Evaluation of sums involving products of Gaussian q-binomial coefficients with applications to Fibonomial sums

Emrah KILIÇ* AND Helmut PRODINGER

Abstract. Sums of products of two Gaussian q-binomial coefficients with a parametric rational weight function are considered. The partial fraction decomposition technique is used to to evaluate the sums in closed form. Interesting applications of these results to certain generalized Fibonomial and Lucanomial sums are provided.

1. Introduction

Define the second order linear sequences $\{U_n\}$ and $\{V_n\}$ for $n \geq 2$ by

$$\begin{aligned} U_n &= pU_{n-1} + U_{n-2}, \quad U_0 = 0,\ U_1 = 1, \\ V_n &= pV_{n-1} + V_{n-2}, \qquad V_0 = 2, \quad V_1 = p. \end{aligned}$$

The Binet forms are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q} \quad \text{and} \quad V_n = \alpha^n + \beta^n = \alpha^n\left(1 + q^n\right)$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$.

When $\alpha = \frac{1+\sqrt{5}}{2}$ (or equivalently $q = (1-\sqrt{5}\,)/(1+\sqrt{5}\,)$), the sequence $\{U_n\}$ is reduced to the Fibonacci sequence $\{F_n\}$ and the sequence $\{V_n\}$ is reduced to the Lucas sequence $\{L_n\}$.

Throughout this paper we will use the following notations: the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1})$ and the Gaussian q-binomial coefficients as

$$\begin{bmatrix} n \\ k \end{bmatrix}_z = \frac{(z;q)_n}{(z;q)_k(z;q)_{n-k}}.$$

When $z = q$, we denote $(q;q)_n$ by $(q)_n$.

2010 *Mathematics Subject Classification.* Primary 11B39; Secondary 05A30.

Key words and phrases. Gaussian q-binomial coefficients, Fibonomial coefficients,Lucanomial Coefficients, Sums identites.

* Corresponding author.

Furthermore, we will use *generalized Fibonomial coefficients*

$$\left\{ {n \atop k} \right\}_U = \frac{U_n U_{n-1} \dots U_{n-k+1}}{U_1 U_2 \dots U_k}$$

with $\left\{ {n \atop 0} \right\}_U = 1$ where U_n is the nth generalized Fibonacci number.

When $U_n = F_n$, the generalized Fibonomial is reduced to the Fibonomial coefficients denoted by $\left\{ {n \atop k} \right\}_F$:

$$\left\{ {n \atop k} \right\}_F = \frac{F_n F_{n-1} \dots F_{n-k+1}}{F_1 F_2 \dots F_k}.$$

The link between the generalized Fibonomial and Gaussian q-binomial coefficients is

$$\left\{ {n \atop k} \right\}_U = \alpha^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q \quad \text{with} \quad q = -\alpha^{-2}.$$

Furthermore, we will use *generalized Lucanomial coefficients*

$$\left\{ {n \atop k} \right\}_V = \frac{V_n V_{n-1} \dots V_{n-k+1}}{V_1 V_2 \dots V_k}$$

with $\left\{ {n \atop k} \right\}_V = 1$ where V_n is the nth generalized Lucas number.

When $V_n = L_n$, the generalized Lucanomial coefficients are reduced to the Lucanomial coefficients denoted by $\left\{ {n \atop k} \right\}_L$:

$$\left\{ {n \atop k} \right\}_L = \frac{L_n L_{n-1} \dots L_{n-k+1}}{L_1 L_2 \dots L_k}.$$

The link between the generalized Lucanomial and Gaussian q-binomial coefficients is

$$\left\{ {n \atop k} \right\}_V = \alpha^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{-q} \quad \text{with} \quad q = -\alpha^{-2}.$$

Recently Kılıç and Prodinger [2, 3] computed various sums including Gaussian q-binomial coefficients with certain rational weight function. A typical example from [3] is

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-1)} = (-1)^n q^{-\frac{n}{2}(3n+1)} \begin{bmatrix} 2n \\ n \end{bmatrix}_q \begin{bmatrix} 3n+1 \\ n \end{bmatrix}_q.$$

From [2], recall that for any positive integer w, any nonzero real number a, nonnegative integer n, integers t and r such that $t + n \geq 0$

and $r \geq -1$,

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(-1)^j q^{\binom{j+1}{2}+jt}}{(aq^j;q^w)_{r+1}}$$
$$= a^{-t}(q;q)_n \Bigg(\sum_{j=0}^{r} \frac{(-1)^j}{(q^w;q^w)_j (q^w;q^w)_{r-j}} \frac{q^{w\binom{j+1}{2}-twj}}{(aq^{wj};q)_{n+1}}$$
$$+ (-1)^{r+1} \sum_{j=0}^{t-r-1} \begin{bmatrix} n+j \\ n \end{bmatrix}_q \begin{bmatrix} t-1-j \\ r \end{bmatrix}_{q^w} q^{w\binom{r+1}{2}+(j-t)rw} a^j \Bigg).$$

In [4], Kılıç and Prodinger are interested to evaluate

$$\sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_U^2 U_{\lambda_1 k + r_1} \dots U_{\lambda_s k + r_s}$$

in closed form where r_i and $\lambda_i \geq 1$ are integers. The authors give a systematic approach to compute this sums. For example, it was shown that for nonnegative n,

$$\sum_{k=0}^{2n} \begin{Bmatrix} 2n \\ k \end{Bmatrix}^2 U_{2k}^2 = \Delta \begin{Bmatrix} 2n \\ n \end{Bmatrix}_{U;2} \frac{U_{2n}^3 U_{2n+1}}{V_{2n-1} V_{2n}},$$

where $\Delta = p^2 + 4$.

Marques and Trojovsky [6] provided various sums including Fibonomial coefficients, Fibonacci and Lucas numbers. For example, for positive integers m and n, they showed that

$$\sum_{j=0}^{4m+2} (-1)^{\frac{j(j+1)}{2}} \begin{Bmatrix} 4m+2 \\ j \end{Bmatrix} L_{2m+1-j} = - \begin{Bmatrix} 4m+2 \\ 4n+3 \end{Bmatrix} \frac{F_{4n+3}}{F_{2m+1}}$$

and

$$\sum_{j=0}^{4m+2} (-1)^{\frac{j(j-1)}{2}} \begin{Bmatrix} 4m \\ j \end{Bmatrix} F_{n+4m-j} = \frac{1}{2} F_{2m+n} \sum_{j=0}^{4m} (-1)^{\frac{j(j-1)}{2}} \begin{Bmatrix} 4m \\ j \end{Bmatrix} L_{2m-j}.$$

Recently the generalized Fibonomial coefficients have taken the interest of several authors. For their properties, we refer to [1,5-8]

In this paper we will compute three types of sums involving products of the Gaussian q-binomial coefficients. They are of the following forms: for any real number a

$$\mathsf{SUM} = \sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{-nk+\binom{k}{2}} (a - q^k),$$

$$\mathsf{SUM} = \sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{-nk+\binom{k}{2}} \frac{1}{q^{-k}-a}$$

and

$$\mathsf{SUM} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q (-1)^k q^{-nk+\binom{k+1}{2}} \frac{a-q^{-k}}{b-q^{-k}}.$$

Then we will present interesting applications of our results to generalized Fibonomial and Lucanomial sums.

2. The Main Results

We start with the first kind of sums:

Theorem 1. *For any real a and $n \geq 0$*

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{-nk+\binom{k}{2}} (a-q^k) = (-1)^n \left[a q^{-\binom{n+1}{2}} - q^{\binom{n+1}{2}} \right].$$

Proof. Rewrite the LHS as

$$\sum_{k=0}^{n} \frac{(1-q^{k+1})\dots(1-q^{k+n})}{(q)_k (q)_{n-k}} (-1)^k q^{-nk+\binom{k}{2}} (a-q^k)$$

or

$$\sum_{k=0}^{n} \frac{(q^{-k}-q^1)\dots(q^{-k}-q^n)}{(q)_k (q)_{n-k}} (-1)^k q^{\binom{k}{2}} (a-q^k).$$

Now set

$$f(z) := \frac{(z-q)\dots(z-q^n)}{(1-z)(1-zq)\dots(1-zq^n)} \Big(a - \frac{1}{z} \Big).$$

Then the partial fraction expansion reads

$$f(z) = \sum_{k=0}^{n} \frac{(q^{-k}-q^1)\dots(q^{-k}-q^n)}{(q)_k (q)_{n-k} (1-zq^k)} (-1)^k q^{\binom{k+1}{2}} (a-q^k) + \frac{C}{z}.$$

If we multiply this by z and then let $z \to \infty$, then we get

$$a(-1)^n q^{-\binom{n+1}{2}} = \sum_{k=0}^{n} \frac{(q^{-k}-q^1)\dots(q^{-k}-q^n)}{(q)_k (q)_{n-k}} (-1)^k q^{\binom{k}{2}} (a-q^k) + C,$$

where

$$C = -(-1)^n q^{\binom{n+1}{2}}.$$

Thus

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{-nk+\binom{k}{2}} (a-q^k) = (-1)^n \left[a q^{-\binom{n+1}{2}} - q^{\binom{n+1}{2}} \right],$$

as claimed. $\square$

As consequences of the result above, we have the following corollaries:

Corollary 1. *For $n \geq 0$, all integers r and m,*

$$\sum_{k=0}^{n} \left\{ {n+k \atop k} \right\}_U \left\{ {n \atop k} \right\}_U (-1)^{kn+\frac{1}{2}k(k+1)} U_{k+rn+m} = (-1)^{\frac{1}{2}n(n-1)} U_{n(n+1+r)+m}.$$

Proof. If we convert the claimed identity into q-notation, it takes the form

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q q^{-nk+\binom{k}{2}} \left(1-q^{k+rn+m}\right)(-1)^k$$
$$= (-1)^n q^{-\binom{n+1}{2}} \left(1-q^{n(n+1+r)+m}\right).$$

Since $(1-q^{k+rn+m}) = q^{rn+m}(q^{-rn-m}-q^k)$, the result follows by taking $a = q^{-rn-m}$ in Theorem 2.1. □

Corollary 2. *For $n \geq 0$, all integers r and m,*

$$\sum_{k=0}^{n} \left\{ {n+k \atop k} \right\}_U \left\{ {n \atop k} \right\}_U U_{nr+m-k} (-1)^{kn+\frac{1}{2}k(k+1)}$$
$$= (-1)^{m+nr-\frac{1}{2}n(n-1)} U_{n(n-r+1)-m}.$$

Proof. If we convert the claim into q-form, then we should prove

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \left(1-q^{nr+m-k}\right) q^{k(1-n)+\binom{k}{2}} (-1)^k$$
$$= (-1)^n q^{m+nr-\frac{1}{2}n(n+1)} \left(1-q^{n(n-r+1)-m}\right).$$

Rewrite the LHS as

$$-\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \left(q^{nr+m}-q^k\right) q^{-nk+\binom{k}{2}} (-1)^k,$$

then the result follows by taking $a = q^{nr+m}$ in Theorem 2.1. □

Corollary 3. *For $n \geq 0$, all integers r and m,*

$$\sum_{k=0}^{n} \left\{ {n+k \atop k} \right\}_U \left\{ {n \atop k} \right\}_U V_{nr+k+m} (-1)^{kn+\frac{1}{2}k(k+1)} = -(-1)^{\frac{1}{2}n(n-1)} V_{m+n(n+r+1)}.$$

Proof. If we convert the claimed identity into q-form, then we need to prove

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \left(1+q^{nr+k+m}\right) q^{-nk+\binom{k}{2}}(-1)^k$$
$$= (-1)^{n+1} q^{-\binom{n+1}{2}} \left(1+q^{nr+m} q^{n(n+1)}\right).$$

Rewrite its LHS as

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \left(1+q^{nr+k+m}\right) q^{-nk+\binom{k}{2}}(-1)^k$$
$$= -q^{nr+m} \sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \left(-q^{-nr-m}-q^k\right) q^{-nk+\binom{k}{2}}(-1)^k.$$

Now the result follows by taking $a = -q^{-nr-m}$ in Theorem 2.1. □

Corollary 4. *For $n \geq 0$, all integers r and m,*

$$\sum_{k=0}^{n} \begin{Bmatrix} n+k \\ k \end{Bmatrix}_U \begin{Bmatrix} n \\ k \end{Bmatrix}_U V_{nr+m-k}(-1)^{kn+\binom{k}{2}} = (-1)^{nr+m-\binom{n}{2}} V_{n(n+1-r)-m}.$$

Proof. In q-form, we have to prove the identity

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \left(1+q^{nr+m-k}\right) q^{-nk+\binom{k}{2}} q^k (-1)^k$$
$$= (-1)^n \left(q^{nr+m} q^{-\binom{n+1}{2}} + q^{\binom{n+1}{2}}\right).$$

The result follows again by taking $a = -q^{nr+m}$ in Theorem 2.1. □

Theorem 2. *For $n \geq 0$ and any real a,*

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{-nk+\binom{k}{2}} \frac{1}{q^{-k}-a} = a^n \frac{(qa^{-1};q)_n}{(a;q)_{n+1}}.$$

Proof. Consider

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{-nk+\binom{k}{2}} \frac{1}{z-a},$$

which we rewrite as

$$\sum_{k=0}^{n} \frac{(q^{-k}-q^1)\dots(q^{-k}-q^n)}{(q)_k (q)_{n-k}} (-1)^k q^{\binom{k}{2}} \frac{1}{z-a}.$$

Now define

$$A(z) := \frac{(z-q)\dots(z-q^n)}{(1-z)(1-zq)\dots(1-zq^n)}\frac{1}{z-a}.$$

The partial fraction decomposition of $A(z)$ takes the form:

$$A(z) = \sum_{k=0}^{n} \frac{(q^{-k}-q^1)\dots(q^{-k}-q^n)}{(q^{-k}-a)\,(q;q)_k(q;q)_{n-k}\,(1-zq^k)}(-1)^k q^{\binom{k+1}{2}} + \frac{F(n,a)}{z-a}.$$

Now we multiply this relation by z and then let $z \to \infty$ and obtain

$$0 = \lim_{z\to\infty}\left(\sum_{k=0}^{n} \frac{(q^{-k}-q^1)\dots(q^{-k}-q^n)(-1)^k q^{\binom{k+1}{2}}}{(q;q)_k(q;q)_{n-k}\,(q^{-k}-a)}\frac{z}{1-zq^k} + \frac{zF(n,a)}{z-a}\right),$$

which gives us the equation

$$0 = \sum_{k=0}^{n} \frac{(q^{-k}-q^1)\dots(q^{-k}-q^n)(-1)^{k-1} q^{\binom{k}{2}}}{(q;q)_k(q;q)_{n-k}\,(q^{-k}-a)} + F\,(n,a)$$

or

$$\sum_{k=0}^{n} \frac{(q^{-k}-q^1)\dots(q^{-k}-q^n)(-1)^{k} q^{\binom{k}{2}}}{(q;q)_k(q;q)_{n-k}\,(q^{-k}-a)} = F(n,a),$$

where

$$\begin{aligned} F(n,a) &= \left.\frac{(z-q)\dots(z-q^n)}{(1-z)(1-zq)\dots(1-zq^n)}\right|_{z=a} \\ &= \frac{(a-q)\,(a-q^2)\dots(a-q^n)}{(1-a)(1-aq)\dots(1-aq^n)} = a^n\frac{(q/a;q)_n}{(a;q)_{n+1}}, \end{aligned}$$

which completes the proof. □

As consequences of the result above, we have the following corollaries:

Corollary 5. *For $n \geq 0$ and $m > 0$,*

$$\sum_{k=0}^{n} \left\{ {n+k \atop k} \right\}_U \left\{ {n \atop k} \right\}_U \frac{1}{U_{n+m-k}}(-1)^{kn+\binom{k+1}{2}}$$
$$= (-1)^{\binom{n}{2}}\frac{1}{U_m}\left\{ {n+m \atop n} \right\}_U^{-1}\left\{ {2n+m \atop n} \right\}_U.$$

Proof. In q-form, we have to prove the corresponding identity:

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \left(1-q^{n+m-k}\right)^{-1} q^{-nk+\binom{k}{2}}(-1)^k$$
$$= (-1)^n q^{-\binom{n+1}{2}}\frac{1}{1-q^m}\begin{bmatrix} n+m \\ n \end{bmatrix}_q^{-1}\begin{bmatrix} 2n+m \\ n \end{bmatrix}_q.$$

Consider its LHS as

$$\begin{aligned}
&\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{1-q^{n+m-k}} q^{-nk+\binom{k}{2}} (-1)^k \\
&= -\frac{1}{q^{n+m}} \sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{(q^{-k}-q^{-n-m})} q^{-nk+\binom{k}{2}} (-1)^k,
\end{aligned}$$

which, by taking $a = q^{-(n+m)}$ in Theorem 2.6, equals

$$\begin{aligned}
&= -q^{-(n+1)(m+n)} \frac{(q^{n+m+1}; q)_n}{(q^{-n-m}; q)_{n+1}} \\
&= (-1)^n q^{-(n+1)(m+n)} q^{-m(n+1)-n(n+1)/2} \frac{(q;q)_{m+n}}{(q;q)_{m-1}} \\
&= (-1)^n q^{-\frac{1}{2}n(n+1)} \frac{1}{1-q^m} \frac{(q;q)_m}{(q;q)_{m+n}} \frac{(q;q)_{2n+m}}{(q;q)_{n+m}} \\
&= (-1)^n q^{-\frac{1}{2}n(n+1)} \frac{1}{1-q^m} \begin{bmatrix} n+m \\ n \end{bmatrix}_q^{-1} \begin{bmatrix} 2n+m \\ n \end{bmatrix}_q,
\end{aligned}$$

as claimed. □

Corollary 6. *For $n > 0$ and $m \geq 1$,*

$$\begin{aligned}
&\sum_{k=0}^{n} \begin{Bmatrix} n+k \\ k \end{Bmatrix}_U \begin{Bmatrix} n \\ k \end{Bmatrix}_U \frac{1}{V_{n+m-k}} (-1)^{kn+\binom{k+1}{2}} \\
&\qquad\qquad = (-1)^{\binom{n}{2}} \frac{1}{V_m} \begin{Bmatrix} n+m \\ m \end{Bmatrix}_V^{-1} \begin{Bmatrix} 2n+m \\ n \end{Bmatrix}_V .
\end{aligned}$$

Proof. We have to prove the corresponding identity in q-form:

$$\begin{aligned}
&\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q q^{-nk+\binom{k}{2}} (-1)^k \frac{1}{(1+q^{n+m-k})} \\
&\qquad\qquad = (-1)^n q^{-\binom{n+1}{2}} \frac{1}{1+q^m} \begin{bmatrix} n+m \\ n \end{bmatrix}_{-q}^{-1} \begin{bmatrix} 2n+m \\ n \end{bmatrix}_{-q} .
\end{aligned}$$

If we take $a = -q^{-n-m}$ in Theorem 2.6, the claimed result follows after some rearrangements. □

Corollary 7. *For $n, m \geq 0$,*

$$\sum_{k=0}^{n} \begin{Bmatrix} n+k \\ k \end{Bmatrix}_U \begin{Bmatrix} n \\ k \end{Bmatrix}_U \frac{1}{V_{n-m-k}} (-1)^{kn+\binom{k+1}{2}} = \frac{1}{2}(-1)^{\binom{m+1}{2}-\binom{n}{2}} \begin{Bmatrix} n \\ m \end{Bmatrix}_V \begin{Bmatrix} 2n-m \\ n \end{Bmatrix}_V.$$

Proof. We should prove the corresponding identity in q-form:

$$\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{1+q^{n-m-k}} (-1)^k q^{-nk+\binom{k}{2}} = \frac{1}{2}(-1)^n q^{\frac{1}{2}m(m+1)-\frac{1}{2}n(n+1)} \begin{bmatrix} n \\ m \end{bmatrix}_{-q} \begin{bmatrix} 2n-m \\ n \end{bmatrix}_{-q}.$$

Consider its LHS as

$$\begin{aligned} &\sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{1+q^{n-m-k}} (-1)^k q^{-nk+\binom{k}{2}} \\ &= q^{m-n} \sum_{k=0}^{n} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{-nk+\binom{k}{2}} \frac{1}{q^{-k}+q^{m-n}}, \end{aligned}$$

which, by taking $a = -q^{m-n}$ in Theorem 2.6, equals

$$\begin{aligned} &= q^{m-n} \left(-q^{m-n}\right)^n \frac{(-q^{n-m+1};q)_n}{(-q^{m-n};q)_{n+1}} \\ &= (-1)^n q^{(n+1)(m-n)} \frac{(-q^{n-m+1};q)_n}{(-q^{m-n};q)_{n+1}} \\ &= (-1)^n q^{(m-n)(n+1)} \frac{(-q;q)_{2n-m}}{(-q;q)_{n-m}} \frac{q^{(n-m)(n-m+1)/2}}{2(-q;q)_{n-m}\,(-q;q)_m} \\ &= \frac{1}{2}(-1)^n q^{\frac{1}{2}m(m+1)-\frac{1}{2}n(n+1)} \frac{(-q;q)_{2n-m}}{(-q;q)_{n-m}\,(-q;q)_n} \frac{(-q;q)_n}{(-q;q)_{n-m}\,(-q;q)_m} \\ &= \frac{1}{2}(-1)^n q^{\frac{1}{2}m(m+1)-\frac{1}{2}n(n+1)} \begin{bmatrix} n \\ m \end{bmatrix}_{-q} \begin{bmatrix} 2n-m \\ n \end{bmatrix}_{-q}, \end{aligned}$$

as claimed. $\square$

Theorem 3. *For $n > 0$, any reals a and b,*

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}n+k-1\\k\end{bmatrix}_q(-1)^k q^{-nk+\binom{k+1}{2}}\frac{a-q^{-k}}{b-q^{-k}}$$
$$= -\left(1-q^n\right)b^{n-1}\left(a-b\right)\frac{\left(q/b;q\right)_{n-1}}{\left(b;q\right)_{n+1}}.$$

Proof. We rewrite the LHS of the claim as

$$(1-q^n)\sum_{k=0}^{n}\frac{(1-q^{k+1})\dots(1-q^{n+k-1})}{(q)_k(q)_{n-k}}q^{\frac{1}{2}k(k-2n+1)}(-1)^k\frac{a-q^{-k}}{b-q^{-k}}$$

or

$$(1-q^n)\sum_{k=0}^{n}\frac{(q^{-k}-q^1)\dots(q^{-k}-q^{n-1})}{(q)_k(q)_{n-k}}q^{\binom{k}{2}}(-1)^k\frac{a-q^{-k}}{b-q^{-k}}.$$

Define

$$A(z)=\frac{(z-q)\dots(z-q^{n-1})}{(1-z)\dots(1-zq^n)}\frac{a-z}{b-z}.$$

Then the partial fraction decomposition reads

$$A(z)=\sum_{k=0}^{n}\frac{q^{-k(n-1)}(1-q^{k+1})\dots(1-q^{n+k-1})q^{\binom{k+1}{2}}(-1)^k}{(q)_k(q)_{n-k}(1-zq^k)}\frac{a-q^{-k}}{b-q^{-k}}+\frac{F}{b-z}.$$

If we multiply this by z and then let $z\to\infty$, we find

$$\begin{aligned}
0&=\sum_{k=0}^{n}\frac{q^{-k(n-1)}(1-q^{k+1})\dots(1-q^{n+k-1})q^{\binom{k+1}{2}}(-1)^k}{(q)_k(-q^k)(q)_{n-k}}\frac{a-q^{-k}}{b-q^{-k}}-F\\
&=\sum_{k=0}^{n}\frac{q^{-k(n-1)}(q)_{n+k-1}q^{\binom{k}{2}}(-1)^{k-1}}{(q)_k(q)_k(q)_{n-k}}\frac{a-q^{-k}}{b-q^{-k}}-F\\
&=\frac{(q)_{n-1}}{(q)_n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}n+k-1\\k\end{bmatrix}_q q^{\frac{1}{2}k(k-2n+1)}(-1)^{k-1}\frac{a-q^{-k}}{b-q^{-k}}-F\\
&=\frac{1}{1-q^n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}n+k-1\\k\end{bmatrix}_q q^{\frac{1}{2}k(k-2n+1)}(-1)^{k-1}\frac{a-q^{-k}}{b-q^{-k}}-F.
\end{aligned}$$

Therefore

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}n+k-1\\k\end{bmatrix}_q q^{\frac{1}{2}k(k-2n+1)}(-1)^k\frac{a-q^{-k}}{b-q^{-k}}=-\left(1-q^n\right)F,$$

where

$$\begin{aligned} F &= \frac{(z-q)\dots(z-q^{n-1})}{(1-z)\dots(1-zq^{n})}\,(a-z)\bigg|_{z=b} \\ &= \frac{(b-q)\dots(b-q^{n-1})}{(1-b)\,(1-zb)\dots(1-bq^{n})}\,(a-b) \\ &= (a-b)\,\frac{b^{n-1}\,(1-q/b)\,(1-q^2/b)\dots(1-q^{n-1}/b)}{(b;q)_{n+1}} \\ &= b^{n-1}\,(a-b)\,\frac{(q/b;q)_{n-1}}{(b;q)_{n+1}}. \end{aligned}$$

Thus we get

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-2n+1)}(-1)^k \frac{a-q^{-k}}{b-q^{-k}} = -\,(1-q^n)\,b^{n-1}\,(a-b)\,\frac{(q/b;q)_{n-1}}{(b;q)_{n+1}},$$

as claimed. $\square$

Corollary 8. *For n, $m>0$*

$$\sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_U \begin{Bmatrix} n+k-1 \\ k \end{Bmatrix}_U (-1)^{kn}\,(-1)^{\frac{1}{2}k(k+1)} \frac{U_{n+1-k}}{U_{n+m+k}} = -(-1)^{\frac{1}{2}n(n+1)} \frac{U_{2n+m+1}}{U_{n+1}} \begin{Bmatrix} n+m-1 \\ m \end{Bmatrix}_U \begin{Bmatrix} 2n+m \\ n+1 \end{Bmatrix}_U^{-1}.$$

Proof. If we convert the claimed identity into q-form, then we have to prove the identity

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-2n+3)}(-1)^k \frac{1-q^{n+1-k}}{1-q^{n+m+k}} = (-1)^{n+1} q^{\frac{1}{2}n(n-1)} \frac{1-q^{2n+m+1}}{1-q^{n+1}} \begin{bmatrix} n+m-1 \\ m \end{bmatrix}_q \begin{bmatrix} 2n+m \\ n+1 \end{bmatrix}_q^{-1}.$$

If we take $a=q^{-n-1}$ and $b=q^{n+m}$ in Theorem 2.10, then we get the claimed identity after some rearrangements. $\square$

References

[1] Horadam AF. Generating functions for powers of a certain generalized sequence of numbers, Duke Math J 1965; 32: 437–446.
[2] Kılıç E, Prodinger H. Evaluation of sums involving Gaussian q-binomial coefficients with rational weight functions, Inter J Number Theory 2016; 12: 495–504.
[3] Kılıç E, Prodinger H. Formulæ related to the q-Dixon formula with applications to Fibonomial sums, Period Math Hung 2015; 70: 216–226.
[4] Kılıç E, Prodinger H. Closed form evaluation of sums containing squares of Fibonomial coefficients, accepted in Math Slovaca
[5] Kılıç E. The generalized Fibonomial matrix, European J Comb 2008; 29: 701–711.
[6] Marques D, Trojovsky P. On Some New Sums of Fibonomial Coefficients, The Fibonacci Quarterly 2012; 50: 155–162.
[7] Seibert J, Trojovsky P. On some identities for the Fibonomial coefficients, Math Slovaca 2005; 55: 9–19.
[8] Trojovsky P. On some identities for the Fibonomial coefficients via generating function. Discrete Appl Math 2007; 155: 2017–2024.

Department of Mathematics, TOBB Economics and Technology University 06560, Ankara, Çankaya, Turkey
E-mail address: `ekilic@etu.edu.tr`

Department of Mathematics, University of Stellenbosch, 7602, Stellenbosch, South Africa
E-mail address: `hproding@sun.ac.za`