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A variant of the reciprocal super Catalan matrix

Abstract: Recently Prodinger [8] considered the reciprocal super Catalan matrix and gave explicit formulæ for its LU-decomposition, the LU-decomposition of its inverse, and obtained some related matrices. For all results, q-analogues were also presented. In this paper, we define and study a variant of the reciprocal super Catalan matrix with two additional parameters. Explicit formulæ for its LU-decomposition, LU-decomposition of its inverse and the Cholesky decomposition are obtained. For all results, q-analogues are also presented.

Keywords: Super Catalan numbers; LU-decomposition; Cholesky decomposition, q-analogues.

2010: 15B05, 05A05.

1 Introduction

For a given sequence $a_0,\ a_1,\ldots,$ the Hankel matrix is defined as

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix},$$

see [6].

Taking certain special number sequences instead of $\{a_n\}$, some authors have defined and studied various combinatorial matrices, see [2, 4, 5, 9]. There are also papers that concentrated on the reciprocals of a given sequence: The Hilbert matrix is defined by

$$\mathcal{H}_n = \left[\frac{1}{i+j+1}\right],\quad 0 \le i,j \le n,\ n = 0,1,\ldots,$$

see [1–3]. As a second example, the Filbert matrix is given by

$$\mathcal{F}_n = \left[\frac{1}{F_{i+j+1}}\right],\quad 0 \le i,j \le n,\ n = 0,1,\ldots,$$

where F_n is the nth Fibonacci number, see [5, 9]. The last one is the reciprocal Pascal matrix with entries

$$\mathcal{R}_n = \left[\binom{i+j}{i}^{-1}\right],\quad 0 \le i,j,$$

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see [10].

Recently Prodinger [8] studied the matrix $\mathcal{A}$ whose entries are the super Catalan numbers, $\frac{(2i)!(2j)!}{i!j!(i+j)!}$. He derived analogues results for the matrix A with entries $\frac{i!j!(i+j)!}{(2i)!(2j)!}$. He also studied q-analogues of these matrices.

The Gaussian q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}},$$

where $(x;q)_n$ is the q-Pochhammer symbol defined by

$$(x;q)_n = (1-x)(1-xq)\dots\left(1-xq^{n-1}\right).$$

Note that

$$\lim_{q\to 1}\begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k},$$

where $\binom{n}{k}$ is the usual binomial coefficient.

We can rewrite the entries of A via the usual binomial coefficients as follows

$$A_{i,j} = \binom{2i}{i}^{-1}\binom{2j}{j}^{-1}\binom{i+j}{i}.$$

In the present paper, we define the matrix $M = [m_{i,j}]$ with

$$m_{i,j} = \binom{2i+r}{i}^{-1}\binom{2j+s}{j}^{-1}\binom{i+j}{i}^{-1}$$

for $0 \le i, j < n$ and nonnegative integers r and s. By the definitions of A and M, note that the matrix M is a variant of the matrix A with two additional parameters. First we derive explicit expressions for the LU-decomposition of M which leads to a formula for the determinant via $\prod_{0\le i<n} U_{i,i}$. Further, we have expressions for the matrices $L^{-1}$and U^{-1}. Similarly we have explicit expressions for the LU-decomposition of $M^{-1} = AB$, and, for A^{-1}, B^{-1}. The latter expressions depend on the size N of the matrix M^{-1}. Via this decomposition, we find that the entries of M^{-1} are integers. We denote the q-analogue of the matrix M by $\mathcal{M}$. We also give formulæ for the q-analogues of all these results. Finally we give expressions for the Cholesky decomposition of M when the matrix is symmetric.

To prove the claimed result, we firstly guess relevant quantities and then use the q-Zeilberger algorithm (for more details, see [7, 11, 12]) to justify relevant equalities.

2 Decomposition of M

For the matrix M, we list here the formulæ that were found; all indices start at $(0,0)$.

$$L_{i,j} = \binom{i}{j}\binom{2j}{j}\binom{2j+r}{j}\binom{i+j}{j}^{-1}\binom{2i+r}{i}^{-1},$$

$$L^{-1}_{i,j} = (-1)^{i+j}\binom{i}{j}\binom{i+j-1}{j}\binom{2j+r}{j}\binom{2i-1}{i}^{-1}\binom{2i+r}{i}^{-1},$$

$$U_{i,j} = (-1)^i \binom{j}{i} \binom{2i-1}{i}^{-1} \binom{i+j}{i}^{-1} \binom{2i+r}{i}^{-1} \binom{2j+s}{j}^{-1},$$

$$U^{-1}_{i,j} = (-1)^i \binom{j}{i} \binom{i+j-1}{i} \binom{2j}{j} \binom{2i+s}{i} \binom{2j+r}{j},$$

$$A_{i,j} = (-1)^{i+j} \binom{i}{j} \binom{N}{i} \binom{N+i}{i} \binom{2i+s}{i} \binom{N}{j}^{-1} \binom{N+j}{j}^{-1} \binom{2j+s}{j}^{-1},$$

$$A^{-1}_{i,j} = \binom{i}{j} \binom{N}{i} \binom{N+i}{i} \binom{2i+s}{i} \binom{N}{j}^{-1} \binom{N+j}{j}^{-1} \binom{2j+s}{j}^{-1},$$

$$B_{i,j} = (-1)^{N+j} \binom{j}{i} \binom{N}{j} \binom{N+j}{j} \binom{2j+r}{j} \binom{2i+s}{i},$$

$$B^{-1}_{i,j} = (-1)^{N+j} \binom{j}{i} \binom{N}{i}^{-1} \binom{N+i}{i}^{-1} \binom{2i+r}{i}^{-1} \binom{2j+s}{j}^{-1}.$$

Since $\det M_N = \prod_{0 \le i < N} U_{i,i}$, we have the following formula

$$\det M_N = (-1)^{\binom{N}{2}} \prod_{0 \le i < N} \binom{2i-1}{i}^{-1} \binom{2i}{i}^{-1} \binom{2i+r}{i}^{-1} \binom{2i+s}{i}^{-1}.$$

Now we give related proofs in some detail. Note that instead of proving that $AB = M^{-1}$ it is more convenient to prove the equivalent $B^{-1}A^{-1} = M$.

$$\begin{aligned}
\sum_k L_{i,k} L^{-1}_{k,j} &= \frac{i!i!i!(i+r)!(2j+r)!}{j!j!j!(j+r)!(2i+r)!} \sum_k (-1)^{k+j} 2k \frac{(k+j-1)!}{(i+k)!(i-k)!(k-j)!} \\
&= \frac{i!i!i!(i+r)!(2j+r)!(2j-1)!}{j!j!j!(j+r)!(2i+r)!(2i)!} \sum_k (-1)^{k+j} 2k \binom{2i}{i+k} \binom{k+j-1}{k-j} \\
&= \frac{i!i!i!(i+r)!(2j+r)!(2j-1)!}{j!j!j!(j+r)!(2i+r)!(2i)!} (2j\,[i=j]) \\
&= [i=j].
\end{aligned}$$

$$\begin{aligned}
&\sum_k U_{i,k} U^{-1}_{k,j} \\
&= \frac{i!i!(i-1)!(i+r)!(2j)!(2j+r)!}{j!j!(j-1)!(j+r)!(2i-1)!(2i+r)!} \sum_k (-1)^{i+k} \frac{(k+j-1)!}{(i+k)!(k-i)!(j-k)!} \\
&= \frac{i!i!(i-1)!(i+r)!(2j)!(2j+r)!}{j!j!(j-1)!(j+r)!(2i-1)!(2i+r)!} \sum_k (-1)^{i+k} \frac{1}{2k} \binom{2k}{i+k} \binom{k+j-1}{j-k} \\
&= \frac{i!i!(i-1)!(i+r)!(2j)!(2j+r)!}{j!j!(j-1)!(j+r)!(2i-1)!(2i+r)!} \left(\frac{1}{2j}\,[i=j]\right) \\
&= [i=j].
\end{aligned}$$

$$\begin{aligned}
&\sum_k A_{i,k}A^{-1}_{k,j}\\
&=\frac{j!j!(N-j)!(j+s)!(N+i)!(2i+s)!}{i!i!(N-i)!(i+s)!(N+j)!(2j+s)!}\sum_k(-1)^{i+k}\frac{1}{(i-k)!(k-j)!}\\
&=\frac{j!j!(N-j)!(j+s)!(N+i)!(2i+s)!}{i!i!(i-j)!(N-i)!(i+s)!(N+j)!(2j+s)!}\sum_k(-1)^{i+k}\binom{i-j}{k-j}\\
&=\frac{j!j!(N-j)!(j+s)!(N+i)!(2i+s)!}{i!i!(i-j)!(N-i)!(i+s)!(N+j)!(2j+s)!}[i=j]\\
&=[i=j]\,.
\end{aligned}$$

$$\begin{aligned}
&\sum_k B_{i,k}B^{-1}_{k,j}\\
&=\frac{j!j!(j+s)!(2i+s)!}{i!i!(i+s)!(2j+s)!}\sum_k(-1)^{j+k}\frac{1}{(k-i)!(j-k)!}\\
&=\frac{j!j!(j+s)!(2i+s)!}{i!i!(j-i)!(i+s)!(2j+s)!}\sum_k(-1)^{j+k}\binom{j-i}{k-i}\\
&=\frac{j!j!(j+s)!(2i+s)!}{i!i!(j-i)!(i+s)!(2j+s)!}[i=j]\\
&=[i=j]\,.
\end{aligned}$$

$$\begin{aligned}
&\sum_k B^{-1}_{i,k}A^{-1}_{k,j}\\
&=\frac{i!i!j!j!(i+r)!(j+s)!}{(2i+r)!(2j+s)!}\sum_k\frac{(-1)^{N+k}(N-i)!(N+k)!(N-j)!}{(N-k)!(k-i)!(N+j)!(k-j)!(N+i)!}\\
&=\frac{i!i!j!j!(i+r)!(j+s)!}{(2i+r)!(2j+s)!(i+j)!}\sum_k(-1)^{N+k}\binom{N-i}{k-i}\binom{N+k}{k-j}\binom{N+i}{i+j}^{-1}\\
&=\frac{i!i!j!j!(i+r)!(j+s)!}{(2i+r)!(2j+s)!(i+j)!}\\
&=M.
\end{aligned}$$

Now we compute an arbitrary entry of AB :

$$\begin{aligned}
&\sum_k A_{i,k}B_{k,j}\\
&=\frac{(2i+s)!(2j+r)!}{i!i!j!j!(i+s)!(j+r)!}\sum_k(-1)^{N+i+j+k}\frac{(N+i)!(N-k)!(N+j)!}{(i-k)!(N+k)!(j-k)!(N-j)!(N-i)!}\\
&=\frac{(2i+s)!(2j+r)!(i+j)!}{i!i!j!j!(i+s)!(j+r)!}\sum_k(-1)^{N+i+j+k}\binom{N+i}{i-k}\binom{N-k}{j-k}\binom{N+j}{i+j}\\
&=\binom{2i+s}{i}\binom{2j+r}{j}\binom{i+j}{i}\sum_k(-1)^{N+i+j+k}\binom{N+i}{i-k}\binom{N-k}{j-k}\binom{N+j}{i+j}.
\end{aligned}$$

3 Decomposition of the matrix $\mathcal{M}$

First we recall that the matrix $\mathcal{M}$ has entries $\begin{bmatrix}2i+r\\ i\end{bmatrix}_q^{-1}\begin{bmatrix}2j+s\\ j\end{bmatrix}_q^{-1}\begin{bmatrix}i+j\\ i\end{bmatrix}_q^{-1}$. Then we have the following formulæ without proofs:

$$L_{i,j}=\begin{bmatrix}i\\ j\end{bmatrix}_q\begin{bmatrix}2j\\ j\end{bmatrix}_q\begin{bmatrix}2j+r\\ j\end{bmatrix}_q\begin{bmatrix}i+j\\ j\end{bmatrix}_q^{-1}\begin{bmatrix}2i+r\\ i\end{bmatrix}_q^{-1},$$

$$L_{i,j}^{-1}=(-1)^{i+j}q^{\binom{i-j}{2}}\begin{bmatrix}i\\ j\end{bmatrix}_q\begin{bmatrix}i+j-1\\ j\end{bmatrix}_q\begin{bmatrix}2j+r\\ j\end{bmatrix}_q\begin{bmatrix}2i-1\\ i\end{bmatrix}_q^{-1}\begin{bmatrix}2i+r\\ i\end{bmatrix}_q^{-1},$$

$$U_{i,j}=(-1)^{i}q^{i(3i-1)/2}\begin{bmatrix}j\\ i\end{bmatrix}_q\begin{bmatrix}2i-1\\ i\end{bmatrix}_q^{-1}\begin{bmatrix}i+j\\ i\end{bmatrix}_q^{-1}\begin{bmatrix}2i+r\\ i\end{bmatrix}_q^{-1}\begin{bmatrix}2j+s\\ j\end{bmatrix}_q^{-1},$$

$$U_{i,j}^{-1}=(-1)^{i}q^{\binom{i+1}{2}-j(j+i)}\begin{bmatrix}j\\ i\end{bmatrix}_q\begin{bmatrix}i+j-1\\ i\end{bmatrix}_q\begin{bmatrix}2j\\ j\end{bmatrix}_q\begin{bmatrix}2i+s\\ i\end{bmatrix}_q\begin{bmatrix}2j+r\\ j\end{bmatrix}_q,$$

$$A_{i,j}=(-1)^{i+j}q^{N(j-i)+\binom{i-j+1}{2}}\begin{bmatrix}i\\ j\end{bmatrix}_q\begin{bmatrix}N\\ i\end{bmatrix}_q\begin{bmatrix}N+i\\ i\end{bmatrix}_q\begin{bmatrix}2i+s\\ i\end{bmatrix}_q\begin{bmatrix}N\\ j\end{bmatrix}_q^{-1}\times\begin{bmatrix}N+j\\ j\end{bmatrix}_q^{-1}\begin{bmatrix}2j+s\\ j\end{bmatrix}_q^{-1},$$

$$A_{i,j}^{-1}=q^{(j-i)(N-i)}\begin{bmatrix}i\\ j\end{bmatrix}_q\begin{bmatrix}N\\ i\end{bmatrix}_q\begin{bmatrix}N+i\\ i\end{bmatrix}_q\begin{bmatrix}2i+s\\ i\end{bmatrix}_q\begin{bmatrix}N\\ j\end{bmatrix}_q^{-1}\begin{bmatrix}N+j\\ j\end{bmatrix}_q^{-1}\begin{bmatrix}2j+s\\ j\end{bmatrix}_q^{-1},$$

$$B_{i,j}=(-1)^{N+j}q^{\binom{j+1}{2}-\binom{N+1}{2}-Nj+i^2}\begin{bmatrix}j\\ i\end{bmatrix}_q\begin{bmatrix}N\\ j\end{bmatrix}_q\begin{bmatrix}N+j\\ j\end{bmatrix}_q\begin{bmatrix}2j+r\\ j\end{bmatrix}_q\begin{bmatrix}2i+s\\ i\end{bmatrix}_q,$$

$$B_{i,j}^{-1}=(-1)^{N+j}q^{\binom{N+1}{2}-\binom{j+1}{2}+i(N-j)}\begin{bmatrix}j\\ i\end{bmatrix}_q\begin{bmatrix}N\\ i\end{bmatrix}_q^{-1}\begin{bmatrix}N+i\\ i\end{bmatrix}_q^{-1}\begin{bmatrix}2i+r\\ i\end{bmatrix}_q^{-1}\begin{bmatrix}2j+s\\ j\end{bmatrix}_q^{-1}.$$

4 Cholesky Decomposition of M

Related with the Cholesky decomposition of matrix $M=CC^T$, we have that for $r=s$

$$C_{i,j}=\mathbf{i}^j\sqrt{2}\binom{i}{j}\binom{2i+r}{i}^{-1}\binom{i+j}{j}^{-1}$$

and

$$C_{i,j}^{-1}=(-1)^j\mathbf{i}^i\sqrt{2}\binom{i}{j}\binom{i+j-1}{j}\binom{2j+r}{j}.$$

Thus we write

$$\begin{aligned}\sum_k C_{i,k}C_{k,j}^{-1} &= \frac{i!i!i!(i+r)!(2j+r)!}{j!j!j!(j+r)!(2i+r)!}\sum_k (-1)^{j-k}2k\frac{(k+j-1)!}{(i-k)!(i+k)!(k-j)!}\\ &= \frac{i!i!i!(i+r)!(2j+r)!(2j-1)!}{j!j!j!(j+r)!(2i+r)!(2i)!}\sum_k (-1)^{j-k}2k\binom{2i}{i+k}\binom{k+j-1}{k-j}\\ &= \frac{i!i!i!(i+r)!(2j+r)!(2j-1)!}{j!j!j!(j+r)!(2i+r)!(2i)!}(2j[i=j])\\ &= [i=j].\end{aligned}$$

The q-analogues of matrix C and its inverse are

$$\mathcal{C}_{i,j} = \mathbf{i}^j q^{j(3j-1)/4}(1+q^j)^{1/2}\begin{bmatrix} i\\ j\end{bmatrix}_q \begin{bmatrix} 2i+r\\ i\end{bmatrix}_q^{-1}\begin{bmatrix} i+j\\ j\end{bmatrix}_q^{-1}$$

and

$$\mathcal{C}_{i,j}^{-1} = (-1)^j\mathbf{i}^i q^{\binom{i-j}{2}-i(3i-1)/4}(1+q^i)^{1/2}\begin{bmatrix} i\\ j\end{bmatrix}_q \begin{bmatrix} i+j-1\\ j\end{bmatrix}_q \begin{bmatrix} 2j+r\\ j\end{bmatrix}_q .$$

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