NEW SUMS IDENTITIES IN WEIGHTED CATALAN TRIANGLE WITH THE POWERS OF GENERALIZED FIBONACCI AND LUCAS NUMBERS

EMRAH KILIÇ AND AYNUR YALÇINER

Abstract. In this paper, we consider a generalized Catalan triangle defined by

$$\frac{k^m}{n}\binom{2n}{n-k}$$

for positive integer m. Then we compute the weighted half binomial sums with the certain powers of generalized Fibonacci and Lucas numbers of the form

$$\sum_{k=0}^{n}\binom{2n}{n+k}\frac{k^m}{n}X_{tk}^r,$$

where X_n either generalized Fibonacci or Lucas numbers, t and r are integers for $1 \leq m \leq 6$. After we describe a general methodology to show how to compute the sums for further values of m.

1. Introduction

Shapiro [6] derived the following triangle similar to Pascal's triangle with entries given by

$$B_{n,k} = \frac{k}{n}\binom{2n}{n-k},$$

which called *Catalan triangle* because the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ are the entries in the first column.

Shapiro derived sums identities from the Catalan triangle. For example, he gave the following identities:

$$\sum_{p=1}^{n}(B_{n,p})^2 = C_{2n-1} \quad \text{and} \quad \sum_{p=1}^{n} B_{n,p}B_{n+1,p} = C_{2n}.$$

We also refer to [5] and references therein for other examples.

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The authors [4] gave also an alternative proof of the identities above and established the following identity:

$$\sum_{p=1}^{n} (pB_{n,p})^2 = (3n-2)\, C_{2(n-1)}.$$

In a somewhat different from the Catalan triangle, Kılıç and Ionascu [2] derived the following result: for any $a \in \mathbb{C} - \{0\}$,

$$\sum_{p=1}^{n} \binom{2n}{n+k} \left(a^k + a^{-k}\right) = \frac{1}{a^n} (a+1)^{2n} + (n+1)\, C_n.$$

The authors also gave applications to the generalized Fibonacci and Lucas sequences, defined by

$$\begin{aligned} U_n &= AU_{n-1} + U_{n-2}, \\ V_n &= AV_{n-1} + V_{n-2}, \end{aligned}$$

where $U_0 = 0,\ U_1 = 1$, and $V_0 = 2,\ V_1 = A$, respectively.

The Binet forms are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n$$

where $\alpha, \beta = (A \pm \sqrt{\Delta})/2$ and $\Delta = A^2 + 4$.

For example, we recall one result from [2]:

$$\sum_{k=0}^{n} \binom{2n}{n+k} U_k^{2r} = \begin{cases} \left(A^2+4\right)^{-r} \left(\binom{2r}{r} 2^{2n-2} + \sum\limits_{t=0}^{r-1} (-1)^{t(n+1)} \binom{2r}{t} V_{r-t}^{2n} \right) & \text{if } r \text{ is even}, \\ \left(A^2+4\right)^{n-r} \sum\limits_{t=0}^{r-1} (-1)^{t(n+1)} \binom{2r}{t} U_{r-t}^{2n} & \text{if } r \text{ is odd}. \end{cases}$$

We define a generalized Catalan triangle by taking m^{th} power of summation index k as follow:

$$D_{n,k}(m) = \frac{k^m}{n} \binom{2n}{n-k}.$$

When $m = 1$, the generalized Catalan triangle is reduced to the usual Catalan triangle $B_{n,p}$.

In [3], the author considered and computed certain binomial sums weighted by the powers of the summation index.

In this paper we consider the sums of the forms: for all nonnegative integer m and $a \in \mathbb{C} \setminus \{0\}$

$$S(n,m,a) := \sum_{k=0}^{n} \binom{2n}{n+k} \frac{k^m}{n} \left(a^k + (-1)^m a^{-k}\right).$$

The sums $S(n,0,a)$ were considered and exactly computed in [2]. We first exactly compute the sums $S(n,1,a)$. Then by using the value of $S(n,1,a)$, we compute $S(n,2,a)$. So we will compute $S(n,m,a)$ by using the value of $S(n,m-1,a)$ for the value of m, $m = 2, ..., 6$. Then we describe a general methodology to compute further values of $S(n,m,a)$ for $m > 6$. Also we present applications of our results.

2. New Sums Identities From the Catalan Triangle

Firstly we compute $S(n,1,a)$. Before it we need to evaluate a partial binomial sums by the following lemma. For partial binomial sums, we may refer to [1].

Lemma 1. *For any nonnegative integer t,*

$$\sum_{j=0}^{t} \binom{2n}{j} \left(1 - \frac{j}{n}\right) = \binom{2n-1}{t}.$$

Proof. (By induction on t) For $t = 0$, the claim is obvious. Suppose that the claim is true for k. We show that the claim is true for $k+1$. Consider

$$\sum_{j=0}^{t+1} \binom{2n}{j} \left(1 - \frac{j}{n}\right) = \binom{2n}{t+1} \left(1 - \frac{t+1}{n}\right) + \sum_{j=0}^{t} \binom{2n}{j} \left(1 - \frac{j}{n}\right),$$

which, by the induction hypothesis, equals

$$\binom{2n}{t+1} \left(1 - \frac{t+1}{n}\right) + \binom{2n-1}{t},$$

which, by using the recursion of the binomial coefficient and the property

$$\binom{n+1}{k+1} = \frac{n+1}{k+1} \binom{n}{k},$$

gives us

$$\begin{aligned}
& \frac{2n}{t+1}\binom{2n-1}{t}\left(1-\frac{t+1}{n}\right)+\binom{2n-1}{t} \\
= & \binom{2n-1}{t}\left[\frac{2n}{t+1}\left(1-\frac{t+1}{n}\right)+1\right] \\
= & \binom{2n-1}{t}\left[\frac{2n}{t+1}-1\right]=\binom{2n}{t+1}-\binom{2n-1}{t} \\
= & \binom{2n-1}{t+1},
\end{aligned}$$

as claimed. □

Now we start with our first result.

Theorem 1. *For* $n>0$

$$S(n,1,a)=\sum_{k=0}^{n}\binom{2n}{n-k}\frac{k}{n}\left(a^{k}-a^{-k}\right)=\frac{1}{a^{n}}(a-1)(a+1)^{2n-1}.$$

Proof. Consider

$$\frac{1}{a-1}\sum_{k=0}^{n}\binom{2n}{n-k}k\left(a^{n+k}-a^{n-k}\right),$$

which equals

$$\begin{aligned}
\sum_{k=0}^{n}\binom{2n}{n-k}ka^{n-k}\left(\frac{a^{2k}-1}{a-1}\right) &= \sum_{k=0}^{n}\binom{2n}{n-k}ka^{n-k}\sum_{j=0}^{2k}a^{j} \\
=\sum_{k=0}^{n}\binom{2n}{n-k}k\sum_{j=0}^{2k}a^{n-k+j} &= \sum_{t=0}^{2n}\sum_{j=0}^{t}\binom{2n}{j}(n-j)\,a^{t},
\end{aligned}$$

which, by Lemma 1, equals

$$\sum_{t=0}^{2n}n\binom{2n-1}{t}a^{t}=n(a+1)^{2n-1},$$

which settles the proof. □

As a result of Theorem 1, by taking $-a$ instead of a, we have the following Corollary:

Corollary 1. *For* $n>0$

$$\sum_{k=0}^{n}\binom{2n}{n+k}(-1)^{k}k\left(a^{k}-a^{-k}\right)=\frac{n(-1)^{n}}{a^{n}}(a+1)(a-1)^{2n-1}.$$

As a variant of the result of Theorem 1, we have that

$$\sum_{k=0}^{n}\binom{2n}{n-k}\frac{k}{n}\left(a^{n+k}-a^{n-k}\right)=(a-1)(a+1)^{2n-1},$$

which is a polynomial in a. Second we give the result:

Corollary 2. *For $n>0$*

$$S(n,2,a)=\frac{1}{a^{n}}(a+1)^{2n-2}\Big(n(a+1)^{2}-2a(2n-1)\Big).$$

Proof. Consider derivation of the RHS of $S(n,1,a)$:

$$\begin{aligned}\frac{d}{da}\sum_{k=0}^{n}\binom{2n}{n+k}\frac{k}{n}\left(a^{k}-a^{-k}\right) &= \sum_{k=0}^{n}\binom{2n}{n+k}\frac{k}{n}\frac{d}{da}\left(a^{k}-a^{-k}\right)\\ &= \sum_{k=0}^{n}\binom{2n}{n+k}\frac{k^{2}}{n}\left(a^{k-1}+a^{-k-1}\right)\\ &= \frac{1}{a}\sum_{k=0}^{n}\binom{2n}{n+k}\frac{k^{2}}{n}\left(a^{k}+a^{-k}\right)\\ &= \frac{1}{a}S(n,2,a).\end{aligned}$$

On the other hand by taking derivation of the LHS of $S(n,1,a)$ gives

$$\begin{aligned}\frac{d}{da}S(n,1,a) &= \frac{d}{da}\left(\frac{1}{a^{n}}(a-1)(a+1)^{2n-1}\right)\\ &= \frac{1}{a^{n+1}}(a+1)^{2n-2}\Big(n(a+1)^{2}-2a(2n-1)\Big).\end{aligned}$$

Thus

$$S(n,2,a)=\frac{1}{a^{n}}(a+1)^{2n-2}\Big(n(a+1)^{2}-2a(2n-1)\Big),$$

as claimed. □

We see that by taking derivation of $S(n,1,a)$, we obtain exact formula for $S(n,2,a)$. The process of taking consecutive derivatives could be continued and so we get

$$S(n,3,a)=a^{-n}(a-1)(a+1)^{2n-3}\Big[n^{2}(a+1)^{2}-a(2n-1)(2n-2)\Big],$$

$$\begin{aligned}S(n,4,a)=a^{-n}(a+1)^{2n-4}&[n^{3}(a+1)^{4}-2a(2n-1)\\ &\times\Big((2n(n-1)+1)(a+1)^{2}-(2n-2)(2n-3)a\Big)],\end{aligned}$$

$$S(n,5,a)=a^{-n}(a-1)(a+1)^{2n-5}[n^4(a+1)^4-a(2n-1)(2n-2)$$
$$\times\Big((2n(n-1)+1)(a+1)^2-(2n-3)(2n-4)a\Big)]$$

and so

$$S(n,6,a)=a^{-n}(a+1)^{2n-6}[n^5(a+1)^6-2a(2n-1)[(a+1)^4$$
$$+(n-1)\left(3n^3-3n^2+4n\right)(a+1)^4-2a(2n-3)$$
$$\times\left((a+1)^2\left(3n^2+5-6n\right)-2a\left(2n^2-9n+10\right)\right)\Big]\Big].$$

Generally taking derivative of $S(n,m,a)$ gives $a^{-1}S(n,m+1,a)$. Since we can't find an operator or a general recursion rule for them, we couldn't derive a closed formula for the further values of $S(n,m,a)$. We leave this problem is an open problem.

3. New Weighted Half Binomial Sums

In this section, we present some applications of our results in order to weighted analogues of the results given [2] including powers of the summation index with the even or odd powers of terms of the generalized binary linear recurrences $\{U_n\}$ and $\{V_n\}$ whose indices are also in arithmetic progressions as well as their alternating analogues. We prove the first result, others could be similarly derived.

Theorem 2. *Let $n\in\mathbb{N}\cup\{0\}$ and $r\in\mathbb{N}$. If r is even,*

$$\sum_{k=0}^{n}\binom{2n}{n+k}k^2U_{tk}^{2r}=\frac{n}{\Delta^r}\binom{2r}{r}2^{2n-2}$$
$$+\frac{n}{\Delta^r}\sum_{j=0}^{r-1}(-1)^{j(tn+1)}\binom{2r}{j}V_{t(r-j)}^{2n-2}\left(nV_{t(r-j)}^2-(-1)^{jt}2(2n-1)\right)$$

and, if r is odd and t is even,

$$\sum_{k=0}^{n}\binom{2n}{n+k}k^2U_{tk}^{2r}=-\frac{n}{\Delta^r}\binom{2r}{r}2^{2n-2}$$
$$+\frac{n}{\Delta^r}\sum_{j=0}^{r-1}(-1)^{j}\binom{2r}{j}V_{t(r-j)}^{2n-2}\left(nV_{t(r-j)}^2-2(2n-1)\right),$$

and, for $n>1$, if r and t are odd,

$$\sum_{k=0}^{n}\binom{2n}{n+k}k^2U_{tk}^{2r}=n\Delta^{n-r-1}$$
$$\times\sum_{j=0}^{r-1}(-1)^{j(tn+1)}\binom{2r}{j}U_{t(r-j)}^{2n-2}\left(n\Delta U_{t(r-j)}^{2}-(-1)^j(4n-2)\right).$$

Proof. Expanding U_{tk}^{2r} by the binomial formula, consider

$$\sum_{k=0}^{n}\binom{2n}{n+k}k^2U_{tk}^{2r}$$
$$=\frac{1}{(\alpha-\beta)^{2r}}\sum_{k=0}^{n}\binom{2n}{n+k}k^2\left[\sum_{j=0}^{2r}(-1)^j\binom{2r}{j}\alpha^{(2r-j)tk}\beta^{jtk}\right].$$

Since $\alpha\beta=-1$, we write

$$\sum_{k=0}^{n}\binom{2n}{n+k}k^2U_{tk}^{2r}=\frac{1}{\Delta^r}\sum_{k=0}^{n}\binom{2n}{n+k}k^2\left[(-1)^{r(1+tk)}\binom{2r}{r}\right.$$
$$\left.+\sum_{j=0}^{r-1}(-1)^{j(1+tk)}\binom{2r}{j}(\alpha^{2(r-j)tk}+\alpha^{-2(r-j)tk})\right],$$

which, by changing summation order and Corollary 2, equals

$$\frac{n}{\Delta^r}\frac{(-1)^r}{2}\binom{2r}{r}S(n,2,(-1)^{tr})$$
$$+\frac{n}{\Delta^r}\sum_{j=0}^{r-1}(-1)^j\binom{2r}{j}S(n,2,(-1)^{jt}\alpha^{2t(r-j)})$$
$$=\frac{n}{\Delta^r}\left(\frac{(-1)^r}{2}\binom{2r}{r}\frac{n\left((-1)^{tr}+1\right)^{2n}}{(-1)^{trn}}\right.$$
$$-\frac{(-1)^r}{2}\binom{2r}{r}(2n-1)\frac{\left((-1)^{tr}+1\right)^{2n-2}}{(-1)^{tr(n-1)}}$$
$$+\sum_{j=0}^{r-1}(-1)^j\binom{2r}{j}[n(-1)^{jtn}((-1)^{jt}\alpha^{t(r-j)}+(-1)^{t(r-j)}\beta^{t(r-j)})^{2n}$$
$$\left.-2(2n-1)(-1)^{jt(n-1)}((-1)^{jt}\alpha^{t(r-j)}+(-1)^{t(r-j)}\beta^{t(r-j)})^{2n-2}]\right)$$

which, by the Binet formulas of $\{U_n\}$ and $\{V_n\}$, gives us the claim. □

Theorem 3. *For $n > 0$*

$$\sum_{k=0}^{n}\binom{2n}{n+k}(-1)^k kU_{tk} = (-1)^n nU_t \begin{cases} \Delta^{n-1}U_{t/2}^{2n-2} & \text{if } t \equiv 0 \pmod 4, \\ V_{t/2}^{2n-2} & \text{if } t \equiv 2 \pmod 4, \end{cases}$$

and, for even t,

$$\sum_{k=0}^{n}\binom{2n}{n+k} kU_{tk} = n\ U_t \begin{cases} V_{t/2}^{2n-2} & \text{if } t \equiv 0 \pmod 4, \\ \Delta^{n-1}U_{t/2}^{2n-2} & \text{if } t \equiv 2 \pmod 4. \end{cases}$$

Theorem 4. *Let $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{N}$. For even r,*

$$\sum_{k=0}^{n}\binom{2n}{n+k} k^2 V_{tk}^{2r} = n\binom{2r}{r}2^{2n-2} + n\sum_{j=0}^{r-1}(-1)^{jtn}\binom{2r}{j}V_{t(r-j)}^{2n-2}\left(nV_{t(r-j)}^2 + (-1)^{jt+1}2(2n-1)\right).$$

For odd r,

(i) For even t,

$$\sum_{k=0}^{n}\binom{2n}{n+k} k^2 V_{tk}^{2r} = n\binom{2r}{r}2^{2n-2} + n\sum_{j=0}^{r-1}\binom{2r}{j}V_{t(r-j)}^{2n-2}\left(nV_{t(r-j)}^2 - 2(2n-1)\right), \quad n \geq 0$$

and for odd t,

$$\sum_{k=0}^{n}\binom{2n}{n+k} k^2 V_{tk}^{2r} = n\Delta^{n-1} \times \sum_{j=0}^{r-1}\binom{2r}{j}(-1)^{jn}U_{t(r-j)}^{2n-2}\left(n\Delta U_{t(r-j)}^2 - (-1)^j 2(2n-1)\right), \ n > 1$$

Theorem 5. *For even $t > 0$,*

$$\sum_{k=0}^{n}\binom{2n}{n+k} k^3 U_{tk} = nU_t \begin{cases} V_{t/2}^{2n-4}\left(n^2V_{t/2}^2 - (2n-1)(2n-2)\right) & \text{if } t \equiv 0 \pmod 4, \\ \Delta^{n-2}U_{t/2}^{2n-4}\left(n^2\Delta U_{t/2}^2 - (2n-1)(2n-2)\right) & \text{if } t \equiv 2 \pmod 4, \end{cases}$$

and, for all integer t,

$$\sum_{k=0}^{n}\binom{2n}{n+k}(-1)^k k^3 U_{tk} = (-1)^n n U_t$$
$$\times \begin{cases} \Delta^{n-2}\ U_{t/2}^{2n-4}\left(n^2\Delta U_{t/2}^2 + (2n-1)(2n-2)\right) & \text{if } t \equiv 0 \pmod 4, \\ V_{t/2}^{2n-4}\left(n^2 V_{t/2}^2 + (2n-1)(2n-2)\right) & \text{if } t \equiv 2 \pmod 4. \end{cases}$$

Theorem 6. *For positive even r,*

$$\sum_{k=0}^{n}\binom{2n}{n+k}k^4 V_{tk}^{2r} = n\binom{2r}{r}2^{2n-3}(3n-1) + n\sum_{j=0}^{r-1}(-1)^{jtn}\binom{2r}{j}V_{t(r-j)}^{2n-4}$$
$$\left[n^3 V_{t(r-j)}^4 + (-1)^{jt+1}2(2n-1)(2n^2-2n+1)V_{t(r-j)}^2 + 4(2n-1)(n-1)\ (2n-3)\right],$$

and, for odd r and even t,

$$\sum_{k=0}^{n}\binom{2n}{n+k}k^4 V_{tk}^{2r} = n\binom{2r}{r}2^{2n-3}(3n-1) + n\sum_{j=0}^{r-1}\binom{2r}{j}V_{t(r-j)}^{2n-4}$$
$$\times\left[n^3 V_{t(r-j)}^4 - 2(2n-1)(2n^2-2n+1)V_{t(r-j)}^2 + 4(2n-1)(n-1)(2n-3)\right],$$

and, for $n > 2$ and odd r, t,

$$\sum_{k=0}^{n}\binom{2n}{n+k}k^4 V_{tk}^{2r} = n\Delta^{n-2}\sum_{j=0}^{r-1}(-1)^{jn}\binom{2r}{j}U_{t(r-j)}^{2n-4}\left[n^3\Delta^2 U_{t(r-j)}^4\right.$$
$$\left. - (-1)^j 2(2n-1)\ (2n^2-2n+1)\Delta U_{t(r-j)}^2 + 4(2n-1)(n-1)(2n-3)\right].$$

Theorem 7. *Let t be a positive even integer. If $t \equiv 0 \pmod 4$,*

$$\sum_{k=0}^{n}\binom{2n}{n+k}k^5 U_{tk} = nU_t V_{t/2}^{2n-6}(n^4 V_{t/2}^4 - (2n-1)(2n-2)$$
$$\times (2n^2-2n+1)V_{t/2}^2 + (2n-1)(2n-2)(2n-3)(2n-4)),$$

and, if $t \equiv 2 \pmod 4$,

$$\sum_{k=0}^{n}\binom{2n}{n+k}k^5 U_{tk} = n\Delta^{n-3}U_t U_{t/2}^{2n-6}(n^4\Delta^2 U_{t/2}^4 - (2n-1)(2n-2)$$
$$\times (2n^2-2n+1)\Delta U_{t/2}^2 + (2n-1)(2n-2)(2n-3)(2n-4).$$

By using the presented results, one can derive many sums formulae similar to the above sums formulae.

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TOBB Economics and Technology University, Mathematics Department 06560 Sogutozu Ankara Turkey

E-mail address: ekilic@etu.edu.tr

Selcuk University, Science Faculty, Department of Mathematics, 42075, Campus, Konya, Turkey

E-mail address: ayalciner@selcuk.edu.tr