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Evaluation of various partial sums of Gaussian q -binomial sums

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Abstract We present three new sets of weighted partial sums of the Gaussian q -binomial coefficients. To prove the claimed results, we will use q -analysis, Rothe's formula and a q -version of the celebrated algorithm of Zeilberger. Finally we give some applications of our results to generalized Fibonomial sums.

Mathematics Subject Classification 11B65 · 05A30

المخلص

نستعرض ثلاث مجموعات جديدة من المجاميع الجزئية لمجاميع جوس ذات q -حدين. ولإثبات النتائج، سوف نستخدم تحليل q ، صيغة روث، ونسخة من نوع q لخوارزمية زيلبرجر المشهورة. وأخيرا نقدم بعض التطبيقات لنتائجنا على مجاميع فيبوناتشي المعممة أحادية الحدود.

1 Introduction

The authors of [2] present some results about the partial sums of rows of Pascal's triangle as well as its alternating analogues. As alternating partial sum of Pascal's triangle, from [2] we have

$$\sum_{k \leq m} \binom{r}{k} (-1)^k = (-1)^m \binom{r-1}{m}, \quad \text{integer } m.$$

The authors note that there is no closed form for the partial sum of a row of Pascal's triangle, $S(m, n) = \sum_{k \leq m} \binom{n}{k}$. However, they give a curious example for the partial sum of the row elements multiplied by their distance from the center:

$$\sum_{k \leq m} \binom{r}{k} \left(\frac{r}{2} - k\right) = \frac{m+1}{2} \binom{r}{m+1}, \quad \text{integer } m.$$

Ollerton [11] considered the same partial sum of a row of Pascal's triangle $S(m, n)$ and developed formulae for the sums by use of generating functions.

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We recall a different result from [6]: for any nonnegative integer t ,

$$\sum_{k \leq t} \binom{2n}{k} \left(1 - \frac{k}{n}\right) = \binom{2n-1}{t}.$$

These kinds of partial sums, as well as certain weighted sums (both alternating, and non alternating) are very rare in the current literature. Calkin [1] proved the following curious identity:

$$\sum_{k=0}^n \left(\sum_{j=0}^k \binom{n}{j}\right)^3 = n \cdot 2^{3n-1} + 2^{3n} - 3n \binom{2n}{n} 2^{n-2}.$$

Hirschhorn [5] established the following two identities on sums of powers of binomial partial sums:

$$\sum_{k=0}^n \sum_{j=0}^k \binom{n}{j} = n \cdot 2^{n-1} + 2^n \quad \text{and} \quad \sum_{k=0}^n \left(\sum_{j=0}^k \binom{n}{j}\right)^2 = n \cdot 2^{2n-1} + 2^{2n} - \frac{n}{2} \binom{2n}{n}.$$

Recently, He [4] gave q -analogues of the results of Hirschhorn as well as a new version of another result. For example, from [4], we recall that

$$\sum_{k=0}^n \sum_{j=0}^k \begin{bmatrix} n \\ j \end{bmatrix}_q q^{mk+\binom{j}{2}} = \frac{(-q^m; q)_n - q^{m(n+1)} (-1; q)_n}{1 - q^m},$$

where the q -Pochhammer symbol is $(x; q)_n = (1-x)(1-xq) \dots (1-xq^{n-1})$ and the Gaussian q -binomial coefficients are

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

Note that

$$\lim_{q \rightarrow 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}.$$

Meanwhile, Guo et al. [3] gave some partial sums including q -binomial coefficients, for example

$$\sum_{k=0}^{2n} (-1)^k \left(\sum_{j=0}^k \begin{bmatrix} 2n \\ j \end{bmatrix}_q\right)^2 = \left(\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q\right) \left(\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q\right).$$

For later use, we recall that one version of the *Cauchy binomial theorem* is given by

$$\sum_{k=0}^n q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = \prod_{k=1}^n (1 + xq^k),$$

and *Rothe's* formula is

$$\sum_{k=0}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = (x; q)_n = \prod_{k=0}^{n-1} (1 - xq^k).$$

Nowadays, there is increasing interest in deriving q -analogues of certain combinatorial statements. This includes, for example, q -analogues of certain identities involving harmonic numbers, we refer to [9, 10].

Recently, Kılıç and Prodinger [7] computed half Gaussian q -binomial sums with certain weight functions. They also gave applications of their results as the generalized Fibonomial sum formulas. For example, they showed that for $n \geq 1$ such that $2n - 1 \geq r$

$$\sum_{k=1}^n \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q (-1)^k q^{\frac{1}{2}(k^2-k(2r+1))} (1+q^k)^{2r+1} = -2^{2r} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

and

$$\sum_{k=0}^n \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q q^{\frac{1}{2}k(k-3)} (1-q^k)^3 = 2 \begin{bmatrix} 2n-3 \\ n-1 \end{bmatrix}_q \frac{(1-q)}{q} (1-q^n)(1-q^{2n-1}).$$

More recently, Kılıç and Prodinger [8] computed three types of sums involving products of the Gaussian q -binomial coefficients. They are of the following forms: for any real number a

$$\sum_{k=0}^n \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{-nk+\binom{k}{2}} (a-q^k),$$

$$\sum_{k=0}^n \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{-nk+\binom{k}{2}} \frac{1}{q^{-k}-a}$$

and

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q (-1)^k q^{-nk+\binom{k+1}{2}} \frac{a-q^{-k}}{b-q^{-k}}.$$

They also presented interesting applications of their results to generalized Fibonomial and Lucanomial sums.

In this paper, inspired by the partial sums given in [2], we present three sets of weighted partial sums of the Gaussian binomial q -binomial coefficients. Before each set, we first give a lemma and then present our main results. To prove the claimed results, we will use Rothe’s formula, q -analysis and a q -version of the celebrated algorithm of Zeilberger. For the sake of simplicity, we will only prove one result from each set.

2 The main results

In this section, we will examine each set in separate subsections. We start with the first set of weighted partial sums.

2.1 First kind of weighted partial sums

We start with the following lemma to use in proving our first set of results.

Lemma 2.1 (i) For odd n ,

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-n)} \mathbf{i}^{k(n-k)} (-1)^k = 0.$$

(ii) For even n ,

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-n-1)} \mathbf{i}^{k(n+k)+k} (1-q^k) = 0.$$



Proof We only give a proof for the first case (i). The second case (ii) is similarly done. For odd n , we write

$$\begin{aligned} \sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix} q^{\binom{k}{2}-kn} (-1)^{\binom{k+1}{2}+kn} &= \sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix} q^{\binom{k}{2}-kn} (-1)^{kn} \left(\frac{i+1}{2} i^k - \frac{i-1}{2} (-i)^k \right) \\ &= \frac{i+1}{2} \sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix} q^{\binom{k}{2}-kn} (-1)^{kn} i^k - \frac{i-1}{2} \sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix} q^{\binom{k}{2}-kn} (-1)^{kn} (-i)^k, \end{aligned}$$

which, by Rothe formula, equals

$$\begin{aligned} \frac{i+1}{2} (-(-1)^n i q^{-n}; q)_{2n+1} - \frac{i-1}{2} ((-1)^n i q^{-n}; q)_{2n+1} \\ = \frac{i+1}{2} \prod_{k=0}^{2n} (1 + (-1)^n i q^{k-n}) - \frac{i-1}{2} \prod_{k=0}^{2n} (1 - (-1)^n i q^{k-n}). \end{aligned}$$

Here we consider two subcases: first, if n is even,

$$\begin{aligned} \frac{i+1}{2} \prod_{k=0}^{2n} (1 + i q^{k-n}) - \frac{i-1}{2} \prod_{k=0}^{2n} (1 - i q^{k-n}) &= 2i \left(\prod_{k=1}^n (i q^{-k} (1 + q^{2k})) - \prod_{k=1}^n (-i q^{-k} (1 + q^{2k})) \right) \\ &= 2i^{1-n} q^{-\binom{n+1}{2}} \prod_{k=1}^n (1 + q^{2k}) [(-1)^n - 1], \end{aligned}$$

which clearly equals 0 for even n . The other subcase, n is odd, can be done in a similar way. □

Theorem 2.2 For $n, m \geq 0$

$$\begin{aligned} \sum_{k=0}^{2n+1} \begin{bmatrix} 2n+2m+4 \\ k+m \end{bmatrix}_q (1 - q^{2n+m+4-k}) (-1)^{\binom{k}{2}+kn} q^{\frac{1}{2}k(k-3)-kn} \\ = q^{-n-1} \begin{bmatrix} 2n+2m+4 \\ m+1 \end{bmatrix}_q \begin{cases} (1 + q^{n+1})(1 - q^{m+n+2}) & \text{if } n \text{ is even,} \\ -(1 - q^{n+1})(1 + q^{m+n+2}) & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Proof Consider the LHS of the claim

$$\begin{aligned} \sum_{k=0}^{2n+1} \begin{bmatrix} 2n+2m+4 \\ k+m \end{bmatrix}_q (1 - q^{2n+m+4-k}) (-1)^{\binom{k}{2}+kn} q^{\frac{1}{2}k(k-3)-kn} \\ = (-1)^{mn - \binom{m+1}{2}} q^{\frac{1}{2}m(m+3)+mn} \\ \times \sum_{k=m}^{2n+m+1} \begin{bmatrix} 2n+2m+4 \\ k \end{bmatrix}_q (1 - q^{2n+2m+4-k}) (-1)^{k(m+n) - \binom{k}{2}} q^{\frac{1}{2}k(k-2(m+n)-3)}. \end{aligned}$$

By Lemma 2.1 (ii), we have

$$\sum_{k=0}^{2n+2m+4} \begin{bmatrix} 2n+2m+4 \\ k \end{bmatrix}_q (1 - q^{2n+2m+4-k}) (-1)^{k(m+n) - \binom{k}{2}} q^{\frac{1}{2}k(k-2(m+n)-3)} = 0.$$

Apart from the constant factor, we consider the LHS of the claim and write

$$\begin{aligned} - \sum_{k=m}^{2n+m+1} \begin{bmatrix} 2n+2m+4 \\ k \end{bmatrix}_q (1 - q^{2n+2m+4-k}) (-1)^{k(m+n) - \binom{k}{2}} q^{\frac{1}{2}k(k-2(m+n)-3)} \\ = \sum_{k=0}^{m-1} \begin{bmatrix} 2n+2m+4 \\ k \end{bmatrix}_q (1 - q^{2n+2m+4-k}) (-1)^{k(m+n) - \binom{k}{2}} q^{\frac{1}{2}k(k-2(m+n)-3)} \end{aligned}$$



$$\begin{aligned}
 &+ \sum_{k=2n+m+2}^{2n+2m+4} \begin{bmatrix} 2n+2m+4 \\ k \end{bmatrix}_q (1 - q^{2n+2m+4-k}) \\
 &\times (-1)^{k(m+n)-\binom{k}{2}} q^{\frac{1}{2}k(k-2(m+n)-3)},
 \end{aligned}$$

which, after some rearrangements, equals

$$\begin{aligned}
 &\sum_{k=0}^{m-1} \begin{bmatrix} 2n+2m+4 \\ k \end{bmatrix}_q (1 - q^{2n+2m+4-k}) (-1)^{k(m+n)-\binom{k}{2}} q^{\frac{1}{2}k(k-2(m+n)-3)} \\
 &+ \sum_{k=0}^{m+2} \begin{bmatrix} 2n+2m+4 \\ m+2-k \end{bmatrix}_q (1 - q^{m-k+2}) \\
 &\times (-1)^{kn+n(m+1)-\binom{k}{2}+\binom{m+1}{2}+1} q^{-\frac{1}{2}(m-k+1)(k+m+2n+2)} \\
 &= \sum_{k=0}^{m-1} \begin{bmatrix} 2n+2m+4 \\ k \end{bmatrix}_q (1 - q^{2n+2m+4-k}) (-1)^{k(m+n)-\binom{k}{2}} q^{\frac{1}{2}k(k-2(m+n)-3)} \\
 &+ \sum_{k=0}^{m+2} \begin{bmatrix} 2n+2m+4 \\ k \end{bmatrix}_q (-1)^{(k+1)(m+n)-\binom{k+1}{2}} q^{-\frac{1}{2}(k-1)(2m-k+2n+4)} (1 - q^k).
 \end{aligned}$$

After further rearrangements, this equals

$$\begin{aligned}
 &(1 - q^{2n+2m+4}) \left(\sum_{k=0}^{m-1} \begin{bmatrix} 2n+2m+3 \\ k \end{bmatrix}_q (-1)^{k(m+n)-\frac{1}{2}k(k-1)} q^{\frac{1}{2}k(k-2m-2n-3)} \right. \\
 &\left. + \sum_{k=0}^{m+2} \begin{bmatrix} 2n+2m+3 \\ k-1 \end{bmatrix}_q (-1)^{(k+1)(m+n)-\binom{k+1}{2}} q^{-\frac{1}{2}(k-1)(2m-k+2n+4)} \right) \\
 &= (1 - q^{2n+2m+4}) \left(\sum_{k=0}^{m-1} \begin{bmatrix} 2n+2m+3 \\ k \end{bmatrix}_q (-1)^{k(m+n)-\binom{k}{2}} q^{\frac{1}{2}k(k-2m-2n-3)} \right. \\
 &\left. + \sum_{k=0}^{m+1} \begin{bmatrix} 2n+2m+3 \\ k \end{bmatrix}_q (-1)^{k(m+n)-\binom{k}{2}+1} q^{\frac{1}{2}k(k-2m-2n-3)} \right) \\
 &= (1 - q^{2n+2m+4}) \left(\sum_{k=0}^{m-1} \begin{bmatrix} 2n+2m+3 \\ k \end{bmatrix}_q (-1)^{k(m+n)-\binom{k}{2}} q^{\frac{1}{2}k(k-2m-2n-3)} \right. \\
 &\left. - \sum_{k=0}^{m+1} \begin{bmatrix} 2n+2m+3 \\ k \end{bmatrix}_q (-1)^{k(m+n)-\binom{k}{2}} q^{\frac{1}{2}k(k-2m-2n-3)} \right),
 \end{aligned}$$

which, by telescoping, equals

$$\begin{aligned}
 &-(1 - q^{2n+2m+4}) \left(\begin{bmatrix} 2n+2m+3 \\ m+1 \end{bmatrix}_q (-1)^{\frac{1}{2}(m+2n)(m+1)} q^{-\frac{1}{2}(m+1)(m+2n+2)} \right. \\
 &\left. + \begin{bmatrix} 2n+2m+3 \\ m \end{bmatrix}_q (-1)^{\frac{1}{2}m(m+2n+1)} q^{-\frac{1}{2}m(m+2n+3)} \right) \\
 &= -(1 - q^{2n+2m+4}) (-1)^{\frac{1}{2}m(m+2n+1)} q^{-\frac{1}{2}m(m+2n+3)} \\
 &\times \left(\begin{bmatrix} 2n+2m+3 \\ m+1 \end{bmatrix}_q (-1)^n q^{-(n+1)} + \begin{bmatrix} 2n+2m+3 \\ m \end{bmatrix}_q \right)
 \end{aligned}$$



$$= -(-1)^{\frac{1}{2}m(m+2n+1)} q^{-\frac{1}{2}(m+1)(m+2n+2)} \begin{bmatrix} 2n+2m+4 \\ m+1 \end{bmatrix}_q$$

$$\times \begin{cases} (1+q^{n+1})(1-q^{m+n+2}) & \text{if } n \text{ is even,} \\ -(1-q^{n+1})(1+q^{m+n+2}) & \text{if } n \text{ is odd.} \end{cases}$$

Thus, we get

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+2m+4 \\ k+m \end{bmatrix}_q (1-q^{2n+m+4-k}) (-1)^{\binom{k}{2}+kn} q^{\frac{1}{2}k(k-3)-kn}$$

$$= q^{-(n+1)} \begin{bmatrix} 2n+2m+4 \\ m+1 \end{bmatrix}_q \begin{cases} (1+q^{n+1})(1-q^{m+n+2}) & \text{if } n \text{ is even,} \\ -(1-q^{n+1})(1+q^{m+n+2}) & \text{if } n \text{ is odd,} \end{cases}$$

as claimed. □

We present the following results without proof.

Theorem 2.3 For $n \geq 0$ and $m \geq 1$,

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n+2m-1 \\ k+m \end{bmatrix}_q q^{\binom{k+1}{2}-kn} (-1)^{kn-\binom{k}{2}} = (-1)^n q^n \begin{bmatrix} 2n+2m-1 \\ m-1 \end{bmatrix}_q,$$

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n+2m \\ k+m \end{bmatrix}_q q^{\binom{k}{2}-kn} (-1)^{\binom{k+1}{2}+kn} (1-q^{k+m}) = (1-q^{2n+2m}) \begin{bmatrix} 2n+2m-1 \\ m-1 \end{bmatrix}_q.$$

Theorem 2.4 For $n, m \geq 0$,

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n+2m+1 \\ k+m \end{bmatrix}_q q^{\frac{1}{2}k(k-2n-1)} (-1)^{kn-\binom{k+1}{2}} = \begin{bmatrix} 2n+2m+1 \\ m \end{bmatrix}_q,$$

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+2m+3 \\ k+m \end{bmatrix}_q q^{\frac{1}{2}k(k-2n-3)} (-1)^{\binom{k}{2}+kn}$$

$$= (-1)^n q^{-n-1} \begin{bmatrix} 2n+2m+4 \\ m+1 \end{bmatrix}_q \begin{cases} (1-q^{n+1})/(1-q^{n+m+2}) & \text{if } n \text{ is odd,} \\ (1+q^{n+1})/(1+q^{n+m+2}) & \text{if } n \text{ is even.} \end{cases}$$

2.2 Second kind of weighted partial sums

Now we continue with our second set of sums. Before that, we need the following lemma which could be proven similar to Lemma 2.1.

Lemma 2.5 (i) If n is odd,

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\frac{1}{2}k(k-n)} = 0.$$

(ii) If n is even,

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-n+1)} (-1)^k (1-q^{n-k}) = 0.$$

We start with the following result.



Theorem 2.6 For $n, m \geq 0$,

$$\sum_{k=0}^{4n+1} \begin{bmatrix} 4n+2m+3 \\ k+m \end{bmatrix}_q (-1)^k q^{\frac{1}{2}k(k-4n-3)} = -q^{-2n-1} \frac{(1-q^{2n+1})(1+q^{2n+m+2})}{(1-q^{m+1})} \begin{bmatrix} 4n+2m+3 \\ m \end{bmatrix}_q.$$

Proof Consider the LHS of the claim, by taking $k-m$ instead of k , we note

$$\begin{aligned} & \sum_{k=0}^{4n+1} \begin{bmatrix} 4n+2m+3 \\ k+m \end{bmatrix}_q (-1)^k q^{-\frac{1}{2}k(4n-k+3)} \\ &= (-1)^m q^{\frac{1}{2}m(m+4n+3)} \sum_{k=m}^{4n+m+1} \begin{bmatrix} 4n+2m+3 \\ k \end{bmatrix}_q (-1)^k q^{\frac{1}{2}k(k-4n-2m-3)} \end{aligned}$$

and by Lemma 2.5 (i),

$$\sum_{k=0}^{4n+2m+3} \begin{bmatrix} 4n+2m+3 \\ k \end{bmatrix}_q (-1)^k q^{\frac{1}{2}k(k-4n-2m-3)} = 0.$$

Thus, we obtain

$$\begin{aligned} & - \sum_{k=m}^{4n+m+1} \begin{bmatrix} 4n+2m+3 \\ k \end{bmatrix}_q (-1)^k q^{\frac{1}{2}k(k-4n-2m-3)} \\ &= \sum_{k=0}^{m-1} \begin{bmatrix} 4n+2m+3 \\ k \end{bmatrix}_q (-1)^k q^{\frac{1}{2}k(k-4n-2m-3)} \\ & \quad + \sum_{k=4n+m+2}^{4n+2m+3} \begin{bmatrix} 4n+2m+3 \\ k \end{bmatrix}_q (-1)^k q^{\frac{1}{2}k(k-4n-2m-3)}, \end{aligned}$$

which, by some rearrangements, equals

$$\begin{aligned} & \sum_{k=0}^{m-1} \begin{bmatrix} 4n+2m+3 \\ k \end{bmatrix}_q (-1)^k q^{\frac{1}{2}k(k-4n-2m-3)} \\ & \quad + \sum_{k=0}^{m+1} \begin{bmatrix} 4n+2m+3 \\ k+4n+m+2 \end{bmatrix}_q (-1)^{k+m} q^{-\frac{1}{2}(m-k+1)(k+m+4n+2)} \\ &= \sum_{k=0}^{m-1} \begin{bmatrix} 4n+2m+3 \\ k \end{bmatrix}_q (-1)^k q^{\frac{1}{2}k(k-4n-2m-3)} \\ & \quad - \sum_{k=0}^{m+1} \begin{bmatrix} 4n+2m+3 \\ k \end{bmatrix}_q (-1)^k q^{\frac{1}{2}k(k-4n-2m-3)}. \end{aligned}$$

By telescoping, this equals

$$\begin{aligned} & - (-1)^m q^{-\frac{1}{2}(m+1)(m+4n+2)} \left(- \begin{bmatrix} 4n+2m+3 \\ m+1 \end{bmatrix}_q + \begin{bmatrix} 4n+2m+3 \\ m \end{bmatrix}_q q^{2n+1} \right) \\ &= - (-1)^m q^{-\frac{1}{2}(m+1)(m+4n+2)} \frac{(q)_{4n+2m+3}}{(q)_{m+1} (q)_{4n+m+3}} \times (-(1-q^{4n+m+3}) + q^{2n+1}(1-q^{m+1})), \end{aligned}$$

which equals

$$\begin{aligned} & (-1)^m q^{-\frac{1}{2}(m+1)(m+4n+2)} \frac{(q)_{4n+2m+3}}{(q)_m (q)_{4n+m+3}} \frac{(1 - q^{2n+1})(1 + q^{2n+m+2})}{1 - q^{m+1}} \\ &= (-1)^m q^{-\frac{1}{2}(m+1)(m+4n+2)} \left[\begin{matrix} 4n + 2m + 3 \\ m \end{matrix} \right]_q \frac{(1 - q^{2n+1})(1 + q^{2n+m+2})}{(1 - q^{m+1})}. \end{aligned}$$

Thus, we derive

$$\begin{aligned} & \sum_{k=0}^{4n+1} \left[\begin{matrix} 4n + 2m + 3 \\ k + m \end{matrix} \right]_q (-1)^k q^{\frac{1}{2}k(k-4n-3)} \\ &= -q^{-2n-1} \frac{(1 - q^{2n+1})(1 + q^{2n+m+2})}{(1 - q^{m+1})} \left[\begin{matrix} 4n + 2m + 3 \\ m \end{matrix} \right]_q, \end{aligned}$$

as claimed. □

Using similar techniques as in the above proof, we obtain the following identities, for all $n, m \geq 0$,

$$\begin{aligned} & \sum_{k=0}^{4n+1} \left[\begin{matrix} 4n + 2m + 4 \\ k + m \end{matrix} \right]_q (-1)^k q^{\frac{1}{2}k(k-4n-3)} (1 - q^{4n+m+4-k}) \\ &= -q^{2n+1} (1 - q^{4n+2m+4}) \frac{(1 - q^{2n+1})(1 + q^{2n+m+2})}{(1 - q^{m+1})} \left[\begin{matrix} 4n + 2m + 3 \\ m \end{matrix} \right]_q, \\ & \sum_{k=0}^{4n+2} \left[\begin{matrix} 4n + 2m + 3 \\ k + m \end{matrix} \right]_q (-1)^k q^{\frac{1}{2}k(k-4n-3)} = \left[\begin{matrix} 4n + 2m + 3 \\ m \end{matrix} \right]_q, \tag{1} \\ & \sum_{k=0}^{4n+1} \left[\begin{matrix} 4n + 2m - 1 \\ k + m \end{matrix} \right]_q (-1)^k q^{\binom{k+1}{2} - 2kn} = q^{2n} \frac{(1 - q^{2n+1})(1 + q^{m+2n})}{1 - q^{m-1}} \left[\begin{matrix} 4n + 2m - 1 \\ m - 2 \end{matrix} \right]_q, \\ & \sum_{k=0}^{4n} \left[\begin{matrix} 4n + 2m - 1 \\ k + m \end{matrix} \right]_q q^{\binom{k+1}{2} - 2kn} (-1)^k = q^{2n} \left[\begin{matrix} 4n + 2m - 1 \\ m - 1 \end{matrix} \right]_q, \\ & \sum_{k=0}^{4n+2} \left[\begin{matrix} 4n + 2m + 4 \\ k + m \end{matrix} \right]_q q^{-\frac{1}{2}k(4n-k+3)} (-1)^k (1 - q^{4n+4+m-k}) = (1 - q^{4n+4+2m}) \left[\begin{matrix} 4n + 2m + 3 \\ m \end{matrix} \right]_q, \\ & \sum_{k=0}^{4n} \left[\begin{matrix} 4n + 2m \\ k + m \end{matrix} \right]_q q^{\frac{1}{2}k(k-4n+1)} (-1)^k (1 - q^{4n+m-k}) \\ &= q^{2n} (1 - q^{4n+2m}) \left[\begin{matrix} 4n + 2m - 1 \\ m - 1 \end{matrix} \right]_q. \end{aligned}$$

2.3 Third kind of weighted partial sums

We start with the following lemma for later use.

Lemma 2.7 For $m > 0$ and integer r ,

$$\begin{aligned} & \sum_{k=0}^{m+r} \left[\begin{matrix} 4n + 2m + 2 \\ k \end{matrix} \right]_q (-1)^k q^{\frac{1}{2}k(k-4n-2m-1)} (1 + q^{2n+m+1-k}) \\ &= (-1)^{m+r} q^{-\frac{1}{2}(m+r)(m+4n-r+1)} (1 + q^{1+m+2n}) \left[\begin{matrix} 4n + 2m + 1 \\ m + r \end{matrix} \right]_q. \end{aligned}$$

Proof Denote the left-hand side of this identity by $S[m, r]$. Using the celebrated algorithm of Zeilberger in Mathematica, we obtain

$$S[m, r] = (-1)^{m+r} q^{-\frac{1}{2}(m+r)(m+4n-r+1)} \frac{1}{(1 - q^{1+m+2n})} \frac{(q)_{4n+2m+2}}{(q)_{4n+m+1-r} (q)_{m+r}},$$

which is easily seen to be the same as the right-hand side of the desired identity. □

Theorem 2.8 For $n \geq 0$ and $m > 0$,

$$\begin{aligned} & \sum_{k=0}^{4n+3} \begin{bmatrix} 4n + 2m + 2 \\ k + m \end{bmatrix}_q q^{\frac{1}{2}k(k-4n-3)} (-1)^k (1 - q^k) \\ &= -q^{2n+1} \frac{(1 - q^{2n+2})(1 - q^m)}{(1 - q^{m+2n+1})} \begin{bmatrix} 4n + 2m + 2 \\ m - 1 \end{bmatrix}_q. \end{aligned}$$

Proof Note that

$$\begin{aligned} & q^{\frac{1}{2}m(m+4n+3)} (-1)^m \sum_{k=m}^{4n+m+3} \begin{bmatrix} 4n + 2m + 2 \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-4n-2m-3)} (-1)^k (1 - q^{k-m}) \\ &= \sum_{k=0}^{4n+3} \begin{bmatrix} 4n + 2m + 2 \\ k + m \end{bmatrix}_q q^{\frac{1}{2}k(k-4n-3)} (-1)^k (1 - q^k) \end{aligned}$$

and also

$$\sum_{k=0}^{4n+2m+2} \begin{bmatrix} 4n + 2m + 2 \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-4n-2m-3)} (-1)^k (1 - q^{k-m}) = 0.$$

Thus, we write

$$\begin{aligned} & - \sum_{k=m}^{4n+m+3} \begin{bmatrix} 4n + 2m + 2 \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-4n-2m-3)} (-1)^k (1 - q^{k-m}) \\ &= \sum_{k=0}^{m-1} \begin{bmatrix} 4n + 2m + 2 \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-4n-2m-3)} (-1)^k (1 - q^{k-m}) \\ &+ \sum_{k=0}^{m-2} \begin{bmatrix} 4n + 2m + 2 \\ k \end{bmatrix}_q (-1)^k q^{\frac{1}{2}(k+1)(k-4n-2m-2)} (1 - q^{m+4n+2-k}) \\ &= \begin{bmatrix} 4n + 2m + 2 \\ m - 1 \end{bmatrix}_q q^{-\frac{1}{2}(m-1)(m+4n+4)} (-1)^{m-1} (1 - q^{-1}) \\ &+ \sum_{k=0}^{m-2} \begin{bmatrix} 4n + 2m + 2 \\ k \end{bmatrix}_q (-1)^k \\ &\times \left[q^{\frac{1}{2}k(k-4n-2m-3)} (1 - q^{k-m}) + q^{\frac{1}{2}(k+1)(k-4n-2m-2)} (1 - q^{m-k+4n+2}) \right] \\ &= \begin{bmatrix} 4n + 2m + 2 \\ m - 1 \end{bmatrix}_q q^{-\frac{1}{2}(m-1)(m+4n+4)} (-1)^{m-1} (1 - q^{-1}) \\ &+ q^{-(2n+m+1)} (1 - q^{2n+1}) \\ &\times \sum_{k=0}^{m-2} \begin{bmatrix} 4n + 2m + 2 \\ k \end{bmatrix}_q (-1)^k q^{\frac{1}{2}k(k-4n-2m-1)} (1 + q^{2n+m+1-k}), \end{aligned}$$

which, by Lemma 2.7 with the case $r = -2$, equals

$$\begin{aligned} & (-1)^{m-1} q^{-\frac{1}{2}(m-1)(m+4n+4)} \\ & \times \left(\begin{bmatrix} 4n + 2m + 2 \\ m - 1 \end{bmatrix}_q (1 - q^{-1}) - (1 - q^{2n+1})(1 + q^{1+m+2n}) \begin{bmatrix} 4n + 2m + 1 \\ m - 2 \end{bmatrix}_q \right) \\ & = (-1)^m q^{-\frac{1}{2}(m-1)(m+4n+4)-1} \\ & \times (1 - q^{2n+2})(1 + q^{m+2n+1})(1 - q^m) \frac{(q)_{4n+2m+1}}{(q)_{m-1} (q)_{4n+m+3}}. \end{aligned}$$

Thus,

$$\sum_{k=0}^{4n+3} \begin{bmatrix} 4n + 2m + 2 \\ k + m \end{bmatrix}_q q^{\frac{1}{2}k(k-4n-3)} (-1)^k (1 - q^k) = -q^{2n+1} \frac{(1 - q^{2n+2})(1 - q^m)}{(1 - q^{m+2n+1})} \begin{bmatrix} 4n + 2m + 2 \\ m - 1 \end{bmatrix}_q,$$

as claimed. □

We present the following results without proof. The proofs use Lemma 2.7 and are similar to that of Theorem 2.8.

Theorem 2.9 For $n \geq 0$ and $m \geq 1$,

$$\sum_{k=0}^{4n+2} \begin{bmatrix} 4n + 2m + 2 \\ k + m \end{bmatrix}_q q^{\frac{1}{2}k(k-4n-3)} (-1)^k (1 - q^k) = (1 - q^{2n+1})(1 + q^{1+m+2n}) \begin{bmatrix} 4n + 2m + 1 \\ m - 1 \end{bmatrix}_q.$$

Theorem 2.10 For $n, m \geq 0$,

$$\begin{aligned} & \sum_{k=0}^{4n+1} \begin{bmatrix} 4n + 2m + 2 \\ k + m \end{bmatrix}_q (-1)^k q^{\frac{1}{2}k(k-4n-3)} (1 - q^k) = -q^{-2n-1} (1 + q^{2n+m+1})(1 - q^{2n+1}) \begin{bmatrix} 4n + 2m + 1 \\ m \end{bmatrix}_q, \\ & \sum_{k=0}^{4n} \begin{bmatrix} 4n + 2m + 2 \\ k + m \end{bmatrix}_q q^{\frac{1}{2}k(k-4n-3)} (-1)^k (1 - q^k) \\ & = q^{-4n-1} \frac{(1 - q^{4n+m+2})(1 - q^{2n})}{(1 - q^{2n+m+1})} \begin{bmatrix} 4n + 2m + 2 \\ m + 1 \end{bmatrix}_q. \end{aligned}$$

3 Applications

In this section, we will give some applications of our results to the generalized Fibonomial coefficients. Define the non-degenerate second-order linear sequences $\{U_n\}$ and $\{V_n\}$ by, for $n > 1$

$$\begin{aligned} U_n &= pU_{n-1} + U_{n-2}, \quad U_0 = 0, \quad U_1 = 1, \\ V_n &= pV_{n-1} + V_{n-2}, \quad V_0 = 2, \quad V_1 = p. \end{aligned}$$

The Binet formulas of $\{U_n\}$ and $\{V_n\}$ are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where $\alpha, \beta = (p \pm \sqrt{p^2 + 4})/2$.

For $n \geq k \geq 1$, define the generalized Fibonomial coefficient by

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_U := \frac{U_1 U_2 \dots U_n}{(U_1 U_2 \dots U_k) \cdot (U_1 U_2 \dots U_{n-k})}$$



with $\left\{ \begin{matrix} n \\ 0 \end{matrix} \right\}_U = \left\{ \begin{matrix} n \\ n \end{matrix} \right\}_U = 1$.

When $p = 1$, we obtain the usual Fibonomial coefficients, denoted by $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_F$.

The link between the generalized Fibonomial and Gaussian q -binomial coefficients is

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_U = \alpha^{n-k} \left[\begin{matrix} n \\ k \end{matrix} \right]_q \quad \text{with } q = -\alpha^{-2}.$$

By taking $q = \beta/\alpha$, the Binet formulae are reduced to the following forms:

$$U_n = \alpha^{n-1} \frac{1 - q^n}{1 - q} \quad \text{and} \quad V_n = \alpha^n (1 + q^n),$$

where $\mathbf{i} = \sqrt{-1} = \alpha\sqrt{q}$.

From each set, we now give one application. From Theorem 2.3, Eq. (1) and Theorem 2.9, we have the following identities by taking $q = \beta/\alpha$, respectively: for $n \geq 0$ and $m \geq 1$,

$$\begin{aligned} \sum_{k=0}^{2n} \left\{ \begin{matrix} 2n + 2m \\ k + m \end{matrix} \right\}_U (-1)^k U_{k+m} &= U_{2n+2m} \left\{ \begin{matrix} 2n + 2m - 1 \\ m - 1 \end{matrix} \right\}_U, \\ \sum_{k=0}^{4n+2} \left\{ \begin{matrix} 4n + 2m + 3 \\ k + m \end{matrix} \right\}_U (-1)^{\frac{1}{2}k(k-1)} &= \left\{ \begin{matrix} 4n + 2m + 3 \\ m \end{matrix} \right\}_U, \\ \sum_{k=0}^{4n+2} \left\{ \begin{matrix} 4n + 2m + 2 \\ k + m \end{matrix} \right\}_U (-1)^{\frac{1}{2}k(k-1)} U_k &= U_{2n+1} V_{1+m+2n} \left\{ \begin{matrix} 4n + 2m + 1 \\ m - 1 \end{matrix} \right\}_U. \end{aligned}$$

As a showcase we will prove the second identity just above. First we convert the second identity into q -form. Thus, it takes the form

$$\sum_{k=0}^{4n+2} \left[\begin{matrix} 4n + 2m + 3 \\ k + m \end{matrix} \right]_q \alpha^{(k+m)(4n+2m+3-(k+m))} (-1)^{\frac{1}{2}k(k-1)} = \left[\begin{matrix} 4n + 2m + 3 \\ m \end{matrix} \right]_q \alpha^{m(4n+2m+3-m)}$$

or

$$\sum_{k=0}^{4n+2} \left[\begin{matrix} 4n + 2m + 3 \\ k + m \end{matrix} \right]_q \alpha^{(k+m)(m-k+4n+3)} (-1)^{\frac{1}{2}k(k-1)} = \left[\begin{matrix} 4n + 2m + 3 \\ m \end{matrix} \right]_q \alpha^{m(4n+m+3)}$$

or

$$\sum_{k=0}^{4n+2} \left[\begin{matrix} 4n + 2m + 3 \\ k + m \end{matrix} \right]_q (-1)^k q^{-\frac{1}{2}k(4n-k+3)} = \left[\begin{matrix} 4n + 2m + 3 \\ m \end{matrix} \right]_q,$$

which was already given as the identity (1).

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