ &= \frac{(1+2m)_{n-m}}{(m+1/2)_{n-m}} = 4^{n-m} \frac{n!(m+n)!}{m!(2n)!}. \end{aligned}$$

Summing up, we have found that

$$S' = (2m)! \binom{2n}{m+n} 4^{n-m} \frac{n!(m+n)!}{m!(2n)!} = 4^{n-m} \binom{n}{m} (2m)!$$

which is equivalent to the first formula.

Let S'' stand for the second sum. It can be analogously reformulated as the following very well–poised ${}_4F_3$-series

$$S'' = (2m)! \binom{2n}{m+n}^2 {}_4F_3 \left[\begin{matrix} 2m, & m-n, & m-n, & 1+m \\ & 1+m+n, & 1+m+n, & m \end{matrix} \,\middle|\, 1 \right].$$

This can be evaluated again by Dougall's sum (2.1) specified with $n \to n-m, a \to 2m, b \to m+\frac{1}{2}, d \to m-n$ as follows

$$\begin{aligned} S'' =&(2m)!\binom{2n}{m+n}^2 \frac{(1+2m)_{n-m}(n+1/2)_{n-m}}{(1+m+n)_{n-m}(m+1/2)_{n-m}} \\ =&\binom{n}{m}^2 \frac{m!(2m)!(4n-2m)!}{(2n)!(2n-m)!} \end{aligned}$$

which is equivalent to the second formula. □

We can even go further to give our last result by evaluating the third sum which is an alternating and includes the cubes of the binomial coefficients:

Theorem 2. *For all positive integers n and m,*

$$\sum_{k=-n}^{n} (-1)^k \binom{2n}{n+k}^3 k^{\underline{m}} k^{\overline{m}} = (-1)^m \frac{(3n-m)!}{(n-m)!^3}.$$

Proof. Denote the sum by S''' in the statement of Theorem 2. By proceeding analogously to the proof of Theorem 1 and using Dougall's identity with $b \to m-n, d \to m-n, n \to n-m$, we get

$$\begin{aligned} S''' =&(-1)^m (2m)!\binom{2n}{m+n}^3 \\ &\times {}_5F_4\left[\begin{matrix} 2m, & m-n, & m-n, & m-n, & 1+m \\ & 1+m+n, & 1+m+n, & 1+m+n, & m \end{matrix} \middle| 1\right] \\ =&(-1)^m (2m)!\binom{2n}{m+n}^3 \frac{(1+2m)_{n-m}(1+2n)_{n-m}}{(1+m+n)_{n-m}(1+m+n)_{n-m}}. \end{aligned}$$

□

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TOBB University of Economics and Technology Mathematics Department 06560 Sögütözü Ankara Turkey

E-mail address: `ekilic@etu.edu.tr`