GENERALIZED BINOMIAL CONVOLUTION OF THE m^{th} POWERS OF THE CONSECUTIVE INTEGERS WITH GENERAL FIBONACCI SEQUENCE

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Abstract. In this paper, we consider Gauthier's generalized convolution and then define its binomial analogue as well as alternating binomial analogue. We formulate these convolutions and give some applications of them.

1. Introduction

The general Fibonacci sequence $\{G_n\}_{n=-\infty}^{+\infty}$ is defined by the recurrence relation

$$G_{n+2} = G_{n+1} + G_n;\ \ G_n = A\alpha^n + B\beta^n,$$

where A, B, α, β are the real numbers and $\alpha + \beta = 1$, $\alpha\beta = -1$.

When $A = -B = (\alpha - \beta)^{-1}$, $G_n = F_n$ (nth Fibonacci number). When $A = B = 1$, $G_n = L_n$ (nth Lucas number).

For $m \geq 0$ and any integers a, b, Gauthier [4] defined the *generalized convolution* of the sequence of powers of the consecutive integers, $\{(a+n)^m\}_{n=-\infty}^{+\infty}$ with the general Fibonacci sequence$\{G_n\}_{n=-\infty}^{+\infty}$ and showed that

$$\begin{aligned}&\sum_{k=0}^{n} (k+a)^m G_{b-k-a}\\ &= \sum_{l=0}^{m} l! \left(c_m^{(l)}(a) G_{b-a+2+l} - c_m^{(l)}(a+n+1) G_{b-a-n+l+1} \right),\end{aligned}$$

where for a an arbitrary variable, the set of coefficients

$$\left\{ c_m^{(l)}(v) : 0 \leq m,\ 0 \leq l \leq m \right\}$$

is the set of Carlitz's weighted Stirling polynomials of the second kind given by the closed formula [2]:

$$c_m^{(l)}(v) = \sum_{k=0}^{m-l} \binom{m}{k} S_{m-k}^{(l)} v^k, \tag{1.1}$$

where $S_m^{(l)}$ are the Stirling numbers of the second kind (for more details see [1]).

Related with summation rules for various types of convolutions and, as a separate area of research, the Stirling numbers and their various generalizations, we refer to the reference list of [4].

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We recall some applications of the results given in [4]:

$$\sum_{k=0}^{n}(n+k)\,G_{n-k}=nG_{n+2}+G_{n+3}-\left[2\,(n+1)\,G_1+G_0\right],$$

$$\sum_{k=0}^{n}kG_{n-k}=G_{n+3}-\left[(n+2)\,G_1+G_0\right].$$

Some authors have also computed various binomial sums by using binomial transformation method, for more details see [8, 9], as well as other techniques such as matrix methods, generating function method and convolution of exponential generating function methods, for more details see [5-10].

In this paper, we consider Gauthier's generalized convolution and then define its binomial analogue, called the generalized binomial convolution, as well as its alternating analogue. We formulate these convolutions and give some applications.

2. A Generalized Binomial Convolution

For $m\geq 0$, a, b integers and A,B real numbers, with $\alpha+\beta=1,\alpha\beta=-1$, define the generalized binomial convolution of the sequence of powers of the consecutive integers, $\{(a+n)^m\}_{n=-\infty}^{+\infty}$ with the general Fibonacci sequence $\{G_n\}_{n=-\infty}^{+\infty}$ and its alternating analogue as follows:

$$\sum_{k=0}^{n}\binom{n}{k}(k+a)^m\,G_{b-k-a}\quad\text{and}\quad\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}(k+a)^m\,G_{b-k-a},$$

respectively.

Now we give closed forms for these binomial convolutions:

Theorem 1. *For $n\geq 0$,*

$$\sum_{k=0}^{n}\binom{n}{k}(k+a)^m\,G_{b-k-a}=\sum_{l=0}^{m}\binom{n}{l}\frac{l!}{n^l}q_m^{(l)}\,(a;n)\,G_{b-a+n-2l},\tag{2.1}$$

where $q_m^{(l)}\,(a;n)$, or briefly $q_m^{(l)}\,(a)$, is given by

$$q_m^{(l)}\,(a)=n^l\sum_{k=0}^{m-l}\binom{m}{k}S_{m-k}^{(l)}a^k,\quad 0\leq l\leq m.$$

Proof. Let $D=x\frac{d}{dx}$ be a differential operator. Consider

$$\sum_{k=0}^{n}\binom{n}{k}x^{k+a}=x^a\,(1+x)^n\,.\tag{2.2}$$

If we act m times on the equation (2.2) with D, for m a nonnegative integer, then we obtain

$$\sum_{k=0}^{n}\binom{n}{k}(k+a)^m\,x^{k+a}=D^m\left(x^a\,(1+x)^n\right).\tag{2.3}$$

To determine $D^m\left(x^a\left(1+x\right)^n\right)$ for any integer a, note that

$$\begin{aligned}
D^0\left(x^a\left(1+x\right)^n\right) &= x^a\left(1+x\right)^n, \\
D^1\left(x^a\left(1+x\right)^n\right) &= x^a\left(a\left(1+x\right)^n+nx\left(1+x\right)^{n-1}\right), \\
D^2\left(x^a\left(1+x\right)^n\right) &= x^a(a^2\left(1+x\right)^n+\left(2a+1\right)nx\left(1+x\right)^{n-1} \\
&\quad +n\left(n-1\right)x^2\left(1+x\right)^{n-2})
\end{aligned}$$

and so the general term being

$$D^m\left(x^a\left(1+x\right)^n\right)=\sum_{l=0}^{m}p_m^{(l)}\left(a\right)x^{a+l}\left(1+x\right)^{n-l}. \tag{2.4}$$

The coefficients $\left\{p_m^{(l)}\left(a\right);0\leq l\leq m\right\}$ may be found as follows: First put $m+1$ in place of m in (2.4) and get

$$D^{m+1}\left(x^a\left(1+x\right)^n\right)=\sum_{l=0}^{m+1}p_{m+1}^{(l)}\left(a\right)x^{a+l}\left(1+x\right)^{n-l}.$$

On the other hand consider the direct action of D on (2.4)

$$\begin{aligned}
&D\left(D^m\left(x^a\left(1+x\right)^n\right)\right) \\
&= D\left(\sum_{l=0}^{m}p_m^{(l)}\left(a\right)x^{a+l}\left(1+x\right)^{n-l}\right) \\
&= \sum_{l=0}^{m}p_m^{(l)}\left(a\right)D\left(x^{a+l}\left(1+x\right)^{n-l}\right) \\
&= \sum_{l=0}^{m}p_m^{(l)}\left(a\right)\left(\left(a+l\right)x^{a+l}\left(1+x\right)^{n-l}+\left(n-l\right)x^{a+l+1}\left(1+x\right)^{n-l-1}\right) \qquad (2.5) \\
&= \sum_{l=0}^{m+1}\left(\left(a+l\right)p_m^{(l)}\left(a\right)+\left(n-l+1\right)p_m^{(l-1)}\left(a\right)\right)x^{a+l}\left(1+x\right)^{n-l},
\end{aligned}$$

where the last line was obtained by shifting the dummy index in the second sum of the second line of (2.5), $l+1\to l$, and by defining $p_m^{(-1)}\left(a\right)\equiv 0$ and $p_m^{(m+1)}\left(a\right)\equiv 0$. From these two equations just above, by equating the coefficients of $x^{a+l}\left(1+x\right)^{n-l}$, we have the following recurrence for the coefficients $p_m^{(l)}$:

$$p_{m+1}^{(l)}\left(a\right)=\left(a+l\right)p_m^{(l)}\left(a\right)+\left(n-l+1\right)p_m^{(l-1)}\left(a\right),\ 0\leq l\leq m+1,$$

where $p_{m=0}^{(l=0)}\left(a\right)\equiv 1$, $p_m^{(-1)}\left(a\right)\equiv 0$, $p_m^{(m+1)}\left(a\right)\equiv 0$.

Now we define a set of new coefficients $q_m^{(l)}$ via $p_m^{(l)}$ as follows:

$$p_m^{(l)}\left(a\right)\equiv\binom{n}{l}\frac{l!}{n^l}q_m^{(l)}\left(a\right),$$

and so we deduce the recurrence relation for the coefficients $q_m^{(l)}$:

$$q_{m+1}^{(l)}\left(a\right)=\left(a+l\right)q_m^{(l)}\left(a\right)+nq_m^{(l-1)}\left(a\right),\ 0\leq l\leq m,$$

where $q_{m=0}^{(l=0)}\left(a\right)\equiv 1, q_m^{(-1)}\left(a\right)\equiv 0$, $q_m^{(m+1)}\left(a\right)\equiv 0$.

The first few polynomials are, for $0 \le m \le 3:$

$$m = 0 : q_0^{(0)}(a) = 1,$$
$$m = 1 : q_1^{(0)}(a) = a, q_1^{(1)}(a) = n,$$
$$m = 2 : q_2^{(0)}(a) = a^2, q_2^{(1)}(a) = 2an + n, q_2^{(2)}(a) = n^2,$$
$$m = 3 : q_3^{(0)}(a) = a^3, q_3^{(1)}(a) = 3a^2n + 3an + n, q_3^{(2)}(a) = 3an^2 + 3n^2,$$
$$q_3^{(3)}(a) = n^3.$$

It is clear from (1.1) that

$$q_m^{(l)}(a) = n^l c_m^{(l)}(a). \tag{2.6}$$

Indeed, for $l = 0$ and $l = m$, it is seen that $q_m^{(0)}(a) = a^m$ and $q_m^{(m)}(a) = n^m$, respectively. For $a = 0$ and $n = 1$, the coefficients $q_m^{(l)}$ has the the same recurrence with the Stirling numbers of second kind. For $n = 1$, the coefficients given in (2.6) are the same with Carlitz's weighted Stirling polynomials of the second kind as a function of the arbitrary variable a. We return to (2.3) and make the following substitution: $x \to x^{-1}$. By (2.4) and (2.3), we write

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(k+a)^m x^{-k-a} &= \sum_{l=0}^{m} p_m^{(l)}(a)\, x^{-a-l}\left(1+x^{-1}\right)^{n-l} \\ &= \sum_{l=0}^{m}\binom{n}{l}\frac{l!}{n^l} q_m^{(l)}(a)\, x^{-a-n}(1+x)^{n-l}, \end{aligned} \tag{2.7}$$

which, multiplying both sides by Ax^b, taking $x = \alpha$ and using $x + 1 = \alpha^2$, gives us

$$\sum_{k=0}^{n}\binom{n}{k}(k+a)^m A\alpha^{b-k-a} = \sum_{l=0}^{m}\binom{n}{l}\frac{l!}{n^l} q_m^{(l)}(a)\, A\alpha^{b-a+n-2l}. \tag{2.8}$$

Finally multiplying both sides of (2.7) by Bx^b but this time set $x = \beta$, $x + 1 = \beta^2$, we get

$$\sum_{k=0}^{n}\binom{n}{k}(k+a)^m B\beta^{b-k-a} = \sum_{l=0}^{m}\binom{n}{l}\frac{l!}{n^l} q_m^{(l)}(a)\, B\beta^{b-a+n-2l} \tag{2.9}$$

Combining (2.8) and (2.9), the desired result is obtained. □

Considering the fact that

$$\sum_{k=0}^{n}(-1)^{k+n}\binom{n}{k}x^{k+a} = x^a(x-1)^n,$$

we have the following result without proof.

Theorem 2. *For $n \ge 0$*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+a)^m G_{b-k-a} = \sum_{l=0}^{m}(-1)^l\binom{n}{l}\frac{l!}{n^l} q_m^{(l)}(a)\, G_{b-a-2n+l}, \tag{2.10}$$

where $q_m^{(l)}(a)$ is defined as before.

3. Some Applications

Now we present some applications of our results. When $m=0$ in (2.1) and (2.10), by $q_0^{(0)}(a)=1$, we get

$$\sum_{k=0}^{n}\binom{n}{k}G_{b-k-a}=G_{b-a+n} \text{ and } \sum_{k=0}^{n}(-1)^k\binom{n}{k}G_{b-k-a}=G_{b-a-2n}.$$

If we take $m=1$ in (2.1) and (2.10), by $q_1^{(0)}(a)=a$, $q_1^{(1)}(a)=n$, we get

$$\sum_{k=0}^{n}\binom{n}{k}(k+a)\,G_{b-k-a}=aG_{b-a+n}+nG_{b-a+n-2}$$

and

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+a)\,G_{b-k-a}=aG_{b-a-2n}-nG_{b-a-2n+1}.$$

When $a=n$ and $b=2n$ in (2.1) and (2.10), we have

$$\sum_{k=0}^{n}\binom{n}{k}(k+n)^m G_{n-k} = \sum_{l=0}^{m}\binom{n}{l}\frac{l!}{n^l}q_m^{(l)}(n)\,G_{2(n-l)}, \quad (3.1)$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+n)^m G_{n-k} = \sum_{l=0}^{m}(-1)^l\binom{n}{l}\frac{l!}{n^l}q_m^{(l)}(n)\,G_{l-n}, \quad (3.2)$$

respectively.

Especially when $m=1$ in (3.1) and (3.2), by $q_1^{(0)}(n)=n$, $q_1^{(1)}(n)=n$, we obtain

$$\sum_{k=0}^{n}\binom{n}{k}(k+n)\,G_{n-k} = n\left(G_{2n}+G_{2(n-1)}\right),$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+n)\,G_{n-k} = n\left(G_{-n}-G_{1-n}\right).$$

Especially for the Fibonacci case, from the results above, we get

$$\sum_{k=0}^{n}\binom{n}{k}(k+n)\,F_{n-k}=nL_{2n-1},$$

and

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+n)\,F_{n-k}=nF_{n+1}.$$

Now, setting $a=0$ and $b=n$ in (2.1) and (2.10) gives the usual number theoretic convolution of m^{th} power of the consecutive integer sequence $\{n^m\}_{n=-\infty}^{\infty}$ with the general Fibonacci sequence $\{G_n\}_{n=-\infty}^{\infty}$:

$$\sum_{k=0}^{n}\binom{n}{k}k^m G_{n-k} = \sum_{l=0}^{m}\binom{n}{l}\frac{l!}{n^l}q_m^{(l)}(0)\,G_{2(n-l)}, \quad (3.3)$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^m G_{n-k} = \sum_{l=0}^{m}(-1)^l\binom{n}{l}\frac{l!}{n^l}q_m^{(l)}(0)\,G_{l-n}, \quad (3.4)$$

respectively.

When $m=1$ in (3.3) and (3.4), by $q_1^{(0)}(0)=0$ and $q_1^{(1)}(0)=n$, we have

$$\sum_{k=0}^{n}\binom{n}{k}kG_{n-k} = nG_{2(n-1)},$$
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}kG_{n-k} = -nG_{1-n}.$$

Similarly, when $m=2$ in (3.3) and (3.4), by $q_2^{(0)}(0)=0$, $q_2^{(1)}(0)=n$ and $q_2^{(2)}(0)=n^2$, we get

$$\sum_{k=0}^{n}\binom{n}{k}k^2G_{n-k} = n\left(G_{2(n-1)}+(n-1)G_{2(n-2)}\right),$$
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^2G_{n-k} = n\left((n-1)G_{2-n}-G_{1-n}\right).$$

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