SOME GAUSSIAN BINOMIAL SUM FORMULÆ WITH APPLICATIONS

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ABSTRACT. We introduce and compute some Gaussian q-binomial sums formulæ. In order to prove these sums, our approach is to use q-analysis, in particular a formula of Rothe, and computer algebra. We present some applications of our results.

1. INTRODUCTION

Let $\{U_n\}$ and $\{V_n\}$ be generalized Fibonacci and Lucas sequences, respectively, whose the Binet forms are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q} \quad \text{and} \quad V_n = \alpha^n + \beta^n = \alpha^n(1+q^n)$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$.

When $\alpha = \frac{1+\sqrt{5}}{2}$ (or equivalently $q = (1-\sqrt{5})/(1+\sqrt{5})$), the sequence $\{U_n\}$ is reduced to the Fibonacci sequence $\{F_n\}$ and the sequence $\{V_n\}$ is reduced to the Lucas sequence $\{L_n\}$.

When $\alpha = 1+\sqrt{2}$ (or equivalently $q = (1-\sqrt{2})/(1+\sqrt{2})$), the sequence $\{U_n\}$ is reduced to the Pell sequence $\{P_n\}$ and the sequence $\{V_n\}$ is reduced to the Pell-Lucas sequence $\{Q_n\}$.

Throughout this paper we will use the following notations: the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1})$ and the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_z = \frac{(q^z;q^z)_n}{(q^z;q^z)_k(q^z;q^z)_{n-k}}.$$

The $z=1$ case will be denoted by $\begin{bmatrix} n \\ k \end{bmatrix}$.

Furthermore, we will use *generalized Fibonomial coefficients*

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{U,t} = \frac{U_{nt}U_{(n-1)t}\dots U_{(n-k+1)t}}{U_tU_{2t}\dots U_{kt}}$$

with $\begin{Bmatrix} n \\ 0 \end{Bmatrix}_{U,t} = 1$ where U_n is the nth generalized Fibonacci number.

In the special case $t=1$, the generalized Fibonomial coefficients are denoted by $\begin{Bmatrix} n \\ k \end{Bmatrix}_U$. When $U_n = F_n$, the generalized Fibonomial reduces to the Fibonomial coefficients denoted by $\begin{Bmatrix} n \\ k \end{Bmatrix}_F$:

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_F = \frac{F_nF_{n-1}\dots F_{n-k+1}}{F_1F_2\dots F_k}.$$

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Similarly, when $U_n = P_n$, the generalized Fibonomial reduces to the Pellnomial coefficients denoted by $\left\{ {n \atop k} \right\}_P$:

$$\left\{ {n \atop k} \right\}_P = \frac{P_n P_{n-1} \dots P_{n-k+1}}{P_1 P_2 \dots P_k}.$$

The link between the generalized Fibonomial and Gaussian q-binomial coefficients is

$$\left\{ {n \atop k} \right\}_{U,t} = \alpha^{tk(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_t \quad \text{with} \quad q = -\alpha^{-2}.$$

For the reader's convenience and later use, we recall Rothe's formula [1, 10.2.2(c)]:

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} x^k = (x;q)_n .$$

We can refer to [2, 3, 4, 5, 6, 7, 8] for various sums of Gaussian q-binomial coefficients and sums of generalized Fibonomial sums with certain weight functions. Recently, the authors of [8, 7] computed certain Fibonomial sums with generalized Fibonacci and Lucas numbers as coefficients. For example, if n and m are both nonnegative integers, then

$$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\} U_{(2m-1)k} = P_{n,m} \sum_{k=1}^{m} \left\{ {2m-1 \atop 2k-1} \right\} U_{(4k-2)n},$$

$$\sum_{k=0}^{2n+1} \left\{ {2n+1 \atop k} \right\} U_{2mk} = P_{n,m} \sum_{k=0}^{m} \left\{ {2m \atop 2k} \right\} U_{(2n+1)2k},$$

$$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\} V_{(2m-1)k} = P_{n,m} \sum_{k=1}^{m} \left\{ {2m-1 \atop 2k-1} \right\} V_{(4k-2)n},$$

$$\sum_{k=0}^{2n+1} \left\{ {2n+1 \atop k} \right\} V_{2mk} = P_{n,m} \sum_{k=0}^{m} \left\{ {2m \atop 2k} \right\} V_{(2n+1)2k},$$

where

$$P_{n,m} = \begin{cases} \prod\limits_{k=0}^{n-m} V_{2k} & \text{if } n \geq m, \\ \prod\limits_{k=1}^{m-n-1} V_{2k}^{-1} & \text{if } n < m; \end{cases}$$

alternating analogues of these sums were also evaluated.

Recently Kılıç and Prodinger [3] computed the following Gaussian q-binomial sums with a parametric rational weight function: For any positive integer w, any nonzero real number a, nonnegative integer n, integers t and r such that $t + n \geq 0$ and $r \geq -1$,

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(-1)^j q^{\binom{j+1}{2}+jt}}{(aq^j;q^w)_{r+1}}$$

$$= a^{-t}(q;q)_n \Bigg(\sum_{j=0}^{r} \frac{(-1)^j}{(q^w;q^w)_j (q^w;q^w)_{r-j}} \frac{q^{w\binom{j+1}{2}-twj}}{(aq^{wj};q)_{n+1}}$$

$$+ (-1)^{r+1} \sum_{j=0}^{t-r-1} \begin{bmatrix} n+j \\ n \end{bmatrix}_q \begin{bmatrix} t-1-j \\ r \end{bmatrix}_{q^w} q^{w\binom{r+1}{2}+(j-t)rw} a^j \Bigg).$$

In this paper we derive some Gaussian q-binomial sums. Then we present some applications of our results.

2. The Main Results

We start with our first result:

Theorem 1. *For any $n \geq 1$,*

$$\sum_{k=1}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix} q^{\frac{1}{2}k(k-1)} \left(1-q^k\right) = (1-q^n) \begin{bmatrix} 2n-1 \\ n \end{bmatrix}$$

and its Fibonomial corollary:

$$\sum_{k=1}^{n} \begin{Bmatrix} 2n \\ n+k \end{Bmatrix}_{U,t} (-1)^{\binom{k}{2}} U_{tk} = U_{tn} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U,t}.$$

Proof. Let

$$S = \sum_{k=-n}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix} q^{\frac{1}{2}k(k-1)} \left(1-q^k\right).$$

Thus

$$S = \sum_{k=-n}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix} q^{\frac{1}{2}k(k+1)} \left(1-q^{-k}\right)$$
$$= \sum_{k=-n}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix} q^{\frac{1}{2}k(k-1)} \left(q^k-1\right) = -S,$$

so $S=0$. Let

$$F(n,m) = \sum_{k=-n}^{m} \begin{bmatrix} 2n \\ n+k \end{bmatrix} q^{\frac{1}{2}k(k-1)} \left(1-q^k\right).$$

We need $-F(n,0)$ to evaluate our sum. Define

$$G(n,m) := -(1-q^n) \begin{bmatrix} 2n-1 \\ n+m \end{bmatrix} q^{m(m+1)/2}.$$

Then we have

$$G(n,m) = F(n,m),$$

which follows from

$$G(n,m) - G(n,m-1) = \begin{bmatrix} 2n \\ n+m \end{bmatrix} q^{\frac{1}{2}m(m-1)} (1-q^n).$$

Therefore our answer is

$$-F(n,0) = -G(n,0) = (1-q^n) \begin{bmatrix} 2n-1 \\ n \end{bmatrix},$$

as claimed.

The Fibonacci corollary follows by first replacing q by q^t and then translating. □

For example, when $t=1$ and $\alpha=1+\sqrt{2}$ (or equivalently $q=\frac{1-\sqrt{2}}{1+\sqrt{2}}$), we have the following Pellnomial-Pell sum identity:

$$\sum_{k=1}^{n}\left\{\begin{matrix}2n\\n+k\end{matrix}\right\}_P(-1)^{\binom{k}{2}}P_k=P_n\left\{\begin{matrix}2n-1\\n\end{matrix}\right\}_P.$$

When $t=3$ and $\alpha=\frac{1+\sqrt{5}}{2}$ (or equivalently $q=\frac{1-\sqrt{5}}{1+\sqrt{5}}$), then we have the following Fibonomial-Fibonacci sum identity:

$$\sum_{k=1}^{n}\left\{\begin{matrix}2n\\n+k\end{matrix}\right\}_{F,3}(-1)^{\binom{k}{2}}F_{3k}=F_{3n}\left\{\begin{matrix}2n-1\\n\end{matrix}\right\}_{F,3}.$$

Our second result is:

Theorem 2. *For all n such that $2n-1\ge r$ we have*

$$\sum_{k=1}^{n}\begin{bmatrix}2n\\n+k\end{bmatrix}(-1)^k q^{\frac12\left(k^2-k(2r+1)\right)}\left(1+q^k\right)^{2r+1}=-2^{2r}\begin{bmatrix}2n\\n\end{bmatrix},$$

and its generalized Fibonomial-Lucas corollary:

$$\sum_{k=1}^{n}\left\{\begin{matrix}2n\\n+k\end{matrix}\right\}_{U,t}(-1)^{\frac{k(k+(-1)^r)}{2}}V_{kt}^{2r+1}=-4^r\left\{\begin{matrix}2n\\n\end{matrix}\right\}_{U,t}.$$

Proof. Define

$$S:=\sum_{k=1}^{n}\begin{bmatrix}2n\\n+k\end{bmatrix}(-1)^k q^{\frac12 k(k-(2r+1))}\left(1+q^k\right)^{2r+1}.$$

Then we write

$$2S=\sum_{k\neq0}\begin{bmatrix}2n\\n+k\end{bmatrix}(-1)^k q^{\frac12 k(k-(2r+1))}\left(1+q^k\right)^{2r+1}$$

and so

$$2S+2^{2r+1}\begin{bmatrix}2n\\n\end{bmatrix}=\sum_{k=-n}^{n}\begin{bmatrix}2n\\n+k\end{bmatrix}(-1)^k q^{\frac12 k(k-(2r+1))}\left(1+q^k\right)^{2r+1}.$$

Consider

$$\begin{aligned}
&\sum_{k=-n}^{n}\begin{bmatrix}2n\\n+k\end{bmatrix}(-1)^k q^{\frac12 k(k-(2r+1))}z^k\\
&=\sum_{k=0}^{2n}\begin{bmatrix}2n\\k\end{bmatrix}(-1)^{k-n}q^{\frac12(k-n)(k-n-(2r+1))}z^{k-n}\\
&=(-1)^n z^{-n}q^{\frac{n^2+n(2r+1)}{2}}\sum_{k=0}^{2n}\begin{bmatrix}2n\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}}(zq^{-n-r})^k\\
&=(-1)^n z^{-n}q^{\binom{n+1}{2}+nr}(zq^{-n-r};q)_{2n},
\end{aligned}$$

according to formula 10.2.2(c) (Rothe's formula) in [1]. In order to obtain our claimed sum S, we use this formula for $z=1,q,q^2,\dots,q^{2r+1}$. Hence they are all 0 provided that $r \leq 2n-1$. Therefore

$$\sum_{k=1}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix} (-1)^k q^{\frac{1}{2}k(k-(2r+1))} \left(1+q^k\right)^{2r+1} = -2^{2r} \begin{bmatrix} 2n \\ n \end{bmatrix},$$

as claimed. □

We can now replace q by q^t to obtain some Fibonomial type corollaries.

As an example, when $t=3$, $r=2$ and $\alpha = \frac{1+\sqrt{5}}{2}$ (or equivalently $q = \frac{1-\sqrt{5}}{1+\sqrt{5}}$), then we have the following Fibonomial-Lucas sum identity:

$$\sum_{k=1}^{n} \left\{ \begin{matrix} 2n \\ n+k \end{matrix} \right\}_{F,3} (-1)^{\binom{k+1}{2}} L_{3t}^5 = -16 \left\{ \begin{matrix} 2n \\ n \end{matrix} \right\}_{F,3}.$$

Our third result is a list of formulæ that can be obtained automatically by using the q-Zeilberger algorithm, in particular the version that was developed at the Risc center in Linz [9].

Theorem 3. *For* $n \geq 1$

$$\sum_{k=0}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix} q^{\frac{1}{2}k(k-(2b+1))} \left(1-q^{(2b+1)k}\right) = \frac{X_b}{q^{\binom{b+1}{2}} \prod_{j=1}^{b}(1+q^{n-j})} (1-q^n) \begin{bmatrix} 2n-1 \\ n \end{bmatrix},$$

and the polynomials X_b *are getting more and more involved.*

We give a list of the first few:

$$\begin{aligned}
X_0 &= 1,\\
X_1 &= 2+q+q^n+2q^{n+1},\\
X_2 &= 2+2q+q^3+2q^n+q^{2n}+3q^{n+1}+3q^{n+2}+2q^{n+3}+2q^{2n+2}+2q^{2n+3},\\
X_3 &= 2+2q+2q^3+q^6\\
&\quad +2q^n+2q^{2n}+q^{3n}+4q^{1+n}+4q^{2+n}+5q^{3+n}+3q^{4+n}+q^{5+n}+2q^{6+n}\\
&\quad +q^{1+2n}+3q^{2+2n}+5q^{3+2n}+4q^{4+2n}+4q^{5+2n}+2q^{6+2n}\\
&\quad +2q^{3+3n}+2q^{5+3n}+2q^{6+3n},\\
X_4 &= 2+2q+2q^3+2q^6+q^{10}\\
&\quad +2q^n+2q^{2n}+2q^{3n}+q^{4n}+4q^{1+n}+4q^{2+n}+6q^{3+n}+6q^{4+n}+4q^{5+n}\\
&\qquad +3q^{6+n}+3q^{7+n}+q^{8+n}+q^{9+n}+2q^{10+n}\\
&\quad +2q^{1+2n}+4q^{2+2n}+7q^{3+2n}+7q^{4+2n}+10q^{5+2n}+7q^{6+2n}+7q^{7+2n}\\
&\qquad +4q^{8+2n}+2q^{9+2n}+2q^{10+2n}\\
&\quad +q^{1+3n}+q^{2+3n}+3q^{3+3n}+3q^{4+3n}+4q^{5+3n}+6q^{6+3n}+6q^{7+3n}\\
&\qquad +4q^{8+3n}+4q^{9+3n}+2q^{10+3n}\\
&\quad +2q^{4+4n}+2q^{7+4n}+2q^{9+4n}+2q^{10+4n}.
\end{aligned}$$

As an example, we state the general Fibonomial-Lucas-Fibonacci instance for $b=1$:

$$\sum_{k=0}^{n} \left\{ \begin{matrix} 2n \\ n+k \end{matrix} \right\}_{U,t} (-1)^{\frac{1}{2}tk(k-3)} U_{3kt} = \frac{\left(2V_{t(n+1)} + (-1)^t V_{t(n-1)}\right) U_{nt}}{(-1)^t V_{(n-1)t}} \left\{ \begin{matrix} 2n-1 \\ n \end{matrix} \right\}_{U,t}.$$

For example, when $\alpha = \left(1+\sqrt{5}\right)/2$ (or equivalently $q = \frac{1-\sqrt{5}}{1+\sqrt{5}}$) and $t=1$, then we have the following Fibonomial-Lucas-Fibonacci sum identity:

$$\sum_{k=0}^{n} \left\{ \begin{matrix} 2n \\ n+k \end{matrix} \right\}_F (-1)^{\frac{1}{2}k(k-3)} F_{3k} = -\frac{L_{n+2}F_n}{L_{n-1}} \left\{ \begin{matrix} 2n-1 \\ n \end{matrix} \right\}_F .$$

We give another Fibonomial-Lucas-Fibonacci corollary (the instance $b=2$); more complicated ones can be obtained by replacing q by q^t and taking larger b's.

$$\begin{aligned} \sum_{k=0}^{n} \left\{ \begin{matrix} 2n \\ n+k \end{matrix} \right\}_U (-1)^{\binom{k}{2}} U_{5k} &= \left(2V_{2n+1} + V_{2n-3} - 2V_{2n+3} + 3(-1)^n V_1 - 2(-1)^n V_3\right) \\ &\qquad \times \frac{U_n}{V_{n-1}V_{n-2}} \left\{ \begin{matrix} 2n-1 \\ n \end{matrix} \right\}_U . \end{aligned}$$

Note that $2V_{2n+1}+V_{2n-3}-2V_{2n+3}$ could still simplified a bit using the recursion, but the recursion depends on α.

For example, when $\alpha = \left(1+\sqrt{5}\right)/2$

$$\sum_{k=0}^{n} \left\{ \begin{matrix} 2n \\ n+k \end{matrix} \right\}_F (-1)^{\binom{k}{2}} F_{5k} = \frac{F_n\left(L_{2n+1} - 4L_{2n} - 5(-1)^n\right)}{L_{n-1}L_{n-2}} \left\{ \begin{matrix} 2n-1 \\ n \end{matrix} \right\}_F .$$

Now we state our next result:

Theorem 4. *For $n \geq 1$*

$$\sum_{k=0}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix} q^{\frac{1}{2}k(k-3)} \left(1-q^k\right)^3 = 2 \begin{bmatrix} 2n-3 \\ n-1 \end{bmatrix} \frac{(1-q)}{q} (1-q^n)\left(1-q^{2n-1}\right),$$

and its Fibonomial-Fibonacci corollary

$$\sum_{k=0}^{n} \left\{ \begin{matrix} 2n \\ n+k \end{matrix} \right\}_{U,t} (-1)^{\frac{1}{2}tk(k-3)} U_{tk}^3 = (-1)^t \, 2U_t U_{tn} U_{t(2n-1)} \left\{ \begin{matrix} 2n-3 \\ n-1 \end{matrix} \right\}_{U,t} .$$

Proof. One can produce a proof similar to our first theorem, but we gain no insight from it; and a computer can prove it without any effort. □

For example, if we take $t=5$ and $\alpha = \frac{1+\sqrt{5}}{2}$ (or equivalently $q = \frac{1-\sqrt{5}}{1+\sqrt{5}}$), then we have the following Fibonomial-Fibonacci sum identity :

$$\sum_{k=0}^{n} \left\{ \begin{matrix} 2n \\ n+k \end{matrix} \right\}_{F,5} (-1)^{\frac{1}{2}k(k-3)} F_{5k}^3 = -2 \left\{ \begin{matrix} 2n-3 \\ n-1 \end{matrix} \right\}_{F,5} F_5 F_{5n} F_{5(2n-1)} .$$

Now we state our next results including the 5^{th} and 7^{th} powers of $\left(1-q^k\right)$:

Theorem 5. *For $n \geq 1$*

$$\sum_{k=0}^{n} \begin{bmatrix} 2n \\ n+k \end{bmatrix} q^{\frac{1}{2}k(k-5)} \left(1-q^k\right)^5 = \frac{2(1-q)^2(1-q^n)^2(1+3q-3q^n-q^{n+1})}{q^3(1+q^{n-1})(1+q^{n-2})} \begin{bmatrix} 2n-1 \\ n \end{bmatrix},$$

and its Fibonomial-Fibonacci corollary

$$\sum_{k=0}^{n} \left\{ \begin{matrix} 2n \\ n+k \end{matrix} \right\}_{U,t} (-1)^{t\binom{k}{2}} U_{tk}^5 = \frac{(-1)^t \, 2U_t^2 U_{tn}^2 \left(U_{t(n+1)} + 3(-1)^t U_{t(n-1)}\right)}{V_{t(n-1)}V_{t(n-2)}} \left\{ \begin{matrix} 2n-1 \\ n \end{matrix} \right\}_{U,t} .$$

Proof. Again, this is best done by a computer. □

For example, when $t=1$ and $\alpha=\left(1+\sqrt{5}\right)/2$, we get the following Fibonomial-Fibonacci corollary:

$$\sum_{k=0}^{n}\begin{Bmatrix}2n\\n+k\end{Bmatrix}_F(-1)^{\binom{k}{2}}F_k^5=\frac{2F_n^2F_{n-3}}{L_{n-1}L_{n-2}}\begin{Bmatrix}2n-1\\n\end{Bmatrix}_F.$$

We also give the next instance; after that, the terms get too involved:

Theorem 6. *For $n\geq 1$*

$$\begin{aligned}\sum_{k=0}^{n}\begin{bmatrix}2n\\n+k\end{bmatrix}q^{\frac{1}{2}k(k-7)}\left(1-q^k\right)^7&=\frac{2(1-q)^3\left(1-q^n\right)^2}{q^6(1+q^{n-1})(1+q^{n-2})(1+q^{n-3})}\begin{bmatrix}2n-1\\n\end{bmatrix}\\&\quad\times(1+4q+9q^2+10q^3+10q^{2n}+9q^{2n+1}+4q^{2n+2}\\&\qquad+q^{2n+3}-5q^n-19q^{n+1}-19q^{n+2}-5q^{n+3}),\end{aligned}$$

and its Fibonomial-Fibonacci-Lucas corollary

$$\begin{aligned}&\sum_{k=0}^{n}\begin{Bmatrix}2n\\n+k\end{Bmatrix}_U(-1)^{\frac{1}{2}k(k-7)}U_k^7\\&\quad=\Big(V_{2n+3}-4V_{2n+1}+9V_{2n-1}-10V_{2n-3}-5\left(-1\right)^nV_3+19\left(-1\right)^nV_1\Big)\\&\quad\times\frac{2U_1^3U_n^2}{5V_{n-1}V_{n-2}V_{n-3}}\begin{Bmatrix}2n-1\\n\end{Bmatrix}_U.\end{aligned}$$

For example, when $\alpha=\left(1+\sqrt{5}\right)/2$, we get

$$\sum_{k=0}^{n}\begin{Bmatrix}2n\\n+k\end{Bmatrix}_F(-1)^{\frac{1}{2}k(k-7)}F_k^7=\frac{2F_n^2(L_{2n-2}+4L_{2n-4}-(-1)^n)}{5L_{n-1}L_{n-2}L_{n-3}}\begin{Bmatrix}2n-1\\n\end{Bmatrix}_F.$$

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