

# The generalized reciprocal super Catalan matrix

Emrah KILIÇ AND Talha ARIKAN\*

ABSTRACT. The reciprocal super Catalan matrix which studied by H. Prodinger is further generalized, introducing two additional parameters. Explicit formulæ are derived for the  $LU$ -decomposition and their inverses, as well as the Cholesky decomposition. The approach is to use  $q$ -analysis and to leave the justification of the necessary identities to the  $q$ -version of Zeilberger's celebrated algorithm.

## 1. INTRODUCTION

As mentioned in [8], there are many combinatorial matrices defined by a given sequence  $\{a_n\}$ . One of them is known as Hankel matrix and defined as follows:

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \vdots \\ \vdots & \vdots & \cdots & \ddots \end{bmatrix}$$

for more details see [6]. Considering some special number sequences instead of  $\{a_n\}$ , there are many special matrices with nice algebraic properties. Also some author, such as [10], studied the Hankel matrix considered the reciprocal sequence of  $\{a_n\}$

$$\begin{bmatrix} \frac{1}{a_0} & \frac{1}{a_1} & \frac{1}{a_2} & \cdots \\ \frac{1}{a_1} & \frac{1}{a_2} & \frac{1}{a_3} & \cdots \\ \frac{1}{a_2} & \frac{1}{a_2} & \frac{1}{a_4} & \vdots \\ \vdots & \vdots & \cdots & \ddots \end{bmatrix}.$$

For the sequence  $\{a_{i,j}\}$  it can be defined a matrix by taking  $(i, j)$ th entries  $a_{i,j}$ . Well-know types of these sequences typically include the

---

1991 *Mathematics Subject Classification.* 15B36, 15A15, 15A23, 11B65.

*Key words and phrases.* Determinant, inverse matrix,  $LU$  factorization, Gaussian  $q$ -binomial coefficient, Zeilberger's algorithm.

\* Corresponding author.

binomial coefficients. As examples, we give the family of Pascal matrices whose entries are defined via the usual binomial coefficients [2, 3]. The Pascal matrices are mainly two kinds: One of them is the left adjusted Pascal matrix  $P_n = (p_{ij})$  and the second one is the right adjusted Pascal matrix  $Q_n = (m_{ij})$ , where

$$p_{ij} = \binom{i}{j} \text{ and } m_{ij} = \binom{i}{n-1-j}, \quad 0 \leq i, j < n.$$

The Gaussian  $q$ -binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}},$$

where  $(x; q)_n$  is the  $q$ -Pochhammer symbol defined by

$$(x; q)_n = (1-x)(1-xq)\dots(1-xq^{n-1}).$$

Note that

$$\lim_{q \rightarrow 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k},$$

where  $\binom{n}{k}$  is the usual binomial coefficient.

We recall that one version of the *Cauchy binomial theorem* is given by

$$\sum_{k=0}^n q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = \prod_{k=1}^n (1+xq^k),$$

and *Rothe's formula* [1] is

$$\sum_{k=0}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = (x; q)_n = \prod_{k=0}^{n-1} (1-xq^k).$$

Recently, Prodinger [8] defined a matrix whose entries are consist of the super Catalan numbers. He also defined its reciprocal analogue as well as their  $q$ -versions that whose  $(i, j)$ th entries are defined by for  $0 \leq i, j < n$

$$\begin{aligned} & \binom{2i}{i}^{-1} \binom{2j}{j}^{-1} \binom{i+j}{i}, \\ & \binom{2i}{i} \binom{2j}{j} \binom{i+j}{i}^{-1}, \\ & \begin{bmatrix} 2i \\ i \end{bmatrix}_q^{-1} \begin{bmatrix} 2j \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} i+j \\ i \end{bmatrix}_q \end{aligned}$$

and

$$\begin{bmatrix} 2i \\ i \end{bmatrix}_q \begin{bmatrix} 2j \\ j \end{bmatrix}_q \begin{bmatrix} i+j \\ i \end{bmatrix}_q^{-1},$$

respectively. Then he gave some algebraic properties of these matrices.

Recently Kılıç et al. define and study a variant of the reciprocal super Catalan matrix with two additional parameters whose entries are defined as

$$\binom{2i+r}{i}^{-1} \binom{2j+s}{j}^{-1} \binom{i+j}{i}.$$

Explicit formulæ for its LU-decomposition, LU decomposition of its inverse and the Cholesky decomposition are obtained. For all results,  $q$ -analogues are also presented.

In this paper, for nonnegative integers  $r$  and  $s$ , we define two  $n \times n$  matrices  $M = [M_{kj}]$  and  $T = [T_{kj}]$  with entries

$$M_{kj} = \binom{k+j}{k} \binom{2k+r}{k}^{-1} \binom{2j+s}{j}^{-1}$$

and

$$T_{kj} = \binom{2k+r}{k} \binom{2j+s}{j} \binom{k+j}{k}^{-1}$$

for  $0 \leq k, j < n$ , respectively.

First, we give the matrices  $\mathcal{M}$  and  $\mathcal{T}$  which are the  $q$ -analogues of the matrices  $M$  and  $T$ , respectively. For both matrices, we derive explicit expressions for the  $LU$ -decomposition which leads to a formula for the determinant via  $\prod_{0 \leq i < n} U_{i,i}$ . Further, we have expressions for  $L^{-1}$  and

$U^{-1}$ , for  $LU$ -decomposition of the inverse matrix and their inverses, and for the Cholesky decomposition when the matrix is symmetric, that is the case  $r = s$ . Afterwards, when  $q \rightarrow 1$ , we get the results for the matrices  $M$  and  $T$ . Our results generalize the results of [8] when the case  $r = s = 0$ .

First we list the result related with the matrix  $\mathcal{M}$  in the next section and secondly prove them in the Section 3. Then we list results related to the matrix  $\mathcal{T}$  and give related proofs in next section. Finally we give the results related to the matrices  $M$  and  $T$  as special cases of the results related with the matrices  $\mathcal{M}$  and  $\mathcal{T}$ . To prove the claimed result, our main tool is to guess relevant quantities and then we will use the  $q$ -version of Zeilberger's celebrated algorithm (for more details see [7, 9]) and Rothe's formula to justify relevant equalities. All identities we will obtain hold for general  $q$  and generalized Fibonomial-Fibonacci

analogue of our results could be obtained by using the application of  $q$ -identities for Fibonacci numbers. We refer to [5] to give an idea.

## 2. THE MATRIX $\mathcal{M}$

We denote matrices  $L$  and  $U$  by  $A$  and  $B$  in  $LU$ -decomposition of any inverse matrix, respectively, that is,  $\mathcal{M}^{-1} = AB$ . For the Cholesky decomposition of a matrix  $G$ , we will use the letter  $C$  such that  $G = CC^T$ .

The matrix  $\mathcal{M}$  is defined with entries for  $0 \leq k, j < n$ ,

$$\mathcal{M}_{kj} = \begin{bmatrix} k+j \\ k \end{bmatrix}_q \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1}.$$

Firstly, we list here the formulas related with matrix  $\mathcal{M}$  that were found for  $0 \leq k, j < n$ :

$$L_{kj} = \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q,$$

$$L_{kj}^{-1} = (-1)^{k+j} q^{\binom{k-j}{2}} \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q,$$

$$U_{kj} = q^{k^2} \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} j \\ k \end{bmatrix}_q,$$

$$U_{kj}^{-1} = (-1)^{k+j} q^{k(k+1)/2-j(j+1)/2-kj} \begin{bmatrix} 2k+s \\ k \end{bmatrix}_q \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q,$$

$$\begin{aligned} A_{kj} &= (-1)^{k+j} q^{k(k+3)/2-j(j+3)/2-n(k-j)} \frac{1-q^{2j+1}}{1-q^{k+j+1}} \begin{bmatrix} n-j-1 \\ k-j \end{bmatrix}_q \begin{bmatrix} 2k+s \\ k \end{bmatrix}_q \\ &\times \begin{bmatrix} k+j \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ s \end{bmatrix}_q^{-1} \begin{bmatrix} j+s \\ s \end{bmatrix}_q, \end{aligned}$$

$$A_{kj}^{-1} = q^{(k-j)(k-n+1)} \begin{bmatrix} k+j \\ k \end{bmatrix}_q \begin{bmatrix} n-j-1 \\ k-j \end{bmatrix}_q \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} 2k+s \\ s \end{bmatrix}_q \begin{bmatrix} k+s \\ s \end{bmatrix}_q^{-1},$$

$$\begin{aligned} B_{kj} &= (-1)^{k+j} q^{(j+1)(j+2)/2-n(k+j+1)+3k(k+1)/2} \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q \begin{bmatrix} n+k \\ k+j+1 \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q \\ &\times \begin{bmatrix} 2k+s \\ s \end{bmatrix}_q \begin{bmatrix} k+s \\ s \end{bmatrix}_q^{-1}, \end{aligned}$$

$$B_{kj}^{-1} = q^{(k+j+1)(n-j-1)} \frac{1 - q^{2j+1}}{1 - q^{n-k}} \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} n+j \\ k+j \end{bmatrix}_q^{-1} \begin{bmatrix} j \\ k \end{bmatrix}_q \\ \times \begin{bmatrix} 2j+s \\ s \end{bmatrix}_q^{-1} \begin{bmatrix} j+s \\ s \end{bmatrix}_q,$$

for  $r = s$ ,

$$C_{kj} = q^{j^2/2} \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} k \\ j \end{bmatrix}_q$$

and

$$\det \mathcal{M} = q^{n(n-1)(2n-1)/6} \prod_{k=0}^{n-1} \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2k+s \\ k \end{bmatrix}_q^{-1}.$$

### 3. PROOFS RELATED WITH THE MATRIX $\mathcal{M}$

For  $L$  and  $L^{-1}$ ,

$$\sum_{j \leq d \leq k} L_{kd} L_{dj}^{-1} = \sum_{j \leq d \leq k} (-1)^{d+j} q^{\binom{d-j}{2}} \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2d+r \\ d \end{bmatrix}_q \begin{bmatrix} k \\ d \end{bmatrix}_q \\ \times \begin{bmatrix} 2d+r \\ d \end{bmatrix}_q^{-1} \begin{bmatrix} d \\ j \end{bmatrix}_q \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q \\ = \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q \sum_{0 \leq d \leq k-j} \begin{bmatrix} k-j \\ d \end{bmatrix}_q (-1)^d q^{\binom{d}{2}}.$$

By Rothe's formula if  $k \neq j$  then we have  $(1; q)_{k-j} = 0$ , and, if  $k = j$ , then the last sum on the RHS of the above equation is equal to 1. Thus we conclude

$$\sum_{j \leq d \leq k} L_{kd} L_{dj}^{-1} = \delta_{k,j},$$

where  $\delta_{k,j}$  is Kronecker delta, as claimed.

For  $U$  and  $U^{-1}$

$$\sum_{k \leq d \leq j} U_{kd} U_{dj}^{-1} = q^{k^2 - \binom{j+1}{2}} \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+r \\ j \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q \\ \times q^{k(2j+k)/2} (-1)^{k+j} \sum_{0 \leq d \leq j-k} \begin{bmatrix} j-k \\ d \end{bmatrix}_q (-1)^d q^{\binom{d+1}{2} + d(k-j)}.$$

By the Cauchy binomial theorem, if  $k \neq j$ , then the last sum on the RHS of the above equation equals to  $\prod_{d=1}^{j-k} (1 - q^{(k-j)+d}) = 0$ . So we have

$$\sum_{k \leq d \leq j} U_{kd} U_{dj}^{-1} = \delta_{k,j},$$

as desired.

For  $LU$ -decomposition, we have to prove that

$$\sum_{0 \leq d \leq \min\{k,j\}} L_{id} U_{dj} = \mathcal{M}_{kj}.$$

Consider,

$$\begin{aligned} \sum_{0 \leq d \leq \min\{k,j\}} L_{kd} U_{dj} &= \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1} (q; q)_k (q; q)_j \\ &\times \sum_{0 \leq d \leq k} q^{d^2} \frac{1}{(q; q)_d^2 (q; q)_{k-d} (q; q)_{j-d}}. \end{aligned}$$

Denote the last sum in the equation just above by  $\text{SUM}_k$ . The Mathematica version of the  $q$ -Zeilberger algorithm [7] produces the recursion

$$\text{SUM}_k = \frac{1 - q^{j+k}}{(1 - q^k)^2} \text{SUM}_{k-1}.$$

Since  $\text{SUM}_0 = (q; q)_k^{-1} (q; q)_j^{-1}$ , we obtain

$$\text{SUM}_k = (q; q)_k^{-1} (q; q)_j^{-1} \begin{bmatrix} k+j \\ k \end{bmatrix}_q.$$

So we get

$$\sum_{0 \leq d \leq \min\{k,j\}} L_{kd} U_{dj} = \mathcal{M}_{kj},$$

which completes the proof.

For  $A$  and  $A^{-1}$ , consider

$$\begin{aligned} \sum_{j \leq d \leq k} A_{kd} A_{dj}^{-1} &= (-1)^k q^{k(k+3)/2 - j + n(j-k)} \frac{(q; q)_{n-j-1}}{(q; q)_{n-k-1}} \\ &\times \begin{bmatrix} 2k+s \\ k \end{bmatrix}_q \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} k \\ j \end{bmatrix}_q \\ &\times \sum_{j \leq d \leq k} \begin{bmatrix} k-j \\ d-j \end{bmatrix}_q (-1)^d q^{d(d-1)/2 - jd} \frac{(q; q)_{d+j}}{(q; q)_{d-j}} \frac{1 - q^{2d+1}}{1 - q^{k+d+1}}. \end{aligned}$$

By the  $q$ -Zeilberger algorithm for the second sum in the last equation, we obtain that it is equal to 0 provided that  $k \neq j$ . If  $k = j$ , it is obvious that  $A_{kk}A_{kk}^{-1} = 1$ . Thus

$$\sum_{k \leq d \leq j} A_{kd}A_{dj}^{-1} = \delta_{k,j},$$

as claimed.

Similarly, we have

$$\sum_{k \leq d \leq j} B_{kd}B_{dj}^{-1} = \delta_{k,j}.$$

For the Cholesky decomposition, we examine the equation

$$\sum_{0 \leq d \leq \min\{k,j\}} C_{kd}C_{jd} = \mathcal{M}_{kj}.$$

Here

$$\sum_{0 \leq d \leq \min\{k,j\}} C_{kd}C_{jd} = \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1} \sum_{0 \leq d \leq \min\{k,j\}} q^{d^2} \begin{bmatrix} k \\ d \end{bmatrix}_q \begin{bmatrix} j \\ d \end{bmatrix}_q.$$

Note that the sum on the RHS of the equation just above is the same as the sum in the  $LU$ -decomposition which was proven before.

For the  $LU$ -decomposition of  $\mathcal{M}^{-1}$ , we should show that  $\mathcal{M}^{-1} = AB$  which is same as  $\mathcal{M} = B^{-1}A^{-1}$ . So it is sufficient to show that

$$\sum_{\max\{k,j\} \leq d \leq n-1} B_{kd}^{-1}A_{dj}^{-1} = \mathcal{M}_{kj}.$$

After some arrangements, we have

$$\begin{aligned} \sum_{\max\{k,j\} \leq d \leq n-1} B_{kd}^{-1}A_{dj}^{-1} &= \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q^{-1} \sum_{j \leq d \leq n-1} q^{(j+k+1)(n-1-d)} \\ &\quad \times \frac{1 - q^{2d+1}}{1 - q^{n-k}} \begin{bmatrix} d \\ k \end{bmatrix}_q \begin{bmatrix} n+d \\ k+d \end{bmatrix}_q^{-1} \begin{bmatrix} d+j \\ d \end{bmatrix}_q \begin{bmatrix} n-j-1 \\ d-j \end{bmatrix}_q, \end{aligned}$$

which, by replacing  $(n-1)$  with  $n$  and apart from the constants factors, equals to

$$\sum_{j \leq d \leq n} q^{(j+k+1)(n-d)} \frac{1 - q^{2d+1}}{1 - q^{n+1-k}} \begin{bmatrix} d \\ k \end{bmatrix}_q \begin{bmatrix} n+1+d \\ k+d \end{bmatrix}_q^{-1} \begin{bmatrix} d+j \\ d \end{bmatrix}_q \begin{bmatrix} n-j \\ d-j \end{bmatrix}_q.$$

Denote this sum by  $\text{SUM}_n$ . The  $q$ -Zeilberger algorithm gives the following recursion provided that  $k \neq n$  and  $j \neq n$

$$\text{SUM}_n = \text{SUM}_{n-1}.$$

So  $\text{SUM}_n = \text{SUM}_j = \begin{bmatrix} k+j \\ k \end{bmatrix}_q$  which completes the proof except the case  $(k, j) = (n-1, n-1)$ , which could be easily checked. Thus proof is completed.

#### 4. THE MATRIX $\mathcal{T}$

In this section, we will give the results related with the matrix  $\mathcal{T}$ . The matrix  $\mathcal{T}$  is defined with entries for  $0 \leq k, j < n$ ,

$$\mathcal{T}_{kj} = \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q \begin{bmatrix} k+j \\ k \end{bmatrix}_q^{-1}.$$

For  $0 \leq k, j < n$ , we have

$$L_{kj} = \begin{bmatrix} 2k+r \\ k+j \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q \begin{bmatrix} k+r \\ j \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} j+r \\ r \end{bmatrix}_q,$$

$$L_{kj}^{-1} = (-1)^{k+j} q^{\binom{k-j}{2}} \frac{1-q^{2k}}{1-q^{k+j}} \begin{bmatrix} k+j \\ k-j \end{bmatrix}_q \begin{bmatrix} 2k+r \\ r \end{bmatrix}_q \begin{bmatrix} k+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+r \\ r \end{bmatrix}_q^{-1} \\ \times \begin{bmatrix} j+r \\ r \end{bmatrix}_q, \text{ for } j \geq 1,$$

$$L_{k0}^{-1} = (-1)^k (1+q^k) q^{\binom{k}{2}} \begin{bmatrix} 2k+r \\ r \end{bmatrix}_q \begin{bmatrix} k+r \\ r \end{bmatrix}_q^{-1} \text{ and } L_{00}^{-1} = 1,$$

$$U_{kj} = (-1)^k q^{k(3k-1)/2} (1+q^k) \begin{bmatrix} 2j+s \\ k+j \end{bmatrix}_q \begin{bmatrix} 2k+r \\ r \end{bmatrix}_q \begin{bmatrix} j-k+s \\ s \end{bmatrix}_q \\ \times \begin{bmatrix} k+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} j+s \\ s \end{bmatrix}_q^{-1} \text{ for } k \geq 1, U_{0j} = \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q,$$

$$U_{kj}^{-1} = (-1)^k q^{k(k+1)/2-j(k+j)} \frac{1-q^j}{1-q^{k+j}} \begin{bmatrix} k+j \\ j-k \end{bmatrix}_q \begin{bmatrix} 2j+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} j+r \\ r \end{bmatrix}_q \\ \times \begin{bmatrix} 2k+s \\ s \end{bmatrix}_q^{-1} \begin{bmatrix} k+s \\ s \end{bmatrix}_q,$$

$$A_{kj} = (-1)^{k+j} q^{(k+1)(k+2)/2-(j+1)(j+2)/2+n(j-k)} \begin{bmatrix} k \\ j \end{bmatrix}_q \begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_q \\ \times \begin{bmatrix} n+j-1 \\ 2j \end{bmatrix}_q^{-1} \begin{bmatrix} k+s \\ s \end{bmatrix}_q \begin{bmatrix} 2k+s \\ s \end{bmatrix}_q^{-1} \begin{bmatrix} j+s \\ s \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ s \end{bmatrix}_q,$$



$$A_{kj}^{-1} = q^{(k-j)(k-n+1)} \begin{bmatrix} k \\ j \end{bmatrix}_q \begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_q \begin{bmatrix} n+j-1 \\ 2j \end{bmatrix}_q^{-1} \begin{bmatrix} k+s \\ s \end{bmatrix}_q \begin{bmatrix} 2k+s \\ s \end{bmatrix}_q^{-1} \\ \times \begin{bmatrix} j+s \\ s \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ s \end{bmatrix}_q,$$

$$B_{kj} = q^{(j+1)(j+2)/2-n(n-1)/2-jn+k^2-1} \begin{bmatrix} n+j-1 \\ 2j \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} 2k+s \\ k \end{bmatrix}_q^{-1} \\ \times \begin{bmatrix} j+r \\ r \end{bmatrix}_q \begin{bmatrix} 2j+r \\ r \end{bmatrix}_q^{-1},$$

$$B_{kj}^{-1} = (-1)^{n+j+1} q^{k-kj-j(j+1)/2+kn+n(n-1)/2} \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ s \end{bmatrix}_q \\ \times \begin{bmatrix} 2k+r \\ r \end{bmatrix}_q \begin{bmatrix} k+r \\ r \end{bmatrix}_q^{-1},$$

for  $r = s$  and  $j \geq 1$ ,

$$C_{kj} = \mathbf{i}^j (1+q)^{j/2} q^{j(3j-1)/4} \begin{bmatrix} 2k+r \\ k+j \end{bmatrix}_q \begin{bmatrix} k+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} k-j+r \\ r \end{bmatrix}_q,$$

where  $\mathbf{i} = \sqrt{-1}$  and for  $j = 0$ ,

$$C_{k0} = \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q$$

and

$$\det \mathcal{T} = (-1)^{\binom{n}{2}} \prod_{k=1}^{n-1} q^{k(3k-1)/2} \begin{bmatrix} 2k+s \\ 2k \end{bmatrix}_q \begin{bmatrix} 2k+r \\ r \end{bmatrix}_q \begin{bmatrix} k+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} k+s \\ s \end{bmatrix}_q^{-1}.$$

## 5. PROOFS RELATED WITH THE MATRIX $\mathcal{T}$

For  $L$  and  $L^{-1}$ , it should be shown

$$\sum_{j \leq d \leq k} L_{kd} L_{dj}^{-1} = \delta_{k,j}.$$

By the definitions of the matrices  $L$  and  $L^{-1}$ , for the case  $j = 0$ , we have

$$\sum_{0 \leq d \leq k} L_{k,d} L_{d,0}^{-1} = L_{k,d} L_{0,0}^{-1} + \sum_{1 \leq d \leq k} L_{k,d} L_{d,0}^{-1}.$$

If  $k = 0$ , we get 1 as  $(0, 0)$ th entry of matrix  $\mathcal{T}$ . If  $k > 0$ ,

$$\begin{aligned} \sum_{1 \leq d \leq k} L_{k,d} L_{d,0}^{-1} &= \sum_{0 \leq d \leq k-1} L_{k,d+1} L_{d+1,0}^{-1} = \sum_{0 \leq d \leq n} L_{n+1,d+1} L_{d+1,0}^{-1} \\ &= \sum_{0 \leq d \leq n} (-1)^{d+1} (1 + q^{d+1}) q^{(d^2+d)/2} \begin{bmatrix} 2n+2+r \\ n+d+2 \end{bmatrix}_q \\ &\quad \times \begin{bmatrix} n+1 \\ d+1 \end{bmatrix}_q \begin{bmatrix} n+1+r \\ d+1 \end{bmatrix}_q^{-1}, \end{aligned}$$

by using  $q$ -Zeilberger algorithm which is equal to  $-\begin{bmatrix} 2n+2+r \\ n+1 \end{bmatrix}_q$ . By changing  $n+1$  with  $k$  again, we get  $-\begin{bmatrix} 2k+r \\ k \end{bmatrix}_q$ . Finally if  $k > 0$ ,

$$\begin{aligned} \sum_{0 \leq d \leq k} L_{k,d} L_{d,0}^{-1} &= \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q + \sum_{1 \leq d \leq k} L_{k,d} L_{d,0}^{-1} \\ &= \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q - \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q = 0, \end{aligned}$$

as desired. For the case  $j > 0$ , we have

$$\begin{aligned} \sum_{j \leq d \leq k} L_{kd} L_{dj}^{-1} &= \sum_{j \leq d \leq k} (-1)^{d+j} q^{\binom{d-j}{2}} \frac{1 - q^{2d}}{1 - q^{d+j}} \begin{bmatrix} 2k+r \\ k+d \end{bmatrix}_q \begin{bmatrix} k \\ d \end{bmatrix}_q \\ &\quad \times \begin{bmatrix} k+r \\ d \end{bmatrix}_q^{-1} \begin{bmatrix} d+j \\ d-j \end{bmatrix}_q \begin{bmatrix} 2j+r \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} j+r \\ r \end{bmatrix}_q. \end{aligned}$$

By the  $q$ -Zeilberger algorithm, we obtain that it is equal to 0 provided that  $k \neq j$ . The case  $k = j$  could be easily checked. So

$$\sum_{j \leq d \leq k} L_{kd} L_{dj}^{-1} = \delta_{k,j},$$

which completes the proof.

Verification of the inverse of  $U$  could be similarly done. Inverses of the matrices  $A$  and  $B$  could be shown as in Section 3.

For  $LU$ -decomposition, we have to prove that

$$\sum_{0 \leq d \leq \min\{k,j\}} L_{kd} U_{dj} = \mathcal{T}_{kj}.$$

The cases  $k = 0$ ,  $0 \leq j < n$  and  $j = 0$ ,  $0 \leq k < n$  could be easily shown. For other cases, consider

$$\begin{aligned}
\sum_{0 \leq d \leq \min\{k,j\}} L_{kd} U_{dj} &= L_{k,0} U_{0,j} + \sum_{1 \leq d \leq \min\{k,j\}} L_{kd} U_{dj} = \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q \\
&+ \sum_{1 \leq d \leq \min\{k,j\}} (-1)^d (1+q^d) q^{(3d-1)d/2} \begin{bmatrix} 2k+r \\ k+d \end{bmatrix}_q \begin{bmatrix} k \\ d \end{bmatrix}_q \\
&\times \begin{bmatrix} k+r \\ d \end{bmatrix}_q^{-1} \begin{bmatrix} 2j+s \\ j+d \end{bmatrix}_q \begin{bmatrix} j-d+s \\ s \end{bmatrix}_q \begin{bmatrix} j+s \\ s \end{bmatrix}_q^{-1} \\
&= \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q + \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q \\
&\times \begin{bmatrix} 2k \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j \\ j \end{bmatrix}_q^{-1} \sum_{1 \leq d \leq \min\{k,j\}} (-1)^d (1+q^d) q^{(3d-1)d/2} \\
&\times \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q \begin{bmatrix} 2j \\ j+d \end{bmatrix}_q.
\end{aligned}$$

Without loss of generality, we may consider  $k < j$ . So consider the sum

$$\text{SUM}_k = \sum_{-k \leq d \leq k} (-1)^d (1+q^d) q^{(3d-1)d/2} \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q \begin{bmatrix} 2j \\ j+d \end{bmatrix}_q.$$

The  $q$ -Zeilberger algorithm gives the recurrence relation:

$$\text{SUM}_k = \frac{(1+q^k)(1-q^{2k-1})}{(1-q^{j+k})} \text{SUM}_{k-1}.$$

Since  $\text{SUM}_0 = 2 \begin{bmatrix} 2j \\ j \end{bmatrix}_q$ , we obtain

$$\text{SUM}_k = 2 \begin{bmatrix} 2k \\ k \end{bmatrix}_q \begin{bmatrix} 2j \\ j \end{bmatrix}_q \begin{bmatrix} k+j \\ k \end{bmatrix}_q^{-1}.$$

Since the summand of the  $\text{SUM}_k$  is symmetric with respect to  $k$  and  $-k$ , we have

$$\sum_{1 \leq d \leq k} (-1)^d (1+q^d) q^{(3d-1)d/2} \begin{bmatrix} 2k \\ k+d \end{bmatrix}_q \begin{bmatrix} 2j \\ j+d \end{bmatrix}_q = \frac{1}{2} \text{SUM}_k - \begin{bmatrix} 2k \\ k \end{bmatrix}_q \begin{bmatrix} 2j \\ j \end{bmatrix}_q.$$

Finally consider

$$\begin{aligned}
\sum_{0 \leq d \leq k} L_{kd} U_{dj} &= \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q + \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q \\
&\times \begin{bmatrix} 2k \\ k \end{bmatrix}_q^{-1} \begin{bmatrix} 2j \\ j \end{bmatrix}_q^{-1} \left( \frac{1}{2} \text{SUM}_k - \begin{bmatrix} 2k \\ k \end{bmatrix}_q \begin{bmatrix} 2j \\ j \end{bmatrix}_q \right) \\
&= \begin{bmatrix} 2k+r \\ k \end{bmatrix}_q \begin{bmatrix} 2j+s \\ j \end{bmatrix}_q \begin{bmatrix} k+j \\ k \end{bmatrix}_q^{-1} = \mathcal{T}_{kj},
\end{aligned}$$

as desired.

For  $LU$ -decomposition of the inverse of the matrix  $\mathcal{T}$ , it could be used similar argument in Section 3.

## 6. THE MATRIX $M$

Recall that the  $n \times n$  matrix  $M = [M_{kj}]$  is defined by for  $0 \leq k, j < n$  and nonnegative integers  $r$  and  $s$ ,

$$M_{kj} = \binom{k+j}{k} \binom{2k+r}{k}^{-1} \binom{2j+s}{j}^{-1}.$$

In the Section 2, by taking  $q \rightarrow 1$ , we get the following results for  $0 \leq k, j < n$ :

$$\begin{aligned}
L_{kj} &= \binom{2k+r}{k}^{-1} \binom{2j+r}{j} \binom{k}{j}, \\
L_{kj}^{-1} &= (-1)^{k+j} \binom{2k+r}{k} \binom{2j+r}{j} \binom{k}{j}, \\
U_{kj} &= \binom{2k+r}{k}^{-1} \binom{2j+s}{j}^{-1} \binom{j}{k}, \\
U_{kj}^{-1} &= (-1)^{k+j} \binom{2k+s}{k} \binom{2j+r}{j} \binom{j}{k}, \\
A_{kj} &= (-1)^{k+j} \frac{1+2j}{k+j+1} \binom{n-j-1}{k-j} \binom{2k+s}{k} \binom{k+j}{k}^{-1} \\
&\quad \times \binom{2j+s}{s}^{-1} \binom{j+s}{s}, \\
A_{kj}^{-1} &= \binom{k+j}{k} \binom{n-j-1}{k-j} \binom{2j+s}{j}^{-1} \binom{2k+s}{s} \binom{k+s}{s}^{-1}, \\
B_{kj} &= (-1)^{k+j} \binom{2j+r}{j} \binom{n+k}{k+j+1} \binom{j}{k} \binom{2k+s}{s} \binom{k+s}{s}^{-1},
\end{aligned}$$

$$B_{kj}^{-1} = \frac{2j+1}{n-k} \binom{2k+r}{k}^{-1} \binom{n+j}{k+j}^{-1} \binom{j}{k} \binom{2j+s}{s}^{-1} \binom{j+s}{s},$$

for  $r = s$ ,

$$C_{kj} = \binom{2k+r}{k}^{-1} \binom{k}{j}$$

and

$$\det \mathcal{M} = \prod_{k=0}^{n-1} \binom{2k+r}{k}^{-1} \binom{2k+s}{k}^{-1}.$$

## 7. THE MATRIX $T$

Recall that the  $n \times n$  matrix  $T = [T_{kj}]$  is defined by for  $0 \leq k, j < n$  and nonnegative integers  $r$  and  $s$ ,

$$T_{kj} = \binom{2k+r}{k} \binom{2j+s}{j} \binom{k+j}{k}^{-1}.$$

In the Section 4, by taking  $q \rightarrow 1$ , we obtain the following results. For  $0 \leq k, j < n$ ,

$$L_{kj} = \binom{2k+r}{k+j} \binom{k}{j} \binom{k+r}{j}^{-1} \binom{2j+r}{r}^{-1} \binom{j+r}{r},$$

for  $j \geq 1$ ,

$$L_{kj}^{-1} = (-1)^{k+j} \frac{2k}{k+j} \binom{k+j}{k-j} \binom{2k+r}{r} \binom{k+r}{r}^{-1} \binom{2j+r}{r}^{-1} \binom{j+r}{r},$$

$$L_{k0}^{-1} = 2(-1)^k \binom{2k+r}{r} \binom{k+r}{r}^{-1} \text{ and } L_{00}^{-1} = 1,$$

for  $k \geq 1$ ,

$$U_{kj} = (-1)^k 2 \binom{2j+s}{k+j} \binom{2k+r}{r} \binom{j-k+s}{s} \binom{k+r}{r}^{-1} \binom{j+s}{s}^{-1}$$

$$\text{and } U_{0j} = \binom{2j+s}{j},$$

$$U_{kj}^{-1} = (-1)^k \frac{j}{k+j} \binom{k+j}{j-k} \binom{2j+r}{r}^{-1} \binom{j+r}{r} \binom{2k+s}{s}^{-1} \binom{k+s}{s},$$

$$A_{kj} = (-1)^{k+j} \binom{k}{j} \binom{n+k-1}{2k} \binom{n+j-1}{2j}^{-1} \binom{2k+s}{s}^{-1} \binom{k+s}{s} \\ \times \binom{2j+s}{s} \binom{j+s}{s}^{-1},$$

$$A_{kj}^{-1} = \binom{k}{j} \binom{n+k-1}{2k} \binom{n+j-1}{2j}^{-1} \binom{2k+s}{s}^{-1} \binom{k+s}{s} \\ \times \binom{2j+s}{s} \binom{j+s}{s}^{-1},$$

$$B_{kj} = \binom{n+j-1}{2j} \binom{j}{k} \binom{2k+s}{k}^{-1} \binom{j+r}{r} \binom{2j+r}{r}^{-1},$$

$$B_{kj}^{-1} = (-1)^{n+j+1} \binom{j}{k} \binom{n+k-1}{2k}^{-1} \binom{2j+s}{s} \binom{2k+r}{r} \binom{k+r}{r}^{-1},$$

for  $r = s$  and  $j \geq 1$ ,

$$C_{kj} = (-2)^{j/2} \binom{2k+r}{k+j} \binom{k+r}{r}^{-1} \binom{k-j+r}{r},$$

for  $j = 0$ ,

$$C_{k0} = \binom{2k+r}{k}.$$

Thus

$$\det \mathcal{T} = (-1)^{\binom{n}{2}} \prod_{k=1}^{n-1} \binom{2k+s}{2k} \binom{2k+r}{r} \binom{k+r}{r}^{-1} \binom{k+s}{s}^{-1}.$$

#### REFERENCES

- [1] Andrews GE, Askey R, Roy R. Special functions. Cambridge: Cambridge University Press, 2000.
- [2] Carlitz L. The characteristic polynomial of a certain matrix of binomial coefficients. *The Fibonacci Quarterly* 1965; 3: 81–89.
- [3] Edelman A, Strang G. Pascal matrices. *Amer. Math. Mont.* 2004; 111: 189–197.
- [4] Kılıç E, Akkus I, Kızılaslan G, A variant of the reciprocal super Catalan matrix. *Special Matrices.* 2015; 3: 163–168.
- [5] Kılıç E, Prodinger H. Variants of the Filbert matrix. *The Fibonacci Quarterly* 2013; 51: 153–162.
- [6] Pan VY, Structured matrices and polynomials. New York, NY, USA: Springer-Verlag, 2001.
- [7] Paule P, Riese A. A Mathematica  $q$ -analogue of Zeilberger's algorithm based on an algebraically motivated approach to  $q$ -hypergeometric telescoping, in special functions,  $q$ -series and related topics, *Fields Inst. Commun.* 1997; 14: 179–210.
- [8] Prodinger H. The reciprocal super Catalan matrix. *Special Matrices.* 2015; 1: 111–117.
- [9] Petkovsek M, Wilf H, Zeilberger D. "A = B". Wellesley, MA: A K Peters, 1996.
- [10] Richardson TM. The Filbert matrix. *The Fibonacci Quarterly* 2001; 39: 268–275.

DEPARTMENT OF MATHEMATICS, TOBB ECONOMICS AND TECHNOLOGY UNIVERSITY 06560, ANKARA, ÇANKAYA, TURKEY

*E-mail address:* `ekilic@etu.edu.tr`

DEPARTMENT OF MATHEMATICS, HACETTEPE UNIVERSITY, ANKARA, ÇANKAYA, TURKEY

*E-mail address:* `tarikan@hacettepe.edu.tr`