DOUBLE BINOMIAL SUMS AND DOUBLE SUMS RELATED WITH CERTAIN LINEAR RECURRENCES OF VARIOUS ORDER

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ABSTRACT

In this paper, we derive new double binomial sums families related with generalized second, third and certain higher order linear recurrences. We also present various parametric generalizations of some results of [1, 2]. Finally we present some interesting double sums including only one binomial coefficient related with certain number sequences of various order. We use generating function methods to prove claimed results.

Keywords: Double sums, double binomial sums, Fibonacci number, Pell number, Tribonacci number, generating function.

1. INTRODUCTION

The second order linear recurrence $\{U_n(p,q)\}$, briefly $\{U_n\}$, is defined by for $n>1$

$$U_n = pU_{n-1} + qU_{n-2}, \tag{1.1}$$

with initials $U_0 = 0$ and $U_1 = 1$.

The third order linear recurrence $\{V_n(p,q,r)\}$, briefly $\{V_n\}$, is defined by for $n>1$

$$V_n = pV_{n-1} + qV_{n-2} + rV_{n-3}, \tag{1.2}$$

with initials $V_{-1} = V_0 = 0$ and $V_1 = 1$.

The fourth order linear recurrence $\{W_n(p,q,r,s)\}$, briefly $\{W_n\}$, is defined by for $n>1$

$$W_n = pW_{n-1} + qW_{n-2} + rW_{n-3} + sW_{n-4}, \tag{1.3}$$

with initials $W_{-2} = W_{-1} = W_0 = 0$ and $W_1 = 1$.

The generating functions of the sequences $\{U_{n+1}\},\{V_{n+1}\}$ and $\{W_{n+1}\}$ are

$$\sum_{n\geq 0} U_{n+1}\, z^n = \frac{1}{1-pz-qz^2}, \quad \sum_{n\geq 0} V_{n+1}\, z^n = \frac{1}{1-pz-qz^2-rz^3},$$

$$\sum_{n\geq 0} W_{n+1}\, z^n = \frac{1}{1-pz-qz^2-rz^3-sz^4}.$$

Kılıç and Prodinger [1] presented some double binomial sums related with the Fibonacci numbers, Pell numbers, Tribonacci numbers and generalized order-*k* Fibonacci numbers. Here we recall some of them from [1]:

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n+i}{2j-1}\binom{n+j}{2i} = F_{4n}, \quad \sum_{i=1}^{n}\sum_{j=1}^{n}\binom{n-i}{j-1}\binom{n-j}{i-1} = F_{2n},$$

$$\sum_{0\leq i\leq j\leq n}\binom{n-i}{j}\binom{j}{i} = P_n, \quad \sum_{0\leq j\leq i\leq n}\binom{n-i}{i-j}\binom{i-j}{j} = T_n,$$

where F_n, P_n and T_n stand for the *n*th Fibonacci, Pell and Tribonacci number, respectively.

Kılıç and Belbechir [2] gave some parametric generalizations of the results of [1] as well as they presented some new kinds of double binomial sums. For example, they computed the generating functions of the following double binomial sums families:

$$\sum_{i,j}\binom{n-i}{sj}\binom{n-sj}{i}t^i u^j,\quad \sum_{i,j}\binom{n}{i+j}\binom{i}{sj}t^i u^j,$$
$$\sum_{i,j}\binom{n-i}{i-sj}\binom{i-sj}{j}t^i u^j,\quad \sum_{i,j}\binom{n-i+sj}{i-sj}t^i u^j.$$

In this paper, firstly we derive new double binomial sums families related with the sequences $\{U_n\},\{V_n\}$ and $\{W_n\}$ as well as we will present generalizations of some results of the works [1, 2]. We also present some interesting double sums including only one binomial coefficient related with the certain number sequences of various order. Generally after each presented result, we will give some related corollaries without proof. We use generating function methods to prove claimed results. As double binomial sums examples, we will present the following new special results:

$$\sum_{0\le i,j\le n}\binom{n-i}{j}\binom{i+j}{j}(-1)^i = F_{n+1},\qquad \sum_{0\le i,j\le n}\binom{n+i}{2j}\binom{i}{j}2^j = F_{3n+1},$$
$$\sum_{0\le i,j\le n}\binom{n-i}{j}\binom{i+j}{j}(-1)^i 2^j = P_{n+1},\qquad \sum_{0\le i,j\le n}\binom{n-i}{j}\binom{2i}{j} = F_{2n+1},$$
$$\sum_{0\le i,j\le n}\binom{n-i}{2j+1}\binom{i}{j}(-1)^i(-2)^j = T_{n+1},\qquad \sum_{0\le i,j\le n}\binom{n-i}{j}\binom{i}{j}2^j = P_{n+1},$$

where F_n, P_n and T_n are the *n*th Fibonacci, Pell and Tribonacci number, respectively.

As double sums with one binomial coefficient, we will present following new special results:

$$\sum_{0\le i,j\le n}\binom{n-i}{i+j} = 2^{n+1} - F_{n+2},\qquad \sum_{0\le i,j\le n}\binom{n-i}{i+j}(-1)^j = F_{n+1},$$
$$\sum_{0\le i,j\le n}\binom{i+j}{i-j} = F_{2n+2},\qquad \sum_{0\le i,j\le n}\binom{i}{j-i} = F_{n+3} - 1$$

and

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{i+j}{2j-1} = \begin{cases} F_n L_{n+3} & \text{if } n \text{ is even,} \\ F_{n+3} L_n & \text{if } n \text{ is odd.} \end{cases}$$

We could refer to [3] for using generating functions in deriving and proving certain combinatorial identities.

2. DOUBLE BINOMIAL SUMS

In this section, we will present some new double binomial sums. Before them, we will need the following Lemma for further use. Note that we will frequently denote the generating function of a sequence $\{a_n\}$ by $A(z)$, that is $A(z) = \sum_{n\ge 0} a_n z^n$. Throughout this paper, we will assume that the parameters c, c_1, c_2 and c_3 are positive integers, the parameters t and k are complex numbers and n, r, r_1 and r_2 are nonnegative integers.

Lemma 2.1 Let α and β be positive integers such that $\beta \ge \alpha$. Then the following identity holds

$$\sum_{n\geq 0}\binom{n+\alpha}{\beta}z^n=\frac{z^{\beta-\alpha}}{(1-z)^{\beta+1}}.$$

Theorem 2.1 The generating function of the sequence $\{s_n\}$ defined by

$$s_n:=\sum_{0\leq i,j\leq n}\binom{n-i}{j}\binom{n-j}{ci+r}t^ik^j$$

is

$$S(z)=\frac{z^r(1-(k+1)z)^{c-r-1}}{(1-(k+1)z)^c-tz^c(1-zk)}.$$

Proof. First, we replace i by $n-i$ and get

$$s_n:=\sum_{0\leq i,j\leq n}\binom{i}{j}\binom{n-j}{cn-ci+r}t^{n-i}k^j.$$

Now we compute its generating function. Consider

$$\begin{aligned} S(z) &= \sum_{n\geq 0}\sum_{0\leq i,j\leq n}\binom{i}{j}\binom{n-j}{cn-ci+r}t^{n-i}k^j\,z^n \\ &= \sum_{n,i,j\geq 0}\binom{n+i}{cn+r}\binom{j+i}{i}t^nk^j\,z^{n+i+j} \\ &= \sum_{n\geq 0}(tz)^n\sum_{i\geq 0}\binom{n+i}{cn+r}z^i\sum_{j\geq 0}\binom{j+i}{i}(kz)^j \\ &= \frac{1}{1-zk}\sum_{n\geq 0}(tz)^n\sum_{i\geq 0}\binom{n+i}{cn+r}\left(\frac{z}{1-zk}\right)^i \end{aligned}$$

which, by Lemma 2.1, equals

$$\frac{z^r}{(1-(k+1)z)^{r+1}}\sum_{n\geq 0}\frac{\left(z^ct(1-zk)\right)^n}{(1-(k+1)z)^{cn}}=\frac{z^r(1-(k+1)z)^{c-r-1}}{(1-(k+1)z)^c-tz^c(1-zk)},$$

as claimed.

Note that the case $c=1$ and $r=0$ could be found in [1].

Corollary 2.1

$$\sum_{0\leq i,j\leq n}\binom{n-i}{j}\binom{n-j}{2i+1}t^ik^j=V_n(2+2k,t-(1+k)^2,-kt),$$

where $\{V_n\}$ is defined by the relation (1.2).

As a consequence of Corollary 2.1, we get

$$\sum_{0\leq i,j\leq n}\binom{n-i}{j}\binom{n-j}{2i+1}(-1)^j=V_n(0,1,1),$$

where the number $V_n(0,1,1)$ is the *(n + 2)*th term of the Padovan sequence which appears as *A000931* in OEIS.

Theorem 2.2 The generating function of the sequence $\{s_n\}$ defined by

$$s_n:=\sum_{0\leq i,j\leq n}\binom{n-i}{j-r_1}\binom{n-j}{i+r_2}t^ik^{j-r_1}$$

is

$$S(z)=\frac{z^{r_1+r_2}}{(1-kz)^{r_1}(1-(k+1)z)^{r_2}(1-(k+t+1)z+tkz^2)}.$$

Proof. Consider

$$
\begin{aligned}
S(z) &= \sum_{n\ge0}\sum_{0\le i,j\le n}\binom{n-i}{j-r_1}\binom{n-j}{i+r_2}t^i k^{j-r_1} z^n \\
&= \sum_{j\ge r_1}\sum_{i\ge0}\sum_{n\ge0}\binom{n+j}{n+r_1}\binom{n+i}{i+r_2}t^i k^{j-r_1} z^{n+j+i} \\
&= \sum_{i\ge0}(zt)^i\sum_{n\ge0}\binom{n+i}{i+r_2}z^n\sum_{j\ge0}\binom{j+n+r_1}{n+r_1}(kz)^j \\
&= \frac{z^{r_1+r_2}}{(1-kz)^{r_1}(1-(k+1)z)^{r_2+1}}\sum_{i\ge0}\left(\frac{zt(1-kz)}{1-(k+1)z}\right)^i \\
&= \frac{z^{r_1+r_2}}{(1-kz)^{r_1}(1-(k+1)z)^{r_2}(1-(k+t+1)z+tkz^2)},
\end{aligned}
$$

as claimed.

The case $r_1 = r_2 = 0$ could be found in [1].

Corollary 2.2

$$\sum_{0\le i,j\le n}\binom{n-i}{j-r_1}\binom{n-j}{i}t^i = \sum_{0\le k_1\le k_2\le\cdots\le k_{r_1}\le n-r_1+1} U_{k_1+1}(2+t,-t),$$

where $\{U_n\}$ is defined as before.

For $t=-1$, we obtain the sums sequence of the sums of the Fibonacci numbers defined by

$$\sum_{0\le i,j\le n}\binom{n-i}{j-r_1}\binom{n-j}{i}(-1)^i = \sum_{0\le k_1\le k_2\le\cdots\le k_{r_1}\le n-r_1+1} F_{k_1+1}.$$

Corollary 2.3

$$\sum_{0\le i,j\le n}\binom{n-i}{j-r_1}\binom{n-j}{i}t^i(-1)^j = \sum_{0\le k_1\le k_2\le\cdots\le k_{r_1}\le n-r_1+1}(-1)^{k_1+k_2+\cdots+k_{r_1}}U_{k_1+1}(t,t),$$

where $\{U_n\}$ is defined as before.

We have the following parametric results without proof.

Theorem 2.3 Define the sequences $\{s_n\}$ and $\{y_n\}$ as shown

$$s_n := \sum_{0\le i,j\le n}\binom{n+i}{j}\binom{n-j}{c_1 i+r}t^i k^j$$

and for any integer c_2,

$$y_n := \sum_{0\le i,j\le n}\binom{n+c_2 i}{j}\binom{n-j}{i}t^i k^j,$$

respectively. The generating functions are

$$S(z) = \frac{(1-kz)z^r}{(1-(k+1)z)^{r+1-c_1}\big((1-kz)(1-(k+1)z)^{c_1}-tz^{c_1}\big)}$$

and

$$Y(z) = \frac{(1-kz)^{c_2}}{(1-kz)^{c_2}(1-(k+1)z)-tz},$$

respectively.

Corollary 2.4 The sequence $\{s_n\}$ defined by

$$s_n \coloneqq \sum_{0\le i,j\le n} \binom{n+i}{j}\binom{n-j}{i} t^i k^j$$

is equal to the sequence $\{U'_n(1+2k+t,-k-k^2)\}$ which satisfies the relation (1.1) with initials $U'_0 = 1$ and $U'_1 = 1+t+k$.

Theorem 2.4 The generating function of $\{s_n\}$ defined by

$$s_n \coloneqq \sum_{0\le i,j\le n} \binom{n-i}{i+cj}\binom{i}{j} t^i k^j$$

is

$$S(z) = \frac{(1-z)^c}{(1-z)^c(1-z-tz^2)-tkz^{c+2}}.$$

Corollary 2.5 The sequence $\{s_n\}$ defined by

$$s_n \coloneqq \sum_{0\le i,j\le n} \binom{n-i}{i+j}\binom{i}{j} t^i k^j$$

is equal to the sequence $\{V'_n(2,t-1,kt-t)\}$ which satisfies the relation (1.2) with initials $V'_0 = V'_1 = 1$ and $V'_2 = 1+t$.

Consequently, for $t=2$ and $k=-1$, we have that

$$\sum_{0\le i,j\le n} \binom{n-i}{i+j}\binom{i}{j} 2^i(-1)^j = V'_n(2,1,0)$$

which is the half of Pell-Lucas sequence (see the sequence *A002203* in OEIS).

Corollary 2.6 The sequence $\{s_n\}$ defined by

$$s_n \coloneqq \sum_{0\le i,j\le n} \binom{n-i}{i+2j}\binom{i}{j} t^i k^j$$

is equal to the sequence $\{W'_n(3,t-3,1-2t,kt+t)\}$ which satisfies the relation (1.3) with initials $W'_0 = W'_1 = 1$, $W'_2 = 1+t$ and $W'_3 = 1+2t$.

Theorem.6 The sequence $\{s_n\}$ defined by

$$s_n \coloneqq \sum_{0\le i,j\le n} \binom{n}{i-j}\binom{n-i}{j} t^i k^j$$

is equal to the sequence $\{U'_n(2t+1,tk-t-t^2)\}$ which satisfies the relation (1.1) with initials $U'_0 = 1$ and $U'_1 = 1+t$.

Proof. Consider

$$\begin{aligned}
S(z) &= \sum_{n\ge 0}\sum_{0\le i,j\le n} \binom{n}{i-j}\binom{n-i}{j} t^i k^j z^n \\
&= \sum_{n\ge 0} z^n \sum_{j\ge 0}\binom{n}{j}(tkz)^j \sum_{i\ge 0}\binom{i+n+j}{n+j}(tz)^i \\
&= \frac{1}{1-tz}\sum_{n\ge 0}\left(\frac{z}{1-tz}\right)^n \sum_{j\ge 0}\binom{n}{j}\left(\frac{tkz}{1-tz}\right)^j
\end{aligned}$$

$$= \quad \frac{1}{1-tz}\sum_{n\geq 0}\left(\frac{z}{1-tz}\right)^n\left(1+\frac{tkz}{1-tz}\right)^n$$
$$= \quad \frac{1-tz}{1-(1+2t)z+(t^2+t-tk)z^2},$$
which is the generating function of the sequence $\{U_n'(2t+1, tk-t-t^2)\}$.

Also as consequences of the result, we have that
$$\sum_{0\leq i,j\leq n}\binom{n}{i-j}\binom{n-i}{j}(-1)^{i+j} = (-1)^n F_{n-1}$$
and for $n \geq 1$
$$\sum_{0\leq i,j\leq n}\binom{n}{i-j}\binom{n-i}{j}(-2)^{i-j} = (-1)^n F_{2n-1}.$$
The case $t = k = 1$ could be found in [2].

Theorem 2.6 The generating function of $\{s_n\}$ defined by
$$s_n := \sum_{0\leq i,j\leq n}\binom{n-i}{c_1 j}\binom{i+j}{c_2 i}t^i k^j$$
is
$$S(z) = \frac{(1-z)^{c_1-1}((1-z)^{c_1}-kz^{c_1})^{c_2-1}}{((1-z)^{c_1}-kz^{c_1})^{c_2}-tk^{c_2-1}z^{c_1c_2-c_1+1}(1-z)^{c_1}}.$$

Corollary 2.7
$$\sum_{0\leq i,j\leq n}\binom{n-i}{j}\binom{i+j}{j}t^i k^j = U_n(t+k+1, -t),$$
where $\{U_n\}$ is defined as before.

Especially, we note the following special cases:
$$\sum_{0\leq i,j\leq n}\binom{n-i}{j}\binom{i+j}{j}(-1)^i = F_{n+1}$$
and
$$\sum_{0\leq i,j\leq n}\binom{n-i}{j}\binom{i+j}{j}(-1)^i 2^j = P_{n+1},$$

where F_n and P_n are the *n*th Fibonacci and Pell number, respectively.

Similar to the above results, we will present the following results without proof.

Theorem 2.7 The generating function of $\{s_n\}$ defined by
$$s_n := \sum_{0\leq i,j\leq n}\binom{n-i}{j}\binom{j+ci}{i}t^i k^j$$
is
$$S(z) = \frac{(1-z-kz)^{c-1}}{(1-z-kz)^c-tz(1-z)^c}.$$
Note that when c = 1 in Theorem 7, we get the Corollary 2.7.

Corollary 2.8 The sequence $\{s_n\}$ defined by

$$s_n := \sum_{0\le i,j\le n} \binom{n-i}{j}\binom{j+2i}{j} t^i k^j$$
is equal to the sequence $\{V_n'(2+2k+t, -(k-1)^2-2t, t)\}$ which satisfies the relation (1.2) with initials $V_0'=1, V_1'=1+t+k$ and $V_2'=(1+k)^2+t(1+3k+t)$.

Theorem 2.8 The generating functions of the sequences $\{s_n\},\{y_n\}$ and $\{w_n\}$ as shown
$$s_n := \sum_{0\le i,j\le n} \binom{n-i}{j}\binom{c_1 j}{i} t^i k^j\,,\; y_n := \sum_{0\le i,j\le n} \binom{n-i}{c_2 j + r_1}\binom{j}{i} t^i k^j$$
and
$$w_n := \sum_{0\le i,j\le n} \binom{n-i}{j}\binom{j}{c_3 i + r_2} t^i k^j$$
are
$$S(z) = \frac{1}{1-z-kz(1+tz)^{c_1}}, \qquad Y(z) = \frac{(1-z)^{c_2-r_1-1} z^{r_1}}{(1-z)^{c_2} - kz^{c_2} - tkz^{c_2+1}}$$
and
$$W(z) = \frac{(1-(k+1)z)^{c_3-r_2-1} z^{r_2}}{(1-(k+1)z)^{c_3} - tk^{c_3} z^{c_3+1}},$$
respectively.

Corollary 2.9
$$\sum_{0\le i,j\le n} \binom{n-i}{2j+1}\binom{j}{i} t^i k^j = V_n(2, k-1, kt),$$
$$\sum_{0\le i,j\le n} \binom{n-i}{j}\binom{2j}{i} t^i k^j = V_{n+1}(k+1, 2kt, kt^2),$$
$$\sum_{0\le i,j\le n} \binom{n-i}{j}\binom{j}{2i+1} t^i k^j = V_n(2+k, -(k+1)^2, k^2 t),$$
where $\{V_n\}$ is defined as before.

Especially, we get
$$\sum_{0\le i,j\le n} \binom{n-i}{j}\binom{j}{2i+1} = F_{2n+1} - 1\,,\; \sum_{0\le i,j\le n} \binom{n-i}{2j+1}\binom{j}{i} (-1)^i = F_{n+2} - 1$$

Theorem 2.9 The generating functions of the sequences $\{s_n\}$ and $\{y_n\}$ defined by
$$s_n := \sum_{0\le i,j\le n} \binom{n-i}{j}\binom{c_1 i + r_1}{j} t^i k^j \text{ and } y_n := \sum_{0\le i,j\le n} \binom{n-i}{c_2 j + r_2}\binom{i}{j} t^i k^j$$
are
$$S(z) = \frac{(1-z)^{c_1-r_1-1}(1-(1-k)z)^{r_1}}{(1-z)^{c_1} - tz(1-(1-k)z)^{c_1}} \text{ and } Y(z) = \frac{(1-z)^{c_2-r_2-1} z^{r_2}}{(1-z)^{c_2}(1-tz) - tkz^{c_2+1}},$$
respectively.

Corollary 2.10

$$\sum_{0\le i,j\le n}\binom{n-i}{j}\binom{i}{j}t^i k^j = U_{n+1}(1+t, tk-t),$$

where $\{U_n\}$ is defined as before.

As consequences, note that

$$\sum_{0\le i,j\le n}\binom{n-i}{j}\binom{i}{j}2^{i-j} = F_{2n+2}\,, \qquad \sum_{0\le i,j\le n}\binom{n-i}{j}\binom{i}{j}2^{j} = P_{n+1}.$$

Corollary 2.11 The sequence $\{s_n\}$ defined by

$$s_n := \sum_{0\le i,j\le n}\binom{n-i}{j}\binom{2i}{j}t^i k^j$$

is equal to the sequence $\{V_n'(2+t, -1-2t(1-k), t(k-1)^2)\}$ which satisfies the relation (1.2) with initials $V_0' = 1, V_1' = 1+t$ and $V_2' = 1+t+2kt+t^2$.

Corollary 2.12

$$\sum_{0\le i,j\le n}\binom{n-i}{2j+1}\binom{i}{j}t^i k^j = V_{n+1}(2+t, -2t-1, t+tk),$$

where $\{V_n\}$ is defined as before.

As consequences, we get

$$\sum_{0\le i,j\le n}\binom{n-i}{2j+1}\binom{i}{j}(-1)^{i+j}2^j = T_{n+1} \text{ and } \sum_{0\le i,j\le n}\binom{n-i}{j}\binom{2i}{j} = F_{2n+1},$$

where T_n and F_n are defined as before.

Theorem 2.10 For positive integer c such that $c \ge 2$, the generating function of $\{s_n\}$ defined by

$$s_n := \sum_{0\le i,j\le n}\binom{n+i}{cj}\binom{j}{i}t^i k^j$$

is

$$S(z) = \frac{(1-z)^{c-1}}{(1-z)^c - tkz^{c-1} - kz^c}.$$

Proof. By some rearrangements and Lemma 2.1, we write

$$\begin{aligned}
S(z) &= \sum_{n\ge 0}\sum_{0\le i,j\le n}\binom{n+i}{cj}\binom{j}{i}t^i k^j z^n \\
&= \sum_{i\ge 0}\sum_{j\ge 0}\sum_{n\ge 0}\binom{n+j+2i}{cj+ci}\binom{j+i}{i}t^i k^{i+j} z^{n+j+i} \\
&= \frac{1}{1-z}\sum_{i\ge 0}\frac{(tkz^{c-1})^i}{(1-z)^{ci}}\sum_{j\ge 0}\binom{j+i}{i}\frac{(kz^c)^j}{(1-z)^{cj}} \\
&= \frac{(1-z)^{c-1}}{(1-z)^c - kz^c}\sum_{i\ge 0}\frac{(tkz^{c-1})^i}{((1-z)^c - kz^c)^i} \\
&= \frac{(1-z)^{c-1}}{(1-z)^c - tkz^{c-1} - kz^c},
\end{aligned}$$

as claimed.

Corollary 2.13 The sequence $\{s_n\}$ defined by

$$s_n := \sum_{0\le i,j\le n} \binom{n+i}{2j}\binom{j}{i} t^i k^j$$

is equal to the sequence $\{U'_n(2+kt, k-1)\}$ which satisfies the relation (1.1) with initials $U'_0 = 1$ and $U'_1 = 1+kt$.

As a consequence, we get

$$\sum_{0\le i,j\le n} \binom{n+i}{2j}\binom{i}{j} 2^j = F_{3n+1}.$$

3. DOUBLE SUMS WITH ONE BINOMIAL COEFFICIENT

In this section, we will present new results on double sums with one binomial coefficient.

Theorem 3.1 The generating function of the sequence $\{s_n\}$ defined by

$$s_n := \sum_{0\le i,j\le n} \binom{n+ci}{j-i} t^i k^j$$

is

$$S(z) = \frac{(1-kz)^{c+1}}{(1-kz)^{c+2} - z(1-kz)^{c+1} + z^2(tk^2+tk) - tkz}.$$

Proof. Consider

$$\begin{aligned}
S(z) &= \sum_{n\ge 0}\sum_{0\le i,j\le n} \binom{n+ci}{j-i} t^i k^j z^n \\
&= \sum_{i\ge 0}\sum_{j\ge 0}\sum_{n\ge 0} \binom{n+(c+1)i+j}{j} t^i k^{i+j} z^{n+j+i} \\
&= \sum_{i\ge 0}\sum_{n\ge 0} t^i k^i z^{n+i} \sum_{j\ge 0} \binom{n+(c+1)i+j}{n+(c+1)i} (kz)^j \\
&= \frac{1}{(1-kz)} \sum_{i\ge 0} \frac{(tkz)^i}{(1-kz)^{(c+1)i}} \sum_{n\ge 0} \frac{z^n}{(1-kz)^n} \\
&= \frac{1}{1-kz-z}\frac{(1-kz)^{c+1}}{(1-kz)^{c+1} - tkz},
\end{aligned}$$

which completes the proof.

Corollary 3.1 The sequence $\{s_n\}$ defined by

$$s_n := \sum_{0\le i,j\le n} \binom{n+i}{j-i} t^i k^j$$

is equal to the sequence $\{V'_n(1+3k+kt, -k(2+3k+t+kt), k^2+k^3)\}$ which satisfies the relation (1.2) with initials $V'_0 = 1, V'_1 = 1+t+kt$ and $V'_2 = (k+1)^2 + kt(1+3k+kt)$.

As consequences of the result, we have that

$$\sum_{0\le i,j\le n} \binom{n+i}{j-i} = F_{2n+3} - 2^n \text{ and } \sum_{0\le i,j\le n} \binom{n+i}{j-i} (-1)^j = (-1)^n F_{2n}.$$

Theorem 3.2 The generating function of $\{s_n\}$ defined by

$$s_n := \sum_{0\le i,j\le n} \binom{n-i}{c_1 i + c_2 j} t^i k^j$$

is

$$S(z) = \frac{(1-z)^{c_1+c_2-1}}{((1-z)^{c_1} - tz^{c_1+1})((1-z)^{c_2} - kz^{c_2})}.$$

Corollary 3.2 The sequence $\{s_n\}$ defined by

$$s_n := \sum_{0\le i,j\le n} \binom{n-i}{i+j} t^i k^j$$

is equal to the sequence $\{V_n'(2+k, t-k-1, -t-kt)\}$ which satisfies the relation (1.2) with initials $V_0' = 1, V_1' = 1+k$ and $V_2' = (k+1)^2 + t$.

Especially, we have the following results:

$$\sum_{0\le i,j\le n} \binom{n-i}{i+j} = 2^{n+1} - F_{n+1}, \qquad \sum_{0\le i,j\le n} \binom{n-i}{i+j} (-1)^j = F_{n+1}.$$

Theorem 3.3 The generating function of $\{s_n\}$ defined by

$$s_n := \sum_{0\le i,j\le n} \binom{i+j}{i-cj} t^i k^j$$

is

$$S(z) = \frac{(1-tz)^c}{(1-z)((1-tz)^{c+1} - kt^c z^c)}.$$

Corollary 3.3 The sequence $\{s_n\}$ defined by

$$s_n := \sum_{0\le i,j\le n} \binom{i+j}{i-j} t^i k^j$$

is equal to the sequence $\{V_n'(1+2t+kt, -t(t+k+2), t^2)\}$ which satisfies the relation (1.2) with initials $V_0' = 1, V_1' = 1+t+tk$ and $V_2' = 1+t+tk+t^2(1+3k+k^2)$.

Corollary 3.4

$$\sum_{0\le i,j\le n} \binom{i+j}{i-j} k^j = U_{n+1}(2+k, -1),$$

where $\{U_n\}$ is defined as before.

Especially, we have the result

$$\sum_{0\le i,j\le n} \binom{i+j}{i-j} = F_{2n+2}.$$

Theorem 3.4 The generating functions of $\{s_n\}$ and $\{y_n\}$ defined by

$$s_n := \sum_{0\le i,j\le n} \binom{c_1 i}{j-i} t^i k^j \text{ and } y_n := \sum_{0\le i,j\le n} \binom{n-i}{c_2(i-j)} t^i k^j$$

are

$$S(z)=\frac{1}{(1-z)(1-ktz(1+kz)^{c_1})}\text{ and } Y(z)=\frac{(1-z)^{c_2-1}}{(1-tkz)((1-z)^{c_2}-tz^{c_2+1})},$$
respectively.

Proof. Consider

$$\begin{aligned}S(z) &= \frac{1}{(1-z)}\sum_{i\geq 0}(tkz)^i\sum_{j\geq 0}\binom{c_1 i}{j}(kz)^j\\ &= \frac{1}{(1-z)}\sum_{i\geq 0}(tkz(1+zk)^{c_1})^i=\frac{1}{(1-z)(1-ktz(1+kz)^{c_1})},\end{aligned}$$

as claimed. The second claim can be similarly proved.

Corollary 3.5

$$\sum_{0\leq i,j\leq n}\binom{i}{j-i}t^ik^j=V_{n+1}(1+kt,k^2t-kt,-k^2t)$$

and

$$\sum_{0\leq i,j\leq n}\binom{2i}{j-i}t^ik^j=W_{n+1}(1+kt,2k^2t-kt,k^3t-2k^2t,-k^3t),$$

where $\{V_n\}$ and $\{W_n\}$ are defined as before.

Especially, for $t=k=1$,

$$\sum_{0\leq i,j\leq n}\binom{i}{j-i}=F_{n+3}-1.$$

Corollary 3.6 The sequence $\{s_n\}$ defined by

$$s_n:=\sum_{0\leq i,j\leq n}\binom{n-i}{2(i-j)}t^ik^j$$

is equal to the sequence $\{W_n'(2+kt,-1-2kt,t+kt,-kt^2)\}$ which satisfies the relation (1.3) with initials $W_0'=1$, $W_1'=1+tk$, $W_2'=1+tk+k^2t^2$ and $W_3'=1+t+tk+k^2t^2+k^3t^3$.

Now we will give an interesting result:

Theorem 3.5 For $n\geq 0$,

$$\sum_{0\leq i,j\leq n}\binom{i+j}{2j-1}=\begin{cases}F_nL_{n+3} & \text{if } n \text{ is even,}\\ F_{n+3}L_n & \text{if } n \text{ is odd,}\end{cases}$$

where F_n and L_n are the *n*th Fibonacci and Lucas number, respectively.

Proof. Consider

$$\begin{aligned}&\sum_{n\geq 0}\sum_{i\geq 0}\sum_{1\leq j\geq n}\binom{i+j}{2j-1}z^n\\ &= \sum_{j\geq 1}\sum_{n\geq 0}\sum_{j-1\leq i\leq n+j}\binom{i+j}{2j-1}z^{n+j}\\ &= \sum_{j\geq 1}\sum_{n\geq 0}\sum_{0\leq i\leq n}\binom{i+2j}{2j-1}z^{n+j}+z^{n+j}\\ &= \sum_{n\geq 0}z^n\sum_{j\geq 1}z^j\sum_{i\geq 1}\binom{i+2j-1}{2j-1}z^{i-1}+\frac{z}{(1-z)^2}\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(1-z)z}\sum_{j\geq 1} z^j\left(\sum_{i\geq 1}\binom{i+2j-1}{2j-1}z^i - 1\right) + \frac{z}{(1-z)^2} \\
&= \frac{1}{(1-z)z}\sum_{j\geq 1}\frac{z^j}{(1-z)^{2j}} - \frac{z}{(1-z)} + \frac{z}{(1-z)^2} \\
&= \frac{1}{1-z}\left(\frac{1}{1-3z-z^2} - \frac{1}{1-z}\right) + \frac{z}{(1-z)^2} \\
&= \frac{3z-z^2}{1-4z+z^2-z^3},
\end{aligned}$$

as claimed.

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