SOME CLASSES OF ALTERNATING WEIGHTED BINOMIAL SUMS

EMRAH KILIÇ

ABSTRACT. In this paper, we consider three classes of generalized alternating weighted binomial sums of the form

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i} f(n,i,k,t)$$

where $f(n,i,k,t)$ will be chosen as $U_{kti}V_{kn-k(t+2)i}$, $U_{kti}V_{kn-kti}$ and $U_{tki}V_{(k+1)tn-(k+2)ti}$. We use the Binet formula and the Newton binomial formula to prove the claimed results. Further we present some interesting examples of our results.

1. Introduction

Define second order linear recurrences $\{U_n\}$ and $\{V_n\}$ as for $n>0$

$$U_n = pU_{n-1}+U_{n-2} \text{ and } V_n = pV_{n-1}+V_{n-2},$$

where $U_0 = 0$, $U_1 = 1$, and, $V_0 = 2$, $V_1 = p$, respectively. If $p = 1$, then $U_n = F_n$ (nth Fibonacci number) and $V_n = L_n$ (nth Lucas number).

The Binet formulæ are

$$U_n = \frac{\alpha^n-\beta^n}{\alpha-\beta} \text{ and } V_n = \alpha^n+\beta^n,$$

where $\alpha,\beta = \left(p\pm\sqrt{\Delta}\right)/2$ and $\Delta = p^2+4$.

Let $A(x)$ and $B(x)$ be the exponential generating functions of sequences $\{a_n\}$ and $\{b_n\}$, that is,

$$A(x) = \sum_{n\geq 0} a_n\frac{x^n}{n!} \text{ and } B(x) = \sum_{n\geq 0} b_n\frac{x^n}{n!}.$$

The convolution of the exponential generating functions is given as

$$A(x)B(x) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}a_k b_{n-k}\right)\frac{x^n}{n!}.$$

Many authors have computed various weighted binomial sums by various methods (for more details, see [1, 3, 6, 7, 8, 9, 10, 11]). One of them is to use the convolution of the exponential generating functions and its direct applications, we recall the followings formulæ from the literature (see [3, 10]):

$$\sum_{i=0}^{n}\binom{n}{i}F_{mi}L_{mn-mi}\,,\ \sum_{i=0}^{n}\binom{n}{i}F_{mi}F_{mn-mi}.$$

2000 *Mathematics Subject Classification.* 05A10, 11B37.

Key words and phrases. Binomial sums, binary linear recurrences.

Meanwhile, for some integers k, t and r such that $k \neq t$ or $k \neq r$, the binomial sums

$$\sum_{i=0}^{n} \binom{n}{i} F_{ti} L_{kni-ri} \text{ and } \sum_{i=0}^{n} \binom{n}{i} (-1)^{i} F_{ti} L_{kni-ri}$$

can't be computed by the convolution of exponential generating functions.

Recently, the authors of [4] give general formulæ for the sums

$$\sum_{h=0}^{n} \binom{n}{h} h^{m} U_{ht}^{2m+\varepsilon} \ , \quad \sum_{h=0}^{n} \binom{n}{h} h^{m} V_{ht}^{2m+\varepsilon},$$
$$\sum_{h=0}^{n} \binom{n}{h} (-1)^{n+h} h^{m} U_{ht}^{2m+\varepsilon} \ , \quad \sum_{h=0}^{n} \binom{n}{h} (-1)^{n+h} h^{m} V_{ht}^{2m+\varepsilon},$$

where t is an positive integer and $\varepsilon \in \{0, 1\}$.

Further the authors of [5] computed the weighted binomial sums

$$\sum_{k=0}^{n} \binom{n}{k} r_{mk} s_{m(tn+k)},$$

where r_n and s_n are the terms of $\{U_n\}$ and $\{V_n\}$ for some positive integers t and m. For example, for odd m,

$$\sum_{i=0}^{n} \binom{n}{i} U_{mi} V_{kmn+mi} = \Delta^{\lfloor \frac{n}{2} \rfloor} U_{m}^{n} \begin{cases} U_{(k+1)mn} & \text{if } n \text{ is even,} \\ V_{(k+1)mn} & \text{if } n \text{ is odd,} \end{cases}$$

and for even m,

$$\sum_{i=0}^{n} \binom{n}{i} V_{mi} V_{kmn+mi} = V_{m}^{n} V_{(k+1)mn} + 2^{n} V_{kmn}.$$

In this paper we consider new three classes of generalized alternating binomial sums that couldn't be derived via the convolution of exponential generating functions :

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^{i} f(n, i, k, t),$$

where $f(n, i, k, t)$ will be chosen as $U_{kti} V_{kn-(t+2)ki}$, $U_{kti} V_{kn-kti}$ and $U_{tki} V_{(k+1)tn-(k+2)ti}$.

These binomial sums (except some special cases of k and t) have not been considered according to our best literature acknowledgement. To compute the claimed sums, our approach is to use Binet formula and the Newton binomial formula. In general, let $\{A_n\}$ and $\{B_n\}$ be two second order linear recurrences whose Binet formulae are

$$A_n = c_1 a_1^n + c_1 a_2^n \text{ and } B_n = d_1 b_1^n + d_2 b_2^n.$$

Then we write for the sum

$$\begin{aligned} \sum_{i=0}^{n} \binom{n}{i} A_{ai} B_{m-bi} &= \sum_{i=0}^{n} \binom{n}{i} \left(c_1 a_1^{ai} + c_1 a_2^{ai} \right) \left(d_1 b_1^{m-bi} + d_2 b_2^{m-bi} \right) \\ &= \sum_{1 \leq i,j \leq 2} c_i d_j b_j^{m} \left(a_i^{a} b_j^{-b} + 1 \right)^{n}. \end{aligned}$$

In computing our sums, we will choose required values of the scalers $a_i, b_i c_i d_i$ for $1 \leq i \leq 2$.

2. The main results

Fist we give a auxiliary lemma and then give our first result.

Lemma 1. *Let t be any integer. i) For odd k,*

$$\begin{aligned}(-1)^t \alpha^{-k(2t+1)} - \alpha^k &= (-1)^{t+1} V_{k(t+1)}\beta^{kt}, \\ (-1)^t \beta^{-k(2t+1)} - \beta^k &= (-1)^{t+1} V_{k(t+1)}\alpha^{kt}.\end{aligned}$$

ii) For even k,

$$\left(\alpha^{-k(2t+1)} - \alpha^k\right) = -\sqrt{\Delta}U_{k(t+1)}\beta^{kt} \text{ and } \left(\beta^{-k(2t+1)} - \beta^k\right) = \sqrt{\Delta}U_{k(t+1)}\alpha^{kt}.$$

Theorem 1. *For $n > 0$, any integer t and odd k,*

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{kti}V_{kn-ki(t+2)} = (-1)^{tn} V_{k(t+1)}^n U_{ktn}$$

and for even k,

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{kti}V_{kn-k(t+2)i} = \begin{cases} \Delta^{\frac{n-1}{2}} \left[2U_k^n - U_{k(t+1)}^n V_{ktn}\right] & \text{if } n \text{ is odd,} \\ \Delta^{\frac{n}{2}} U_{k(t+1)}^n U_{ktn} & \text{if } n \text{ is even.} \end{cases}$$

Proof. First assume that k is an odd integer. We write by recalling $\alpha\beta = -1$

$$\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{kti}V_{kn-k(t+2)i} \\
&= \frac{1}{\alpha-\beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \left[\left(\alpha^{kn-2ki} - \beta^{kn-2ki}\right) - (-1)^{ti}\left(\alpha^{kn-2ki(t+1)} - \beta^{kn-2ki(t+1)}\right)\right] \\
&= \frac{\alpha^{kn}}{\alpha-\beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \alpha^{-2ki} - \frac{\beta^{kn}}{\alpha-\beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \beta^{-2ki} \\
&- \frac{1}{\alpha-\beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^{i(t+1)} \left(\alpha^{kn-2ki(t+1)} - \beta^{kn-2ki(t+1)}\right) \\
&= \frac{1}{\alpha-\beta} \left[\left(\alpha^k - \alpha^{-k}\right)^n - \left(\beta^k - \beta^{-k}\right)^n\right] \\
&- \frac{1}{\alpha-\beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^{i(t+1)} \left(\alpha^{kn-2ki(t+1)} - \beta^{kn-2ki(t+1)}\right),
\end{aligned}$$

which, since $\alpha^k - \alpha^{-k} = \beta^k - \beta^{-k}$ for odd k, equals

$$\begin{aligned}
&- \frac{1}{\alpha-\beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^{i(t+1)} \left[\alpha^{kn-2ki(t+1)} - \beta^{kn-2ki(t+1)}\right] \\
&= -\frac{1}{\alpha-\beta} \left[\alpha^{kn} \sum_{i=0}^{n} \binom{n}{i} (-1)^{i(t+1)} \alpha^{-2ki(t+1)} - \beta^{kn} \sum_{i=0}^{n} \binom{n}{i} (-1)^{i(t+1)} \beta^{-2ki(t+1)}\right] \\
&= -\frac{1}{\alpha-\beta} \left[\left(\alpha^k - (-1)^t \alpha^{-k(2t+1)}\right)^n - \left(\beta^k - (-1)^t \beta^{-k(2t+1)}\right)^n\right],
\end{aligned}$$

which, Lemma 1 (i) and the Binet formula, equals

$$-\frac{1}{\alpha-\beta}\left[(-1)^{tn}V_{k(t+1)}^{n}\beta^{ktn}-(-1)^{tn}V_{k(t+1)}^{n}\alpha^{ktn}\right]=(-1)^{tn}V_{k(t+1)}^{n}U_{ktn},$$

as claimed.

Now we consider the case k is even. We write

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}U_{kti}V_{kn-ki(t+2)}\\&=\frac{1}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\left[\left(\alpha^{k(n-2i)}-\beta^{k(n-2i)}\right)-\left(\alpha^{kn-2ki(t+1)}-\beta^{kn-2ki(t+1)}\right)\right],\end{aligned}$$

which, after some arrangements similar to the previous one, equals

$$\alpha^{kn}\left(1-\alpha^{-2k}\right)^{n}-\beta^{kn}\left(1-\beta^{-2k}\right)^{n}-\alpha^{kn}\left(1-\alpha^{-2k(t+1)}\right)^{n}+\beta^{kn}\left(1-\beta^{-2k(t+1)}\right)^{n},$$

which, Lemma 1 (ii), equals

$$\begin{aligned}&\alpha^{kn}\left(1-\alpha^{-2k}\right)^{n}-\beta^{kn}\left(1-\beta^{-2k}\right)^{n}-\alpha^{kn}\left(1-\alpha^{-2k(t+1)}\right)^{n}+\beta^{kn}\left(1-\beta^{-2k(t+1)}\right)^{n}\\&=\frac{1}{\alpha-\beta}U_{k}^{n}\Delta^{\frac{n}{2}}\left[1-(-1)^{n}\right]+\frac{1}{\alpha-\beta}U_{k(t+1)}^{n}\Delta^{\frac{n}{2}}\left[(-1)^{n}\alpha^{ktn}-\beta^{ktn}\right]\\&=U_{k}^{n}\Delta^{\frac{n-1}{2}}\left[1-(-1)^{n}\right]+\Delta^{\frac{n-1}{2}}U_{k(t+1)}^{n}\left[(-1)^{n}\alpha^{ktn}-\beta^{ktn}\right],\end{aligned}$$

which gives the claimed result according to the case of n. □

For example, we have

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}U_{10i}V_{5(n-4i)}=V_{15}^{n}U_{10n},\qquad\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}U_{-3i}V_{3n-3i}=-V_{0}^{n}U_{3n}$$

and

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}F_{2i}L_{2(n-3i)}=\begin{cases}5^{(n-1)/2}\left(2-3^{n}L_{2n}\right)&\text{if } n \text{ is odd,}\\5^{\frac{n}{2}}3^{n}F_{2n}&\text{if } n \text{ is even.}\end{cases}$$

Lemma 2. *Let t be an integer.*

i) For odd k,

$$\begin{aligned}(-1)^{t}\alpha^{k(1-2t)}-\alpha^{k}&=(-1)^{t}U_{kt}\beta^{k(t-1)}\sqrt{\Delta},\\(-1)^{t}\beta^{k(1-2t)}-\beta^{k}&=(-1)^{t+1}U_{kt}\alpha^{k(t-1)}\sqrt{\Delta}.\end{aligned}$$

ii) For even k,

$$\alpha^{k(1-2t)}-\alpha^{k}=-U_{kt}\beta^{k(t-1)}\sqrt{\Delta},\quad\beta^{k(1-2t)}-\beta^{k}=U_{kt}\alpha^{k(t-1)}\sqrt{\Delta}.$$

Theorem 2. *For $n\geq 0$ and for odd k,*

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}U_{kti}V_{k(n-ti)}=U_{kt}^{n}\begin{cases}(-1)^{t}V_{kn(t-1)}\Delta^{\frac{n-1}{2}}&\text{if } n \text{ is odd,}\\U_{kn(t-1)}\Delta^{\frac{n}{2}}&\text{if } n \text{ is even,}\end{cases}$$

and for even k,

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}U_{kti}V_{kn-kti}=U_{kt}^{n}\begin{cases}-\Delta^{\frac{n-1}{2}}V_{k(t-1)n}&\text{if } n \text{ is odd,}\\\Delta^{\frac{n}{2}}U_{k(t-1)n}&\text{if } n \text{ is even.}\end{cases}$$

Proof. First assume that k is odd. We write

$$\begin{aligned}
&\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}U_{kti}V_{kn-kti}\\
&=\frac{1}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\left[\left(\alpha^{kn}-\beta^{kn}\right)-(-1)^{it}\left(\alpha^{kn-2ikt}-\beta^{kn-2ikt}\right)\right]\\
&=-\frac{1}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}(-1)^{i(t+1)}\left(\alpha^{kn-2ikt}-\beta^{kn-2ikt}\right)\\
&=-\frac{\alpha^{kn}}{\alpha-\beta}\left(1+(-1)^{t+1}\alpha^{-2kt}\right)^{n}+\frac{\beta^{kn}}{\alpha-\beta}\left(1+(-1)^{t+1}\beta^{-2kt}\right)^{n}\\
&=\frac{1}{\alpha-\beta}\left(\beta^{k}+(-1)^{t+1}\beta^{k(1-2t)}\right)^{n}-\frac{1}{\alpha-\beta}\left(\alpha^{k}+(-1)^{t+1}\alpha^{k(1-2t)}\right)^{n},
\end{aligned}$$

which, by Lemma 2 (ii), equals

$$\frac{1}{\alpha-\beta}U_{kt}^{n}\Delta^{\frac{n}{2}}\left[(-1)^{tn}\alpha^{kn(t-1)}-(-1)^{n}(-1)^{tn}\beta^{kn(t-1)}\right],$$

which gives the claim according to the case of n.

The case k is even could be obtained similarly. □

For example, for $k=1$, $t=2$ and the Fibonacci-Lucas case:

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}F_{2i}L_{n-2i}=\begin{cases}5^{\frac{n-1}{2}}V_{n} & \text{if } n \text{ is odd},\\ 5^{\frac{n}{2}}U_{n} & \text{if } n \text{ is even.}\end{cases}$$

Lemma 3. *i) For odd k and t,*

$$\begin{aligned}
\alpha^{t(k-1)}-\alpha^{t(k+1)} &= -V_{t}\alpha^{tk}, \quad \beta^{t(k-1)}-\beta^{t(k+1)}=-V_{t}\beta^{tk},\\
\alpha^{-t(k+1)}+\alpha^{t(k+1)} &= \beta^{-t(k+1)}+\beta^{t(k+1)}=V_{(k+1)t}.
\end{aligned}$$

ii) For odd k and even t,

$$\begin{aligned}
\alpha^{t(k-1)}-\alpha^{t(k+1)} &= -U_{t}\alpha^{tk}\sqrt{\Delta}, \quad \beta^{t(k-1)}-\beta^{t(k+1)}=U_{t}\beta^{tk}\sqrt{\Delta},\\
\alpha^{-t(k+1)}-\alpha^{t(k+1)} &= -U_{(k+1)t}\sqrt{\Delta}, \quad \beta^{-t(k+1)}-\beta^{t(k+1)}=U_{(k+1)t}\sqrt{\Delta}.
\end{aligned}$$

iii) For even k and t,

$$\begin{aligned}
\alpha^{t(k-1)}-\alpha^{(k+1)t}=-U_{t}\alpha^{kt}\sqrt{\Delta}, \quad \beta^{t(k-1)}-\beta^{(k+1)t}=U_{t}\beta^{kt}\sqrt{\Delta},\\
\alpha^{-t(k+1)}-\alpha^{t(k+1)}=-U_{(k+1)t}\sqrt{\Delta}, \quad \beta^{-t(k+1)}-\beta^{t(k+1)}=U_{(k+1)t}\sqrt{\Delta}.
\end{aligned}$$

iv) For even k and odd t,

$$\begin{aligned}
\alpha^{t(k-1)}-\alpha^{(k+1)t} &= -V_{t}\alpha^{kt}, \quad \beta^{t(k-1)}-\beta^{(k+1)t}=-V_{t}\beta^{kt},\\
\alpha^{-t(k+1)}-\alpha^{t(k+1)} &= -V_{(k+1)t}, \quad \beta^{-t(k+1)}-\beta^{t(k+1)}=-V_{(k+1)t}.
\end{aligned}$$

Similar to the previous results, we give the following result without proof.

Theorem 3. *For odd k,*

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}U_{tki}V_{(k+1)tn-(k+2)ti}$$
$$=\begin{cases} V_t^n U_{ktn} \text{ if } t \text{ is odd,} \\ \begin{cases} \Delta^{\frac{n-1}{2}}\left(U_t^n V_{ktn}-2U_{(k+1)t}^n\right) & \text{if } n \text{ is odd,} \\ \Delta^{\frac{n}{2}}U_t^n U_{ktn} & \text{if } n \text{ is even,}\end{cases}, \text{ if } t \text{ is even,}\end{cases}$$

and for even k,

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}U_{tki}V_{(k+1)tn-(k+2)ti}$$
$$=\begin{cases} V_t^n U_{ktn} \text{ if } t \text{ is odd,} \\ \Delta^{\frac{n-1}{2}}\left(U_t^n V_{ktn}-2U_{(k+1)t}^n\right) \quad \text{if } n \text{ is odd,} \quad \text{if } t \text{ is even.} \\ \Delta^{\frac{n}{2}}U_t^n U_{ktn} \quad \text{if } n \text{ is even,}\end{cases}$$

As consequences of the Lemma 1, we have the following results:

Theorem 4. *For even m,*

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}V_{2mi}=\begin{cases} \Delta^{\frac{n}{2}}V_{nm} & \text{if } n \text{ is even,} \\ -\Delta^{\frac{n+1}{2}}U_m^n U_{nm} & \text{if } n \text{ is odd.}\end{cases}$$

Proof. By Lemma 1 (ii), for even k, we have

$$\alpha^{-k(2t+1)}-\alpha^{k}=-\sqrt{\Delta}U_{k(t+1)}\beta^{kt} \text{ and } \beta^{-k(2t+1)}-\beta^{k}=\sqrt{\Delta}U_{k(t+1)}\alpha^{kt}$$

or

$$1-\alpha^{-2k(t+1)}=\sqrt{\Delta}U_{k(t+1)}\alpha^{-k}\beta^{kt} \text{ and } 1-\beta^{-2k(t+1)}=-\sqrt{\Delta}U_{k(t+1)}\beta^{-k}\alpha^{kt}$$

and write

$$\begin{aligned}\left(1-\alpha^{-2k(t+1)}\right)^{n} &= \Delta^{\frac{n}{2}}U_{k(t+1)}^{n}\alpha^{-kn}\beta^{knt}, \\ \left(1-\beta^{-2k(t+1)}\right)^{n} &= (-1)^{n}\Delta^{\frac{n}{2}}U_{k(t+1)}^{n}\beta^{-kn}\alpha^{knt},\end{aligned}$$

or

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\alpha^{-2k(t+1)i} &= \Delta^{\frac{n}{2}}U_{k(t+1)}^{n}\alpha^{-kn}\beta^{knt}, \\ \sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\beta^{-2k(t+1)i} &= (-1)^{n}\Delta^{\frac{n}{2}}U_{k(t+1)}^{n}\beta^{-kn}\alpha^{knt}.\end{aligned}$$

By adding these equalities side by side, we obtain

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\left(\alpha^{-2k(t+1)i}+\beta^{-2k(t+1)i}\right)=\Delta^{\frac{n}{2}}U_{k(t+1)}^{n}\left(\alpha^{-kn}\beta^{knt}+(-1)^{n}\beta^{-kn}\alpha^{knt}\right),$$

or since $V_{-k}=(-1)^{k}V_{k}$,

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}V_{-2k(t+1)i} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{i}V_{2k(t+1)i}$$

$$
\begin{aligned}
&= \Delta^{\frac{n}{2}} U_{k(t+1)}^{n} \left[(-1)^{n} \beta^{-kn} \alpha^{knt} + \alpha^{-kn} \beta^{knt} \right] \\
&= U_{k(t+1)}^{n} \begin{cases} \Delta^{\frac{n}{2}} V_{kn(t+1)} & \text{if } n \text{ is even,} \\ (-1)^{k+1} \Delta^{\frac{n+1}{2}} U_{kn(t+1)} & \text{if } n \text{ is odd,} \end{cases}
\end{aligned}
$$

which, by taking m instead of $k(t+1)$ for even k, completes the proof. □

Similar to the proof method of Theorem just above, we have the following results without proof by using Lemma 1 (i).

Theorem 5. *For any integer m and $n \geq 0$,*

$$
\sum_{i=0}^{n} \binom{n}{i} (-1)^{im} V_{2mi} = (-1)^{mn} V_m^n V_{nm},
$$

and

$$
\sum_{i=0}^{n} \binom{n}{i} (-1)^{im} U_{2mi} = (-1)^{nm} V_m^n U_{nm}.
$$

3. Acknowledgements

The author would like to thank the referee for a number of helpful suggestions.

References

[1] Carlitz, L. – *Some classes of Fibonacci sums*, The Fibonacci Quarterly 16 (5) (1978) 411–426.

[2] Graham, R. L.; Knuth, D. E.; Patashnik, O. – *Concrete Mathematics:* A Foundation for Computer Science. Addison-Wesley, 1992.

[3] Haukkanen, P. – *Formal power series for binomial sums of sequences of numbers*, The Fibonacci Quarterly 31 (1) (1993) 28–31.

[4] Kılıç, E.; Ulutaş Y. T.; Ömür, N. – *Formulas for weighted binomial sums with the powers of terms of binary recurrences*, Miskolc Math. Notes, 13 (1) (2012), 53–65.

[5] Kılıç, E.; Ömür, N. – *Some weighted binomial sums by generating function methods*, Integers, 2013, #A27.

[6] Kılıç, E.; Ömür, N.; Ulutaş Y. T. – *Binomial sums whose coefficients are products of terms of binary sequences*, Utilitas Mathematica 84 (2011), 45-52.

[7] Kılıç, E.; Ionascu, E. J. – *Certain binomial sums with recursive coefficients*, The Fibonacci Quarterly 48 (2) (2010), 161-167.

[8] Kılıç E.; Irmak, N. – *Binomial identities involving the generalized Fibonacci type polynomials*, Ars Combinatoria 98 (2011), 129-134.

[9] Koshy, T. – *Fibonacci and Lucas numbers with applications.* Pure and Applied Mathematics, Wiley-Interscience, New York, 2001.

[10] Prodinger, H. – *Some information about the Binomial transform*, The Fibonacci Quarterly 32 (5) (1994) 412-415.

[11] Vajda, S. – *Fibonacci & Lucas numbers, and the golden section.* John Wiley & Sons, Inc., New York, 1989.

TOBB University of Economics and Technology Mathematics Department 06560 Ankara Turkey

E-mail address: ekilic@etu.edu.tr