FORMULAS FOR BINOMIAL SUMS INCLUDING POWERS OF FIBONACCI AND LUCAS NUMBERS

Emrah KILIÇ¹, Ilker AKKUS², Neşe ÖMÜR, Yücel Türker ULUTAŞ³

Recently Prodinger [3] proved general expansion formulas for sums of powers of Fibonacci and Lucas numbers. In this paper, we will prove general expansion formulas for binomial sums of powers of Fibonacci and Lucas numbers.

Keywords: Fibonacci and Lucas numbers, binomial sums, powers.
MSC2010: 11B39, 05A19.

1. Introduction

The Fibonacci numbers \( F_n \) and Lucas numbers \( L_n \) are defined by the following recursions: for \( n > 0 \),

\[
F_{n+1} = F_n + F_{n-1} \quad \text{and} \quad L_{n+1} = L_n + L_{n-1},
\]

where \( F_0 = 0, F_1 = 1 \) and \( L_0 = 2, L_1 = 1 \), respectively.

If the roots of the characteristic equation \( x^2 - x - 1 = 0 \) are \( \alpha \) and \( \beta \), then the Binet formulas for them are

\[
F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n.
\]

Wiemann and Cooper [5] mentioned about some conjectures of Melham for the sum:

\[
\sum_{k=1}^{n} F_{2m+1}^{2m+1}.
\]

Ozeki [2] considered Melham’s sum and then he gave an explicit expansion for Melham’s sum as a polynomial in \( F_{2n+1} \).

¹TOBB Economics and Technology University, Mathematics Department, 06560 Ankara, Turkey, e-mail: ekilic@etu.edu.tr
²Kırıkkale University, Faculty of Arts and Sciences, Department of Mathematics, 71450 Yahşihan, Kırıkkale, Turkey, e-mail: iakkus.tr@gmail.com
³Kocaeli University, Department of Mathematics, 41380 Izmit, Turkey, e-mail: neseomur@kocaeli.edu.tr, turkery@kocaeli.edu.tr
In general, Prodinger [3] derived the general formula for the sum:

\[ \sum_{k=0}^{n} F_{2k+\varepsilon}^{2m+\varepsilon} + \varepsilon_{2k} + \delta, \]

where \( \varepsilon, \delta \in \{0, 1\} \), as well as the evaluations of the corresponding sums for Lucas numbers.

In [6], the authors gave formulas for the alternating analogues of sums of Melham for Fibonacci and Lucas numbers of the forms

\[ \sum_{k=1}^{n} (-1)^k F_{2k+\varepsilon}^{2m+\varepsilon} + \varepsilon_{2k} + \delta \]

\[ \sum_{k=1}^{n} (-1)^k L_{2k+\varepsilon}^{2m+\varepsilon}, \]

where \( \varepsilon, \delta \in \{0, 1\} \).

In this paper, we consider certain binomial sums given by

\[ \sum_{k=0}^{n} \binom{n}{k} F_{2k+\varepsilon}^{2m+\varepsilon} + \varepsilon_{2k+\delta} t, \]

\[ \sum_{k=0}^{n} \binom{n}{k} L_{2k+\varepsilon}^{2m+\varepsilon}, \]

\[ \sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{2k+\varepsilon}^{2m+\varepsilon} + \varepsilon_{2k+\delta} t, \]

\[ \sum_{k=0}^{n} \binom{n}{k} (-1)^k L_{2k+\varepsilon}^{2m+\varepsilon}, \]

where \( t \) is a positive integer and \( \varepsilon, \delta \in \{0, 1\} \).

Throughout this paper, we use the indicator function \([S]\) defined by 1 if the statement \( S \) is true or 0 otherwise.

We recall some facts for the readers convenience in [4]: For any real numbers \( m \) and \( n \),

\[ (m + n)^t = \sum_{i=0}^{t/2-1} \binom{t}{i} (mn)^{t/2} (m^{t-2i} + n^{t-2i}) + \binom{t}{t/2} (mn)^{t/2} [t \text{ is even}] \] (1)

and

\[ (m - n)^t = \sum_{i=0}^{t/2-1} \binom{t}{i} (mn)^{t/2} (-1)^i (m^{t-2i} + n^{t-2i}) + \binom{t}{t/2} (mn)^{t/2} (-1)^{t/2} [t \text{ is even}]. \] (2)

From [7], we have the following result:

**Lemma 1.1.** Let \( r \) and \( s \) be arbitrary integers. Then

**i)**

\[ \sum_{i=0}^{n} \binom{n}{i} F_{r+2si} = \begin{cases} 5^{(n-1)/2} F_s^n F_{sn+r} & \text{if } n \text{ is odd}, \\
5^{n/2} F_s^n F_{sn+r} & \text{if } n \text{ is even}, \\
L_s^n F_{sn+r} & \text{if } s \text{ is even}. \end{cases} \] (3)
Formulas for Binomial Sums Including Powers of Fibonacci and Lucas Numbers

\[ \sum_{i=0}^{n} \binom{n}{i} L_{r+2si} = \begin{cases} 5^{(n+1)/2} F_{s}^{n} F_{sn+r} & \text{if } n \text{ is odd}, s \text{ is odd}, \\ 5^{n/2} F_{s}^{n} L_{sn+r} & \text{if } n \text{ is even}, s \text{ is odd}, \\ L_{s}^{n} L_{sn+r} & \text{if } s \text{ is even}. \end{cases} \quad (4) \]

\[ \begin{align*} \sum_{i=0}^{n} \binom{n}{i} (-1)^i F_{r+2si} & = \begin{cases} 5^{n/2} F_{s}^{n} F_{sn+r} & \text{if } n \text{ is even}, s \text{ is odd}, \\ -5^{(n-1)/2} F_{s}^{n} L_{sn+r} & \text{if } n \text{ is odd}, s \text{ is even}, \\ (-1)^n L_{s}^{n} F_{sn+r} & \text{if } s \text{ is odd}. \end{cases} \quad (5) \end{align*} \]

\[ \begin{align*} \sum_{i=0}^{n} \binom{n}{i} (-1)^i L_{r+2si} & = \begin{cases} 5^{n/2} F_{s}^{n} L_{sn+r} & \text{if } n \text{ is even}, s \text{ is odd}, \\ -5^{(n+1)/2} F_{s}^{n} F_{sn+r} & \text{if } n \text{ is odd}, s \text{ is even}, \\ (-1)^n L_{s}^{n} L_{sn+r} & \text{if } s \text{ is odd}. \end{cases} \quad (6) \end{align*} \]

2. Some Binomial Sums for Fibonacci Numbers

Here we consider binomial and alternating binomial sums of powers of Fibonacci numbers.

**Theorem 2.1.** i) For \( t > 0 \),

\[ \sum_{k=0}^{n} \binom{n}{k} F_{2kt}^{2m} = \frac{1}{5^{m}} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} L_{2(m-i)t}^{n} L_{2(m-i)t}^{2n} + \frac{1}{5^{m}} \binom{2m}{m} (-1)^m 2^n. \]

ii) For odd \( t > 0 \),

\[ \sum_{k=0}^{n} \binom{n}{k} F_{2kt}^{2m+1} = \begin{cases} 5^{(n-2m)/2} \sum_{i=0}^{m} (-1)^i \binom{2m+1}{i} F_{(2m-2i+1)t}^{n} F_{(2m-2i+1)t}^{2n} & \text{if } n \text{ is even}, \\ 5^{(n-2m-1)/2} \sum_{i=0}^{m} (-1)^i \binom{2m+1}{i} F_{(2m-2i+1)t}^{n} L_{(2m-2i+1)t}^{2n} & \text{if } n \text{ is odd}, \end{cases} \]

and for even \( t > 0 \),

\[ \sum_{k=0}^{n} \binom{n}{k} F_{2kt}^{2m+1} = \frac{1}{5^{m}} \sum_{i=0}^{m} (-1)^i \binom{2m+1}{i} L_{(2m-2i+1)t}^{n} F_{(2m-2i+1)t}^{2n} \]
Proof. \(i\) From the Binet formulas of \(\{F_n\}\) and \(\{L_n\}\), and by (2), we write

\[
\sum_{k=0}^{n} \binom{n}{k} F_{2kt}^{2m} = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{\alpha^{2kt} - \beta^{2kt}}{\alpha - \beta} \right)^{2m}
\]

\[
= \frac{1}{(\alpha - \beta)^{2m}} \sum_{k=0}^{n} \binom{n}{k} \left( \sum_{i=0}^{m-1} \binom{2m}{i} (-1)^i \left( \alpha^{2(2m-2i)tk} + \beta^{2(2m-2i)tk} \right) \right)
+ \left( \frac{2m}{m} \right)(-1)^{m}
\]

\[
= \frac{1}{5^m} \left( \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{k=0}^{n} \binom{n}{k} L_{2(2m-2i)tk} + \left( \frac{2m}{m} \right) (-1)^{m} \sum_{k=0}^{n} \binom{n}{k} \right),
\]

which, by (4) in Lemma 1.1 and since \(\sum_{k=0}^{n} \binom{n}{k} = 2^n\), equivalent to

\[
\sum_{k=0}^{n} \binom{n}{k} F_{2kt}^{2m} = \frac{1}{5^m} \left( \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} L_{2(2m-2i)tk} + \left( \frac{2m}{m} \right) (-1)^{m} 2^n \right),
\]
as claimed.

\(ii\) Consider

\[
\sum_{k=0}^{n} \binom{n}{k} F_{2kt}^{2m+1} = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{\alpha^{2kt} - \beta^{2kt}}{\alpha - \beta} \right)^{2m+1}
\]

\[
= \frac{1}{(\alpha - \beta)^{2m+1}} \sum_{k=0}^{n} \binom{n}{k} \left( \sum_{i=0}^{m} \binom{2m+1}{i} (-1)^i \left( \alpha^{2(2m+1-2i)tk} - \beta^{2(2m+1-2i)tk} \right) \right)
\]

\[
= \frac{1}{5^m} \sum_{i=0}^{m} (-1)^i \binom{2m+1}{i} \sum_{k=0}^{n} \binom{n}{k} F_{2(2m+1-2i)kt},
\]

which, by taking \(s = (2m + 1 - 2i) t\) and \(r = 0\) in (3) in Lemma 1.1, gives the claimed results. \(\square\)

Following the proof way of Theorem 2.1, we have the following result without proof:
Theorem 2.2. i) For \( t > 0 \),

\[
\sum_{k=0}^{n} \binom{n}{k} F_{2m}^{2m}
\]

\[
\frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^i (2m+i) L_{(2m-2i)t}^n L_{(2m-2i)t(n+1)} + \frac{1}{5^m} \left(\frac{2m}{m}\right) (-1)^{t+1} m^{2n}.
\]

ii) For odd \( t > 0 \),

\[
\sum_{k=0}^{n} \binom{n}{k} F_{2m+1}^{2m+1}
\]

\[
= \begin{cases} 
5^{(n-2m)/2} \sum_{i=0}^{m} \binom{2m+1}{i} F_{(2m+1-2i)t}^n F_{(2m+1-2i)t(n+1)} & \text{if } n \text{ is even}, \\
5^{(n-2m-1)/2} \sum_{i=0}^{m} \binom{2m+1}{i} F_{(2m-2i+1)t}^n L_{(2m-2i+1)t(n+1)} & \text{if } n \text{ is odd},
\end{cases}
\]

and, for even \( t > 0 \),

\[
\sum_{k=0}^{n} \binom{n}{k} F_{2m+1}^{2m+1} = \frac{1}{5^m} \sum_{i=0}^{m} \binom{2m+1}{i} (-1)^i L_{(2m+1-2i)t}^n F_{(2m+1-2i)t(n+1)}. 
\]

Theorem 2.3. i) For \( t > 0 \),

\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{2m}^{2m}
\]

\[
= \begin{cases} 
5^{(n-2m)/2} \sum_{i=0}^{m-1} (-1)^i (2m+i) F_{(2m-2i)t}^n L_{(2m-2i)t(n+1)} \\
+ \frac{1}{5^m} \left(\frac{2m}{m}\right) (-1)^{m(t+1)} [n = 0] & \text{if } n \text{ is even}, \\
-5^{(n-2m+1)/2} \sum_{i=0}^{m-1} (-1)^i (2m+i) F_{(2m-2i)t}^n F_{(2m-2i)t(n+1)} & \text{if } n \text{ is odd},
\end{cases}
\]

ii) For odd \( t > 0 \),

\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{2m+1}^{2m+1} = \frac{(-1)^n}{5^m} \sum_{i=0}^{m} \binom{2m+1}{i} L_{(2m+1-2i)t}^n F_{(2m+1-2i)t(n+1)}. 
\]
and, for even $t > 0$,

$$
\sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{(2k+1)t}^{2m+1} =
\begin{cases}
5 \frac{(n-2m)}{2} \sum_{i=0}^{m} (-1)^i \binom{2m+1}{i} F_{(2m+1-2i)t}^m F_{(2m+1-2i)t(n+1)} & \text{if } n \text{ is even}, \\
-5 \frac{(n-2m-1)}{2} \sum_{i=0}^{m} (-1)^i \binom{2m+1}{i} F_{(2m+1-2i)t}^m L_{(2m+1-2i)t(n+1)} & \text{if } n \text{ is odd}.
\end{cases}
$$

Proof. i) For $t > 0$, by (2), consider

$$
\sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{(2k+1)t}^{2m+1} =
\frac{1}{(\alpha - \beta)^{2m}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \sum_{i=0}^{m-1} \binom{2m}{i} (-1)^i (t+1)^{(i+1)}
\times \left( \alpha (2m-2i)t(2k+1) + \beta (2m-2i)(2k+1) \right)
+ \binom{2m}{m} (-1)^m \frac{1}{(\alpha - \beta)^{2m}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k (-1)^{(2k+1)tm}
= \sum_{i=0}^{m-1} \binom{2m}{i} (-1)^i (t+1) \sum_{k=0}^{n} \binom{n}{k} (-1)^k L_{(2m-2i)t(2k+1)}
+ \frac{1}{5^m} \binom{2m}{m} (-1)^m (t+1) \sum_{k=0}^{n} \binom{n}{k} (-1)^k,
$$

which, by taking $s = r = (2m - 2i) t$ in (6) in Lemma 1.1 and $\sum_{k=0}^{n} \binom{n}{k} (-1)^k = [n = 0]$, gives the claimed result.
ii) For \( t > 0 \), by (2), consider
\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{2k+1}^{2m+1} t^{2m+1}
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left( \frac{\alpha (2k+1)t - \beta (2k+1)t}{\alpha - \beta} \right)^{2m+1}
\]
\[
= \frac{1}{(\alpha - \beta)^{2m+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \sum_{i=0}^{m} \binom{2m+1}{i} (-1)^i t^{2m+1}
\]
\[
\times \left( \alpha (2m+1-2i)(2k+1) - \beta (2m+1-2i)(2k+1) \right)
\]
\[
= \frac{1}{5^m} \sum_{i=0}^{m} \binom{2m+1}{i} (-1)^i \sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{2m+1-2i}(2k+1)
\]
which, by taking \( s = r = (2m+1-2i)t \) in (5) in Lemma 1.1, gives the claimed result.

Following the proof way of Theorem 2.3, we have the following result:

**Theorem 2.4.**

i) For \( t > 0 \),
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} F_{2k}^{2m}
\]
\[
= \left\{ \begin{array}{ll}
5 \frac{(n-2m)}{2} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} F_{2(m-i)t}^{n} L_{2(m-i)t}^{2m} & \text{if } n \text{ is even}, \\
+ \frac{1}{5^m} \binom{2m}{m} (-1)^m [n = 0] & \\
5 \frac{(n-2m+1)}{2} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} F_{2(m-i)t}^{n} F_{2(m-i)t}^{n} & \text{if } n \text{ is odd}
\end{array} \right.
\]

ii) For odd \( t > 0 \),
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} F_{2k+1}^{2m+1} = (-1)^n \frac{1}{5^m} \sum_{i=0}^{m} (-1)^i \binom{2m+1}{i} L_{(2m-2i+1)t}^{n} F_{(2m-2i+1)t}^{n},
\]
and for even \( t > 0 \),
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} F_{2k}^{2m+1}
\]
\[
= \left\{ \begin{array}{ll}
5 \frac{(n-2m)}{2} \sum_{i=0}^{m} (-1)^i \binom{2m+1}{i} F_{(2m-2i+1)t}^{n} F_{(2m-2i+1)t}^{n} & \text{if } n \text{ is even}, \\
-5 \frac{(n-2m-1)}{2} \sum_{i=0}^{m} (-1)^i \binom{2m+1}{i} F_{(2m-2i+1)t}^{n} L_{(2m-2i+1)t}^{n} & \text{if } n \text{ is odd}
\end{array} \right.
\]
3. Binomial Sums and Binomial Alternating Sums for Lucas Numbers

Now we consider binomial and alternating binomial sums of powers of Lucas numbers.

**Theorem 3.1.** i) For $t > 0$,

$$\sum_{k=0}^{n} \binom{n}{k} L_{2kt}^{2m} = \sum_{i=0}^{m-1} \binom{2m}{i} L_{2(m-i)t}^{n} L_{2(m-i)tn} + \binom{2m}{m} 2^n.$$

ii) For even $t > 0$,

$$\sum_{k=0}^{n} \binom{n}{k} L_{2kt}^{2m+1} = \sum_{i=0}^{m} \binom{2m+1}{i} L_{(2m-2i+1)t}^{n} L_{(2m-2i+1)tn},$$

and, for odd $t > 0$,

$$\sum_{k=0}^{n} \binom{n}{k} L_{2kt}^{2m+1} = \begin{cases} 
5^{n/2} \sum_{i=0}^{m} \binom{2m+1}{i} F_{(2m-2i+1)t}^{n} L_{(2m-2i+1)tn} & \text{if } n \text{ is even,} \\
5^{(n+1)/2} \sum_{i=0}^{m} \binom{2m+1}{i} F_{(2m-2i+1)t}^{n} F_{(2m-2i+1)tn} & \text{if } n \text{ is odd.} 
\end{cases}$$

**Proof.** i) For $t > 0$, by the Binet formula of $\{L_n\}$ and (1), we write

$$\sum_{k=0}^{n} \binom{n}{k} L_{2kt}^{2m}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (\alpha^{2kt} + \beta^{2kt})^{2m}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left( \sum_{i=0}^{m-1} \binom{2m}{i} (\alpha^{2(2m-2i)tk} + \beta^{2(2m-2i)tk}) + \binom{2m}{m} (\alpha\beta)^{2ktm} \right)$$

$$= \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{k=0}^{n} \binom{n}{k} L_{2(2m-2i)tk} + \binom{2m}{m} \sum_{k=0}^{n} \binom{n}{k},$$

which, by taking $s = (2m-2i)t$ and $r = 0$ in (4) in Lemma 1.1, gives the claimed result.
For \( t > 0 \), by the Binet formula of \( \{L_n\} \) and (1), we write

\[
\sum_{k=0}^{n} \binom{n}{k} L_{2kt}^{2m+1} = \sum_{k=0}^{n} \binom{n}{k} (\alpha^{2kt} + \beta^{2kt})^{2m+1} = \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{m} \binom{2m+1}{i} \left(\alpha^{2(2m+1-2i)tk} + \beta^{2(2m+1-2i)tk}\right) = \sum_{i=0}^{m} \binom{2m+1}{i} \sum_{k=0}^{n} \binom{n}{k} L_{2(2m+1-2i)tk}
\]

which, by taking \( s = (2m + 1 - 2i) t \) and \( r = 0 \) in (4) in Lemma 1.1, gives the claimed result. \( \square \)

Following the proof way of the previous Theorems, we give the following results without proof:

**Theorem 3.2.**

i) For \( t > 0 \),

\[
\sum_{k=0}^{n} \binom{n}{k} L_{(2k+1)t}^{2m} = \sum_{i=0}^{m-1} \binom{2m}{i} (-1)^i \binom{n}{m-i} L_{2(m-i)t}^{2} L_{2(m-i)t(n+1)}^{2} + \binom{2m}{m} (-1)^m 2^n.
\]

ii) For even \( t > 0 \),

\[
\sum_{k=0}^{n} \binom{n}{k} L_{(2k+1)t}^{2m+1} = \sum_{i=0}^{m} \binom{2m+1}{i} \binom{n}{2m+1-i} L_{(2m-2i+1)t}^{2} L_{(2m-2i+1)t(n+1)}^{2}.
\]

and, for odd \( t > 0 \),

\[
\sum_{k=0}^{n} \binom{n}{k} L_{(2k+1)t}^{2m+1} \begin{cases} 5 \binom{m}{i} \binom{2m+1}{i} (-1)^i F_{(2m-2i+1)t(n+1)}^n L_{(2m-2i+1)t}^{2} & \text{if } n \text{ is even}, \\ 5 \binom{m}{i} \binom{2m+1}{i} (-1)^i F_{(2m-2i+1)t(n+1)}^n F_{(2m-2i+1)t}^{(n+1)} & \text{if } n \text{ is odd}. \end{cases}
\]
Theorem 3.3. i) For \( t > 0 \),

\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^k L_{2kt}^{2m} \sum_{k=0}^{n} \binom{n}{k} (-1)^k L_{2kt}^{2m} + \left(\binom{2m}{m}\right) [n = 0] \quad \text{if } n \text{ is even,}
\]

\[
-5 \frac{n+1}{2} \sum_{i=0}^{\left(\frac{n+1}{2}\right)} \binom{2m+1}{i} F_{(2m+1-2i)t}^{n} F_{(2m+1-2i)t}^{n} \quad \text{if } n \text{ is odd.}
\]

ii) For odd \( t > 0 \),

\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^k L_{2kt}^{2m+1} = (-1)^n \sum_{i=0}^{m} \binom{2m+1}{i} L_{(2m+1-2i)t}^{n} L_{(2m+1-2i)t}^{n}.
\]

and, for even \( t > 0 \),

\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^k L_{2kt}^{2m+1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k L_{2kt}^{2m+1} + \left(\binom{2m}{m}\right) [n = 0] \quad \text{if } n \text{ is even,}
\]

\[
-5 \frac{n+1}{2} \sum_{i=0}^{\left(\frac{n+1}{2}\right)} \binom{2m+1}{i} F_{(2m+1-2i)t}^{n} F_{(2m+1-2i)t}^{n} \quad \text{if } n \text{ is odd.}
\]

Theorem 3.4. i) For \( t > 0 \),

\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^k L_{(2k+1)t}^{2m} \sum_{k=0}^{n} \binom{n}{k} (-1)^k L_{(2k+1)t}^{2m} + \left(\binom{2m}{m}\right) [n = 0] \quad \text{if } n \text{ is even,}
\]

\[
-5 \frac{n+1}{2} \sum_{i=0}^{\left(\frac{n+1}{2}\right)} \binom{2m+1}{i} F_{(2m+1-2i)t}^{n} F_{(2m+1-2i)t}^{n} \quad \text{if } n \text{ is odd.}
\]

ii) For odd \( t > 0 \),

\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^k L_{(2k+1)t}^{2m+1} = (-1)^n \sum_{i=0}^{m} \binom{2m+1}{i} (-1)^i L_{(2m+1-2i)t}^{n} L_{(2m+1-2i)t}^{n}.
\]
and, for even $t > 0$,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{L^{2m+1}}{(2k+1)t}$$

$$= \begin{cases} 
5 \sum_{i=0}^{m} \binom{2m+1}{i} \frac{F^n}{(2m+1-2i)t} L^{(2m+1-2i)t(n+1)} & \text{if } n \text{ is even}, \\
-5 \sum_{i=0}^{(n+1)/2} \binom{2m+1}{i} \frac{F^n}{(2m+1-2i)t} F^{(2m+1-2i)t(n+1)} & \text{if } n \text{ is odd}.
\end{cases}$$

**REFERENCES**