ON BINOMIAL DOUBLE SUMS WITH FIBONACCI AND LUCAS NUMBERS-I

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Abstract. In this paper, we compute various binomial double sums involving the generalized Fibonacci and Lucas numbers as well as their alternating analogous.

1. Introduction

Define second order linear recurrences $\{U_n, V_n\}$ as for $n > 0$

$$U_n = pU_{n-1} + U_{n-2},$$
$$V_n = pV_{n-1} + V_{n-2},$$

where $U_0 = 0,\ U_1 = 1$, and $V_0 = 2,\ V_1 = p$, resp. If $p = 1$, then $U_n = F_n$ (nth Fibonacci number) and $V_n = L_n$ (nth Lucas number). For various properties of these sequences and their generalizations, we could refer to [4, 5, 15].

The Binet formulæ are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n,$$

where $\alpha,\ \beta = (p \pm \sqrt{\triangle})/2$ and $\triangle = p^2 + 4$.

By the Binet formulae of U_n and V_n, for later use one can see that

$$U_{-n} = (-1)^{n+1} U_n \text{ and } V_{-n} = (-1)^n V_n.$$

There are many types of identities involving sums of products of binomial coefficients and Fibonacci or Lucas numbers (for more details see [1, 2, 14, 16]). For example from [1], we recall that

$$\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n},\ \sum_{k=0}^{n} \binom{n}{k} F_{4k} = 3^n F_{2n},$$

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} F_{5k} = 5^n F_{2n}, \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} F_{6k} = 8^n F_{2n},$$

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$$\sum_{k=0}^{n}\binom{n}{k}(-2)^k F_{2k}=(-1)^n F_{3n},\ \sum_{k=0}^{n}\binom{n}{k}(-2)^k F_{5k}=(-1)^n 5^n F_{3n}\ .$$

Meanwhile many authors have computed various weighted binomial sums by various methods (for more details, see [12, 13]). For example, in [13], the authors studied the sums have the forms

$$\sum_{i=0}^{n}\binom{n}{i}T_{k(a+bi)}T_{k(c+di)} \quad \text{and} \quad \sum_{i=0}^{n}\binom{n}{i}(-1)^i T_{k(a+bi)}T_{k(c+di)},$$

where T_n is either U_n or V_n.

It is assumed that the reader is familiar with the basic facts about binomial sums, the Binomial theorem, combinatorial summation formulæ, etc. (we could refer to [3]).

Kılıç et. al. [8] proved general expansion formulæ for binomial sums of powers of Fibonacci and Lucas numbers as shown

$$\sum_{k=0}^{n}\binom{n}{k}F_{(2k+\delta)t}^{2m+\varepsilon}\ ,\quad \sum_{k=0}^{n}\binom{n}{k}L_{(2k+\delta)t}^{2m+\varepsilon}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k F_{(2k+\delta)t}^{2m+\varepsilon},\ \sum_{k=0}^{n}\binom{n}{k}(-1)^k L_{(2k+\delta)t}^{2m+\varepsilon},$$

where t is a positive integer and $\delta,\varepsilon\in\{0,1\}$.

In [11] Kılıç and Ionascu established some identities containing sums of binomials with coefficients satisfying third order linear recursive relations. For example, we recall one result from [11]: for any $a\in\mathbb{C}\setminus\{0\}$,

$$\sum_{k=0}^{n}\binom{2n}{n+k}(a^k+a^{-k})=\frac{1}{a^n}(a+1)^{2n}+\binom{2n}{n}.$$

Khan and Kwong [6] studied two kinds of binomial sums

$$\sum_{h=0}^{n}h^m\binom{n}{h}U_h \text{ and } \sum_{h=0}^{n}(-1)^{n+h}\ h^m\binom{n}{h}U_h$$

and then express them in terms of two associated sequences.

Kılıç and Arıkan [9] derived new double binomial sums families related with generalized second, third and certain higher order linear recurrences. For example,

$$\sum_{1\le i,j\le n}\binom{n-j}{j}\binom{i+j}{j}(-1)^i=F_{n+1}$$

and

$$\sum_{1\le i,j\le n}\binom{i}{j-1}=F_{n+3}-1.$$

Kılıç and Belbachir [10] derived various double binomial sums and binomial sums with complex coefficients related with the sequences $\{U_n, V_n\}$. For example, they showed that

$$\sum_{i,j} \binom{n-i}{j} \binom{n-j}{i} = F_{2n+2}.$$

Recently, Kılıç [7] considered and computed three classes of generalized alternating weighted binomial sums of the forms

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i \ f(n,i,k,t),$$

where $f(n,i,k,t)$ is $U_{kti}V_{kn-k(t+2)i}$, $U_{kti}V_{kn-kti}$ and $U_{tki}V_{(k+1)tn-(k+2)ti}$.

Much recently, Kılıç and Arıkan [9] also considered and computed various interesting families of binomial sums namely *binomial-double-sums* including double sums and one binomial coefficient. For example they showed that

$$\sum_{0\le i,j\le n} \binom{n+i}{j-i} = F_{2n+3} - 2^n, \quad \sum_{0\le i,j\le n} \binom{n+i}{j-i} (-1)^j = (-1)^n F_{2n}$$

$$\sum_{0\le i,j\le n} \binom{i+j}{i-j} = F_{2n+2} \quad \text{and} \quad \sum_{0\le i,j\le n} \binom{i}{j-i} = F_{n+3} - 1.$$

These are the first interesting examples of double sums with one binomial coefficient.

In this paper, inspiring from the results of [9] about double sums with one binomial coefficient, we shall consider new kinds of binomial-double-sums families with general Fibonacci and Lucas numbers.

2. Binomial-Double-Sums with the Generalized Fibonacci And Lucas Numbers

First we give some auxiliary lemmas before our main results.

Lemma 1. *For any real numbers x and y such that $x(1+y) \neq 1$.*

$$\sum_{0\le i,j\le k} \binom{i}{j} x^i y^j = \frac{(x+xy)^{k+1} - 1}{x+xy-1} \ .$$

Proof. By the Binomial theorem and some properties of sigma notation, we write

$$\sum_{0\le i,j\le k} \binom{i}{j} x^i y^j \quad = \quad \sum_{0\le i\le k} x^i \sum_{0\le j\le i} \binom{i}{j} y^j = \sum_{i=0}^{k} x^i \, (1+y)^i$$

$$= \sum_{i=0}^{k} (x(1+y))^i = \frac{(x+xy)^{k+1}-1}{x+xy-1},$$

as claimed. □

From [7] we have the following result:

Lemma 2. *Let t be any integer.*

(i) For odd k,

$$\begin{aligned}(-1)^t \alpha^{k(1-2t)} - \alpha^k &= (-1)^t U_{kt}\beta^{k(t-1)}\sqrt{\Delta},\\ (-1)^t \beta^{k(1-2t)} - \beta^k &= (-1)^{t+1} U_{kt}\alpha^{k(t-1)}\sqrt{\Delta}.\end{aligned}$$

(ii) For even k,

$$\alpha^{k(1-2t)} - \alpha^k = -U_{kt}\beta^{k(t-1)}\sqrt{\Delta},\ \ \beta^{k(1-2t)} - \beta^k = U_{kt}\alpha^{k(t-1)}\sqrt{\Delta}.$$

Now we shall give our first result:

Theorem 1. *Let t and r be odd integers.*

a)For nonnegative even k,

$$\sum_{0\le i,j\le k}\binom{i}{j}U_{ri+2tj} = \frac{\Delta^{\frac{k}{2}}U_t^{k+1}\left[V_{(t+r)(k+1)} + \Delta U_t U_{k(t+r)}\right] - U_t V_{t+r}}{\Delta U_t^2 + \Delta U_t U_{t+r} - 1}$$

and

$$\sum_{0\le i,j\le k}\binom{i}{j}V_{ri+2tj}$$
$$= \frac{\Delta^{\frac{k}{2}+1}U_t^{k+1}\left[U_t V_{k(t+r)} + U_{(t+r)(k+1)}\right] + \Delta U_t U_{t+r} - 2}{\Delta U_t^2 + \Delta U_t U_t - 1}.$$

b)For positive odd k,

$$\sum_{0\le i,j\le k}\binom{i}{j}U_{ri+2tj} = \frac{\Delta^{\frac{k+1}{2}}U_t^{k+1}\left[U_t V_{k(t+r)} + U_{(t+r)(k+1)}\right] - U_t V_{t+r}}{\Delta U_t^2 + \Delta U_t U_{t+r} - 1}$$

and

$$\sum_{0\le i,j\le k}\binom{i}{j}V_{ri+2tj}$$
$$= \frac{\Delta^{\frac{k+1}{2}}U_t^{k+1}[\Delta U_t U_{k(t+r)} + V_{(t+r)(k+1)}] + \Delta U_t U_{t+r} - 2}{\Delta U_t^2 + \Delta U_t U_{t+r} - 1}.$$

Proof. We only prove the first identity. The others could be similarly proven. By the Binet formula, we write that for odd r,

$$\sum_{0\le i,j\le k}\binom{i}{j}U_{ri+2tj} = \frac{1}{\alpha-\beta}\sum_{0\le i,j\le k}\binom{i}{j}\left(\alpha^{ri+2tj} - \beta^{ri+2tj}\right)$$

$$= \frac{1}{\alpha-\beta}\left[\sum_{0\le i,j\le k}\binom{i}{j}\alpha^{ri+2tj} - \sum_{0\le i,j\le k}\binom{i}{j}\beta^{ri+2tj}\right],$$

which, by Lemma 1, equals

$$\frac{1}{\alpha-\beta}\left[\frac{\left(\alpha^r+\alpha^{r+2t}\right)^{k+1}-1}{\alpha^r+\alpha^{r+2t}-1} - \frac{\left(\beta^r+\beta^{r+2t}\right)^{k+1}-1}{\beta^r+\beta^{r+2t}-1}\right].$$

Also, by Lemma 2 (i), if k and t are odd, then we write

$$\begin{aligned} -\alpha^{k(1-2t)} - \alpha^k &= -U_{kt}\beta^{k(t-1)}\sqrt{\Delta}, \\ -\beta^{k(1-2t)} - \beta^k &= U_{kt}\alpha^{k(t-1)}\sqrt{\Delta}. \end{aligned}$$

Hence write

$$\alpha^{k(1-2t)} + \alpha^k = \alpha^{k-2kt} + \alpha^k = U_{kt}\beta^{k(t-1)}\sqrt{\Delta}.$$

Thus

$$\alpha^{k+s} + \alpha^k = U_{-\frac{s}{2}}\beta^{-\frac{s}{2}-k}\sqrt{\Delta} = U_{\frac{s}{2}}\beta^{-\frac{s}{2}-k}\sqrt{\Delta},$$

where $s = -2kt$. Therefore, by taking $k = r$ and $s = t$, we write

$$\alpha^r + \alpha^{r+t} = U_{\frac{t}{2}}\beta^{-\frac{t}{2}-r}\sqrt{\Delta}$$

for odd r and $t = -2kt = 2t$. Namely,

$$\alpha^r + \alpha^{r+2t} = U_t\beta^{-t-r}\sqrt{\Delta},$$

and similarly,

$$\beta^r + \beta^{r+2t} = -U_t\alpha^{-t-r}\sqrt{\Delta}.$$

Hence,

$$\begin{aligned} &\frac{1}{\alpha-\beta}\left[\frac{\left(\alpha^r+\alpha^{r+2t}\right)^{k+1}-1}{\alpha^r+\alpha^{r+2t}-1} - \frac{\left(\beta^r+\beta^{r+2t}\right)^{k+1}-1}{\beta^r+\beta^{r+2t}-1}\right] \\ &= \frac{1}{\sqrt{\Delta}}\left[\frac{(U_t\beta^{-t-r}\sqrt{\triangle})^{k+1}-1}{U_t\beta^{-t-r}\sqrt{\triangle}-1} - \frac{(-U_t\alpha^{-t-r}\sqrt{\triangle})^{k+1}-1}{-U_t\alpha^{-t-r}\sqrt{\triangle}-1}\right] \\ &= \frac{1}{\sqrt{\Delta}}\left[\frac{U_t^{k+1}\beta^{(-t-r)(k+1)}\Delta^{\frac{k+1}{2}}-1}{U_t\beta^{-t-r}\sqrt{\triangle}-1} - \frac{-U_t^{k+1}\alpha^{(-t-r)(k+1)}\Delta^{\frac{k+1}{2}}-1}{-U_t\alpha^{-t-r}\sqrt{\triangle}-1}\right] \\ &= \frac{1}{\sqrt{\Delta}}\left[\frac{U_t^{k+1}\beta^{(-t-r)(k+1)}\Delta^{\frac{k+1}{2}}-1}{U_t\beta^{-t-r}\sqrt{\triangle}-1} - \frac{U_t^{k+1}\alpha^{(-t-r)(k+1)}\Delta^{\frac{k+1}{2}}+1}{U_t\alpha^{-t-r}\sqrt{\triangle}+1}\right], \end{aligned}$$

which equals

$$\begin{aligned} &\frac{1}{\sqrt{\Delta}\left(U_t\beta^{-t-r}\sqrt{\triangle}-1\right)\left(U_t\alpha^{-t-r}\sqrt{\triangle}+1\right)} \\ &\times\left[\left(U_t^{k+1}\beta^{(-t-r)(k+1)}\Delta^{\frac{k+1}{2}}-1\right)\left(U_t\alpha^{-t-r}\sqrt{\triangle}+1\right)\right. \end{aligned}$$

$$-\left(U_t^{k+1}\alpha^{(-t-r)(k+1)}\Delta^{\frac{k+1}{2}}+1\right)\left(U_t\beta^{-t-r}\sqrt{\triangle}-1\right)\Big].$$

By recalling $\alpha\beta=-1$ and after some rearrangement, consider the statement in the numerator of the last equation

$$\begin{aligned}
&\left(U_t^{k+1}\beta^{(-t-r)(k+1)}\Delta^{\frac{k+1}{2}}-1\right)\left(U_t\alpha^{-t-r}\sqrt{\triangle}+1\right)\\
&-\left(U_t^{k+1}\alpha^{(-t-r)(k+1)}\Delta^{\frac{k+1}{2}}+1\right)\left(U_t\beta^{-t-r}\sqrt{\triangle}-1\right)\\
&=U_t^{k+2}\beta^{(-t-r)k}\Delta^{\frac{k}{2}+1}+U_t^{k+1}\beta^{(-t-r)(k+1)}\Delta^{\frac{k+1}{2}}-U_t\alpha^{-t-r}\sqrt{\triangle}-1\\
&-U_t^{k+2}\alpha^{(-t-r)k}\Delta^{\frac{k}{2}+1}+U_t^{k+1}\alpha^{(-t-r)(k+1)}\Delta^{\frac{k+1}{2}}-U_t\beta^{-t-r}\sqrt{\triangle}+1\\
&=U_t^{k+2}U_{k(t+r)}\Delta^{\frac{k}{2}+1}\sqrt{\triangle}+U_t^{k+1}\Delta^{\frac{k+1}{2}}V_{(t+r)(k+1)}-U_tV_{t+r}\sqrt{\triangle}.
\end{aligned}$$

And now consider the statement in the denominator of the equation

$$\begin{aligned}
&\left(U_t\beta^{-t-r}\sqrt{\triangle}-1\right)\left(U_t\alpha^{-t-r}\sqrt{\triangle}+1\right)\\
&=U_t^2\Delta+U_t\beta^{-t-r}\sqrt{\triangle}-U_t\alpha^{-t-r}\sqrt{\triangle}-1\\
&=U_t^2\Delta+U_t\sqrt{\triangle}(\beta^{-t-r}-\alpha^{-t-r})-1\\
&=U_t^2\Delta+U_t\Delta U_{t+r}-1.
\end{aligned}$$

Thus we write

$$\sum_{0\le i,j\le k}\binom{i}{j}U_{ri+2tj}=\frac{\Delta^{\frac{k}{2}+1}U_t^{k+2}U_{k(t+r)}+\Delta^{\frac{k}{2}}U_t^{k+1}V_{(t+r)(k+1)}-U_tV_{t+r}}{\Delta U_t^2+\Delta U_tU_{t+r}-1},$$

as claimed. □

From [7], we have the following result:

Lemma 3. *Let t be any integer.*

(i) For odd k,

$$\begin{aligned}
(-1)^t\alpha^{-k(2t+1)}-\alpha^k &= (-1)^{t+1}V_{k(t+1)}\beta^{kt},\\
(-1)^t\beta^{-k(2t+1)}-\beta^k &= (-1)^{t+1}V_{k(t+1)}\alpha^{kt}.
\end{aligned}$$

(ii) For even k,

$$\alpha^{-k(2t+1)}-\alpha^k=-\sqrt{\Delta}U_{k(t+1)}\beta^{kt},\ \ \beta^{-k(2t+1)}-\beta^k=\sqrt{\Delta}U_{k(t+1)}\alpha^{kt}.$$

We have the following result without proof that could be proven by Lemmas 1 and 3.

Theorem 2. *For any integer t and odd r,*

$$\sum_{0\le i,j\le k}\binom{i}{j}U_{ri+4tj}=\frac{V_{2t}U_{2t+r}-V_{2t}^{k+1}[V_{2t}U_{k(2t+r)}+U_{(2t+r)(k+1)}]}{1-V_{2t}^2-V_{2t}V_{2t+r}},$$

$$\sum_{0\leq i,j\leq k}\binom{i}{j}V_{ri+4tj}=\frac{2-V_{2t}^{k+1}[V_{2t}V_{k(2t+r)}+V_{(2t+r)(k+1)}]-V_{2t}V_{2t+r}}{1-V_{2t}^{2}-V_{2t}V_{2t+r}}.$$

3. Alternating Binomial Sums For The Generalized Fibonacci And Lucas Numbers

In this section, we present certain alternating binomial double sums including the generalized Fibonacci and Lucas numbers. First we give a consequence of Lemma 1 by taking $-x$ instead of x : For any real numbers x and y such that $x(1+y)\neq -1$

$$\sum_{0\leq i,j\leq k}\binom{i}{j}(-1)^{i}x^{i}y^{j}=\frac{(-1)^{k}\left(x+xy\right)^{k+1}+1}{x+xy+1}. \tag{3.1}$$

Theorem 3. *Let t and r be odd integers.*

a)For nonnegative even k,

$$\sum_{0\leq i,j\leq k}\binom{i}{j}(-1)^{i}U_{ri+2tj}=\frac{\Delta^{\frac{k}{2}}U_{t}^{k+1}[\Delta U_{t}U_{k(t+r)}-V_{(t+r)(k+1)}]+U_{t}V_{t+r}}{\Delta U_{t}^{2}-\Delta U_{t}U_{t+r}-1}$$

and

$$\sum_{0\leq i,j\leq k}\binom{i}{j}(-1)^{i}V_{ri+2tj}$$

$$=\frac{\Delta^{\frac{k+2}{2}}U_{t}^{k+1}[U_{t}V_{k(t+r)}-U_{(t+r)(k+1)}]-\Delta U_{t}U_{t+r}-2}{\Delta U_{t}^{2}-\Delta U_{t}U_{t+r}-1}.$$

b)For positive odd k,

$$\sum_{0\leq i,j\leq k}\binom{i}{j}(-1)^{i}U_{ri+2tj}=\frac{\Delta^{\frac{k+1}{2}}U_{t}^{k+1}[U_{t}V_{k(t+r)}-U_{(t+r)(k+1)}]+U_{t}U_{t+r}}{-\Delta U_{t}^{2}+\Delta U_{t}U_{t+r}+1}$$

and

$$\sum_{0\leq i,j\leq k}\binom{i}{j}(-1)^{i}V_{ri+2tj}$$

$$=\frac{\Delta^{\frac{k+1}{2}}U_{t}^{k+1}[\Delta U_{t}U_{k(t+r)}-V_{(t+r)(k+1)}]+\Delta U_{t}U_{t+r}+2}{-\Delta U_{t}^{2}+\Delta U_{t}U_{t+r}+1}.$$

Proof. We only prove the first identity. The others could be similarly proven. Assume that r is an odd integer. Thus we write

$$\sum_{0\leq i,j\leq k}\binom{i}{j}(-1)^{i}U_{ri+2tj}=\frac{1}{\alpha-\beta}\sum_{0\leq i,j\leq k}\binom{i}{j}(-1)^{i}(\alpha^{ri+2tj}-\beta^{ri+2tj})$$

$$= \frac{1}{\alpha-\beta}\left[\sum_{0\le i,j\le k}\binom{i}{j}(-1)^i\alpha^{ri+2tj} - \sum_{0\le i,j\le k}\binom{i}{j}(-1)^i\beta^{ri+2tj}\right],$$

which, by (3.1), equals

$$\frac{1}{\alpha-\beta}\left[\frac{(-1)^k\left(\alpha^r+\alpha^{r+2t}\right)^{k+1}+1}{\alpha^r+\alpha^{r+2t}+1} - \frac{(-1)^k\left(\beta^r+\beta^{r+2t}\right)^{k+1}+1}{\beta^r+\beta^{r+2t}+1}\right]$$
$$= \frac{1}{\alpha-\beta}\left[\frac{(U_t\beta^{-t-r}\sqrt{\triangle})^{k+1}+1}{U_t\beta^{-t-r}\sqrt{\triangle}+1} - \frac{(-U_t\ \alpha^{-t-r}\sqrt{\triangle})^{k+1}+1}{-U_t\alpha^{-t-r}\sqrt{\triangle}+1}\right]$$

which equals

$$\frac{1}{\sqrt{\triangle}\left(U_t\beta^{-t-r}\sqrt{\triangle}+1\right)\left(-U_t\alpha^{-t-r}\sqrt{\triangle}+1\right)}$$
$$\times\left[\left(U_t^{k+1}\beta^{(-t-r)(k+1)}\Delta^{\frac{k+1}{2}}+1\right)\left(-U_t\alpha^{-t-r}\sqrt{\triangle}+1\right)\right.$$
$$\left.-\left(-U_t^{k+1}\alpha^{(-t-r)(k+1)}\Delta^{\frac{k+1}{2}}+1\right)\left(U_t\beta^{-t-r}\sqrt{\triangle}+1\right)\right].$$

By $\alpha\beta=-1$ and some rearrangement, we write

$$\left(U_t^{k+1}\beta^{(-t-r)(k+1)}\Delta^{\frac{k+1}{2}}+1\right)\left(-U_t\alpha^{-t-r}\sqrt{\triangle}+1\right)$$
$$-\left(-U_t^{k+1}\alpha^{(-t-r)(k+1)}\Delta^{\frac{k+1}{2}}+1\right)\left(U_t\beta^{-t-r}\sqrt{\triangle}+1\right)$$
$$= -U_t^{k+2}U_{k(t+r)}\Delta^{\frac{k+2}{2}}\sqrt{\triangle}+U_t^{k+1}\Delta^{\frac{k+1}{2}}V_{(t+r)(k+1)}-U_tV_{t+r}\sqrt{\triangle}.$$

and

$$\left(U_t\beta^{-t-r}\sqrt{\triangle}+1\right)\left(-U_t\alpha^{-t-r}\sqrt{\triangle}+1\right)$$
$$= -U_t^2\Delta+U_t\sqrt{\triangle}(\beta^{-t-r}-\alpha^{-t-r})+1$$
$$= -U_t^2\Delta+U_tU_{t+r}\Delta+1.$$

Finally we write

$$\sum_{0\le i,j\le k}\binom{i}{j}(-1)^iU_{ri+2tj}$$
$$= \frac{\Delta^{\frac{k+2}{2}}U_t^{k+2}U_{k(t+r)}-\Delta^{\frac{k}{2}}U_t^{k+1}V_{(t+r)(k+1)}+U_tV_{t+r}}{\Delta U_t^2-\Delta U_tU_{t+r}-1},$$

as claimed. □

We have the following result without proof that could be proven by Eq. (3.1) and Lemma 3.

Theorem 4. *For $k > 0$, any integer t and odd r*

$$\sum_{0\leq i,j\leq k}\binom{i}{j}(-1)^i U_{ri+4tj}$$
$$= \frac{(-1)^{k+1}V_{2t}^{k+1}[V_{2t}U_{k(2t+r)} - U_{(2t+r)(k+1)}] - V_{2t}U_{2t+r}}{1 - V_{2t}^2 + V_{2t}V_{2t+r}}$$

and

$$\sum_{0\leq i,j\leq k}\binom{i}{j}(-1)^i V_{ri+4tj}$$
$$= \frac{(-1)^{k+1}V_{2t}^{k+1}[V_{(2t+r)(k+1)} - V_{2t}V_{k(2t+r)}] + V_{2t}V_{2t+r} + 2}{1 - V_{2t}^2 + V_{2t}V_{2t+r}}.$$

Now we give another consequence of Lemma 1 by taking $-y$ instead of y : For any real numbers x and y such that $x(1-y) \neq 1$

$$\sum_{0\leq i,j\leq k}\binom{i}{j}(-1)^j x^i y^j = \frac{(x-xy)^{k+1}-1}{x-xy-1}. \tag{3.2}$$

One can similarly obtain the following results by using Eq. (3.2)

Theorem 5. *For any integer t and odd r,*

a) For nonnegative even k,

$$\sum_{0\leq i,j\leq k}\binom{i}{j}(-1)^j U_{ri+4tj}$$
$$= \frac{\Delta^{\frac{k}{2}}U_{2t}^{k+1}[\Delta U_{2t}U_{k(2t+r)} + V_{(2t+r)(k+1)}] - U_{2t}V_{2t+r}}{\Delta U_{2t}^2 + \Delta U_{2t}U_{2t+r} + 1}$$

and

$$\sum_{0\leq i,j\leq k}\binom{i}{j}(-1)^j V_{ri+4tj}$$
$$= \frac{\Delta^{\frac{k+2}{2}}U_{2t}^{k+1}[U_{2t}V_{k(2t+r)} + U_{(2t+r)(k+1)}] + \Delta U_{2t}U_{2t+r} + 2}{\Delta U_{2t}^2 + \Delta U_{2t}U_{2t+r} + 1}.$$

b) For positive odd k,

$$\sum_{0\leq i,j\leq k}\binom{i}{j}(-1)^j U_{ri+4tj}$$
$$= -\frac{\Delta^{\frac{k+1}{2}}U_{2t}^{k+1}[U_{2t}V_{k(2t+r)} + U_{(2t+r)(k+1)}] + U_{2t}V_{2t+r}}{\Delta U_{2t}^2 + \Delta U_{2t}U_{2t+r} + 1}$$

and

$$\sum_{0\le i,j\le k}\binom{i}{j}(-1)^j V_{ri+4tj}$$
$$=-\frac{\Delta^{\frac{k+1}{2}}U_{2t}^{k+1}[\Delta U_{2t}U_{k(2t+r)}+V_{(2t+r)(k+1)}]-\Delta U_{2t}U_{2t+r}-2}{\Delta U_{2t}^2+\Delta U_{2t}U_{2t+r}+1}.$$

Theorem 6. *For any integer t and odd r,*

$$\sum_{0\le i,j\le k}\binom{i}{j}(-1)^j U_{ri+2tj}$$
$$=\frac{(-1)^k V_t^{k+1}[V_tU_{k(t+r)}+U_{(t+r)(k+1)}]-V_tU_{t+r}}{V_t^2+V_tV_{t+r}+1}$$

and

$$\sum_{0\le i,j\le k}\binom{i}{j}(-1)^j V_{ri+2tj}$$
$$=\frac{(-1)^k V_t^{k+1}[V_tV_{k(t+r)}+V_{(t+r)(k+1)}]+V_tV_{t+r}+2}{V_t^2+V_tV_{t+r}+1}.$$

Theorem 7. *For any integer t and even r,*

a) For nonnegative even k,

$$\sum_{0\le i,j\le k}\binom{i}{j}(-1)^j U_{ri+4tj}$$
$$=\frac{\Delta^{\frac{k}{2}}U_{2t}^{k+1}[\Delta U_{2t}U_{k(2t+r)}-V_{(2t+r)(k+1)}]+U_{2t}V_{2t+r}}{\Delta U_{2t}^2-\Delta U_{2t}U_{2t+r}-1}$$

and

$$\sum_{0\le i,j\le k}\binom{i}{j}(-1)^j V_{ri+4tj}$$
$$=\frac{\Delta^{\frac{k+2}{2}}U_{2t}^{k+1}[U_{2t}V_{k(2t+r)}-U_{(2t+r)(k+1)}]-\Delta U_{2t}U_{2t+r}-2}{\Delta U_{2t}^2-\Delta U_{2t}U_{2t+r}-1}.$$

b) For positive odd k,

$$\sum_{0\le i,j\le k}\binom{i}{j}(-1)^j U_{ri+4tj}$$
$$=\frac{\Delta^{\frac{k+1}{2}}U_{2t}^{k+1}[U_{2t}V_{k(2t+r)}-U_{(2t+r)(k+1)}]-U_{2t}V_{2t+r}}{-\Delta U_{2t}^2+\Delta U_{2t}U_{2t+r}+1}$$

and

$$\sum_{0\leq i,j\leq k}\binom{i}{j}(-1)^{j}V_{ri+4tj}$$
$$=\frac{\Delta^{\frac{k+1}{2}}U_{2t}^{k+1}[\Delta U_{2t}U_{k(2t+r)}+V_{(2t+r)(k+1)}]-\Delta U_{2t}U_{2t+r}-2}{\Delta U_{2t}^{2}-\Delta U_{2t}U_{2t+r}-1}.$$

We give another consequence of Lemma 1 by taking $-x$ instead of x and$-y$ instead of y : For any real numbers x and y such that $x(1-y)\neq -1$

$$\sum_{0\leq i,j\leq k}\binom{i}{j}(-1)^{i+j}x^{i}y^{j}=\frac{(-1)^{k}\left(x-xy\right)^{k+1}+1}{x-xy+1}. \tag{3.3}$$

By using Eq. (3.3), similar to the previous results, we have the following results without proof.

Theorem 8. *For any integer t and odd integer r,*
a)For nonnegative even k,

$$\sum_{0\leq i,j\leq k}\binom{i}{j}(-1)^{i+j}U_{ri+4tj}$$
$$=\frac{\Delta^{\frac{k}{2}}U_{2t}^{k+1}[\Delta U_{2t}U_{k(2t+r)}-V_{(2t+r)(k+1)}]+U_{2t}V_{2t+r}}{\Delta U_{2t}^{2}-\Delta U_{2t}U_{2t+r}+1}$$

and

$$\sum_{0\leq i,j\leq k}\binom{i}{j}(-1)^{i+j}V_{ri+4tj}$$
$$=\frac{\Delta^{\frac{k+2}{2}}U_{2t}^{k+1}[U_{2t}V_{k(2t+r)}-U_{(2t+r)(k+1)}]-\Delta U_{2t}U_{2t+r}+2}{\Delta U_{2t}^{2}-\Delta U_{2t}U_{2t+r}+1}.$$

b)For positive odd k,

$$\sum_{0\leq i,j\leq k}\binom{i}{j}(-1)^{i+j}U_{ri+4tj}$$
$$=\frac{\Delta^{\frac{k+1}{2}}U_{2t}^{k+1}[U_{2t}V_{k(2t+r)}-U_{(2t+r)(k+1)}]+U_{2t}V_{2t+r}}{\Delta U_{2t}^{2}-\Delta U_{2t}U_{2t+r}+1}$$

and

$$\sum_{0\leq i,j\leq k}\binom{i}{j}(-1)^{i+j}V_{ri+4tj}$$
$$=\frac{\Delta^{\frac{k+1}{2}}U_{2t}^{k+1}[\Delta U_{2t}U_{k(2t+r)}-V_{(2t+r)(k+1)}]-\Delta U_{2t}U_{2t+r}+2}{\Delta U_{2t}^{2}-\Delta U_{2t}U_{2t+r}+1}.$$

Theorem 9. *For odd integers t and r,*

$$\sum_{0\le i,j\le k}\binom{i}{j}(-1)^{i+j}U_{ri+2tj}=\frac{V_t^{k+1}[V_tU_{k(t+r)}-U_{(t+r)(k+1)}]+V_tU_{t+r}}{V_t^2-V_tV_{t+r}+1}$$

and

$$\sum_{0\le i,j\le k}\binom{i}{j}(-1)^{i+j}V_{ri+2tj}=\frac{V_t^{k+1}[V_tV_{k(t+r)}-V_{(t+r)(k+1)}]-V_tV_{t+r}+2}{V_t^2-V_tV_{t+r}+1}.$$

Theorem 10. *For any integer t and even r,*

a)For nonnegative even k,

$$\sum_{0\le i,j\le k}\binom{i}{j}(-1)^{i+j}U_{ri+4tj}=\frac{\Delta^{\frac{k}{2}}U_{2t}^{k+1}[\Delta U_{2t}U_{k(2t+r)}+V_{(2t+r)(k+1)}]-U_{2t}V_{2t+r}}{\Delta U_{2t}^2+\Delta U_{2t}U_{2t+r}-1}$$

and

$$\sum_{0\le i,j\le k}\binom{i}{j}(-1)^{i+j}V_{ri+4tj}=\frac{\Delta^{\frac{k+2}{2}}U_{2t}^{k+1}[U_{2t}V_{k(2t+r)}+U_{(2t+r)(k+1)}]+\Delta U_{2t}U_{2t+r}-2}{\Delta U_{2t}^2+\Delta U_{2t}U_{2t+r}-1}.$$

b)For positive odd k,

$$\sum_{0\le i,j\le k}\binom{i}{j}(-1)^{i+j}U_{ri+4tj}=\frac{\Delta^{\frac{k+1}{2}}U_{2t}^{k+1}[U_{2t}V_{k(2t+r)}+U_{(2t+r)(k+1)}]-U_{2t}V_{2t+r}}{\Delta U_{2t}^2+\Delta U_{2t}U_{2t+r}-1}$$

and

$$\sum_{0\le i,j\le k}\binom{i}{j}(-1)^{i+j}V_{ri+4tj}=\frac{\Delta^{\frac{k+1}{2}}U_{2t}^{k+1}[\Delta U_{2t}U_{k(2t+r)}+V_{(2t+r)(k+1)}]+\Delta U_{2t}U_{2t+r}-2}{\Delta U_{2t}^2+\Delta U_{2t}U_{2t+r}-1}.$$

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